

## BIPOLAR NEUTROSOPHIC GRAPH STRUCTURES

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**Abstract.** In this research study, we introduce the concept of bipolar single-valued neutrosophic graph structures. We discuss certain notions of bipolar single-valued neutrosophic graph structures with examples. We present some methods of construction of bipolar single-valued neutrosophic graph structures. We also investigate some of their prosperities.

*Key words and Phrases:* Graph structure, bipolar single-valued neutrosophic graph structure, operations.

**Abstrak.** Pada penelitian ini, kami memperkenalkan konsep struktur graf neutrosifik bipolar bernilai tunggal. Kami mengkaji ide-ide tertentu dari struktur graf neutrosifik bipolar bernilai tunggal beserta contoh-contohnya. Kami menyajikan beberapa metode konstruksi struktur graf neutrosifik bipolar bernilai tunggal. Kami juga memeriksa beberapa sifat-sifat mereka.

*Kata kunci:* Striktur graf, struktur graf neutrosifik bipolar bernilai tunggal, operasi

### 1. INTRODUCTION

Fuzzy graph theory has a number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. In 1973, Kauffmann [13] illustrated the notion of fuzzy graphs based on Zadeh's fuzzy relations [24]. Rosenfeld [16] discussed several basic graph-theoretic concepts, including bridges,

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cut-nodes, connectedness, trees and cycles. Later on, Bhattacharya [8] gave some remarks on fuzzy graphs. In 1994, Mordeson and Chang-Shyh [14] defined some operations on fuzzy graphs. The complement of fuzzy graph was defined in [14]. Further, this concept was discussed by Sunitha and Vijayakumar [20]. Akram described bipolar fuzzy graphs in 2011 [1]. Akram and Shahzadi [5] described the concept of neutrosophic soft graphs with applications. Dinesh and Ramakrishnan [12] introduced the concept of the fuzzy graph structure and investigated some related properties. Akram and Akmal [4] proposed the notion of bipolar fuzzy graph structures. On the other hand, Dhavaseelan et al. [10] defined strong neutrosophic graphs. Akram and Sarwar [3] portrayed bipolar neutrosophic graphs with applications. Akram and Shahzadi [5] introduced the notion of neutrosophic soft graphs with applications. Akram [2] introduced the notion of single-valued neutrosophic planar graphs. Representation of graphs using intuitionistic neutrosophic soft sets was discussed in [6]. Single-valued neutrosophic minimum spanning tree and its clustering method were studied by Ye [22]. In this research study, we introduce the concept of bipolar single-valued neutrosophic graph structures. We discuss certain notions of bipolar single-valued neutrosophic graph structures with examples. We present some methods of construction of bipolar single-valued neutrosophic graph structures. We also investigate some of their prosperities.

## 2. BIPOLAR SINGLE-VALUED NEUTROSOPHIC GRAPH STRUCTURES

Smarandache [19] introduced neutrosophic sets as a generalization of fuzzy sets and intuitionistic fuzzy sets. A neutrosophic set has three constituents: truth-membership, indeterminacy-membership and falsity-membership, in which each membership value is a real standard or non-standard subset of the unit interval  $]0^-, 1^+[$ . In real-life problems, neutrosophic sets can be applied more appropriately by using the single-valued neutrosophic sets defined by Smarandache [19] and Wang et al [21].

**Definition 2.1.** [19] A *neutrosophic set*  $N$  on a non-empty set  $V$  is an object of the form

$$N = \{(v, T_N(v), I_N(v), F_N(v)) : v \in V\}$$

where,  $T_N, I_N, F_N : V \rightarrow ]0^-, 1^+[$  and there is no restriction on the sum of  $T_N(v)$ ,  $I_N(v)$  and  $F_N(v)$  for all  $v \in V$ .

**Definition 2.2.** [21] A *single-valued neutrosophic set*  $N$  on a non-empty set  $V$  is an object of the form

$$N = \{(v, T_N(v), I_N(v), F_N(v)) : v \in V\}$$

where,  $T_N, I_N, F_N : V \rightarrow [0, 1]$  and sum of  $T_N(v)$ ,  $I_N(v)$  and  $F_N(v)$  is confined between 0 and 3 for all  $v \in V$ .

Deli et al. [9] defined bipolar neutrosophic sets a generalization of bipolar fuzzy sets. They also studied some operations and applications in decision making problems.

**Definition 2.3.** [9] A *bipolar single-valued neutrosophic set* on a non-empty set  $V$  is an object of the form

$$B = \{(v, T_B^P(v), I_B^P(v), F_B^P(v), T_B^N(v), I_B^N(v), F_B^N(v)) : v \in V\}$$

where,  $T_B^P, I_B^P, F_B^P : V \rightarrow [0, 1]$  and  $T_B^N, I_B^N, F_B^N : V \rightarrow [-1, 0]$ . The positive values  $T_B^P(v), I_B^P(v), F_B^P(v)$  denote the truth, indeterminacy and falsity membership values of an element  $v \in V$ , whereas negative values  $T_B^N(v), I_B^N(v), F_B^N(v)$  indicates the implicit counter property of truth, indeterminacy and falsity membership values of an element  $v \in V$ .

**Definition 2.4.** A *bipolar single-valued neutrosophic graph* on a non-empty set  $V$  is a pair  $G = (B, R)$ , where  $B$  is a bipolar single-valued neutrosophic set on  $V$  and  $R$  is a bipolar single-valued neutrosophic relation in  $V$  such that

$$\begin{aligned} T_R^P(bd) &\leq T_B^P(b) \wedge T_B^P(d), & I_R^P(bd) &\leq I_B^P(b) \wedge I_B^P(d), & F_R^P(bd) &\leq F_B^P(b) \vee F_B^P(d), \\ T_R^N(bd) &\geq T_B^N(b) \vee T_B^N(d), & I_R^N(bd) &\geq I_B^N(b) \vee I_B^N(d), & F_R^N(bd) &\geq F_B^N(b) \wedge F_B^N(d), \end{aligned}$$

for all  $b, d \in V$ .

We now define bipolar single-valued neutrosophic graph structure.

**Definition 2.5.**  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$  is called *bipolar single-valued neutrosophic graph structure (BSVNGS)* of graph structure  $\check{G}_s = (V, V_1, V_2, \dots, V_m)$  if  $B = \langle b, T^P(b), I^P(b), F^P(b), T^N(b), I^N(b), F^N(b) \rangle$  and  $B_k = \langle (b, d), T_k^P(b, d), I_k^P(b, d), F_k^P(b, d), T_k^N(b, d), I_k^N(b, d), F_k^N(b, d) \rangle$  are bipolar single-valued neutrosophic (BSVN) sets on  $V$  and  $V_k$ , respectively, such that

$$\begin{aligned} T_k^P(b, d) &\leq \min\{T^P(b), T^P(d)\}, & I_k^P(b, d) &\leq \min\{I^P(b), I^P(d)\}, \\ F_k^P(b, d) &\leq \max\{F^P(b), F^P(d)\}, & T_k^N(b, d) &\geq \max\{T^N(b), T^N(d)\}, \\ I_k^N(b, d) &\geq \max\{I^N(b), I^N(d)\}, & F_k^N(b, d) &\geq \min\{F^N(b), F^N(d)\}, \end{aligned}$$

for all  $b, d \in V$ . Note that  $0 \leq T_k^P(b, d) + I_k^P(b, d) + F_k^P(b, d) \leq 3$ ,  
 $-3 \leq T_k^N(b, d) + I_k^N(b, d) + F_k^N(b, d) \leq 0$  for all  $(b, d) \in V_k$ .

**Example 2.6.** Consider graph structure (GSR)  $\check{G}_s = (V, V_1, V_2)$  such that  $V = \{b_1, b_2, b_3, b_4\}$ ,  $V_1 = \{b_1b_3, b_1b_2, b_3b_4\}$ ,  $V_2 = \{b_1b_4, b_2b_3\}$ . By defining bipolar single-valued neutrosophic sets  $B, B_1$  and  $B_2$  on  $V, V_1$  and  $V_2$ , respectively, we can draw a bipolar SVNGS as depicted in Fig. 1.

**Definition 2.7.** Let  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$  be a BSVNGS of GS  $\check{G}_s$ . If  $\check{H}_{bn} = (B', B'_1, B'_2, \dots, B'_m)$  is a BSVNGS of  $\check{G}_s$  such that

$$\begin{aligned} T'^P(b) &\leq T^P(k), & I'^P(b) &\leq I^P(b), & F'^P(b) &\geq F^P(b), \\ T'^N(b) &\geq T^P(k), & I'^N(b) &\geq I^P(b), & F'^N(b) &\leq F^N(b), \end{aligned}$$

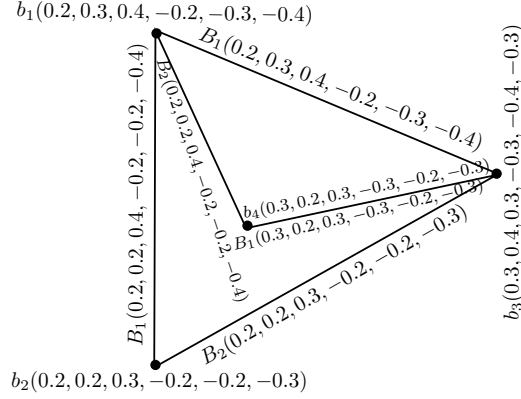


FIGURE 1. A bipolar single-valued neutrosophic graph structure

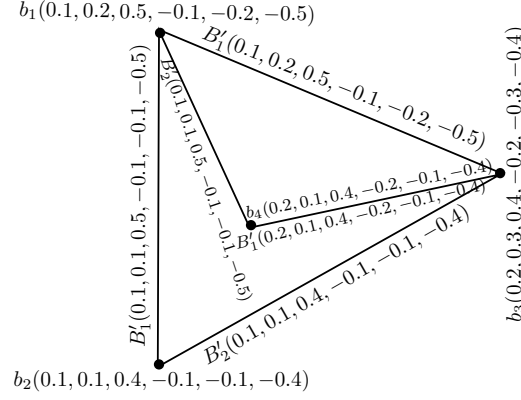


FIGURE 2. A BSVN subgraph structure

$$\begin{aligned}
 T_k^P(b, d) &\leq T_k^P(b, d), I_k^P(b, d) \leq I_k^P(b, d), F_k^P(b, d) \geq F_k^P(b, d), \\
 T_k^N(b, d) &\geq T_k^N(b, d), I_k^N(b, d) \geq I_k^N(b, d), F_k^N(b, d) \leq F_k^N(b, d), \\
 &\forall b \in V \text{ and } (b, d) \in V_k, k = 1, 2, \dots, m.
 \end{aligned}$$

Then  $\check{H}_{bn}$  is named as a *bipolar single-valued neutrosophic (BSVN) subgraph structure* of BSVNGS  $\check{G}_{bn}$ .

**Example 2.8.** Consider a BSVNGS  $\check{H}_{bn} = (B', B'_1, B'_2)$  of GS  $\check{G}_s = (V, V_1, V_2)$  as depicted in Fig. 2. Routine calculations indicate that  $\check{H}_{bn}$  is BSVN subgraph-structure of BSVNGS  $\check{G}_{bn}$ .

**Definition 2.9.** A BSVNGS  $\check{H}_{bn} = (B', B'_1, B'_2, \dots, B'_m)$  is called a *BSVN induced subgraph-structure* of BSVNGS  $\check{G}_{bn}$  by  $Q \subseteq V$  if

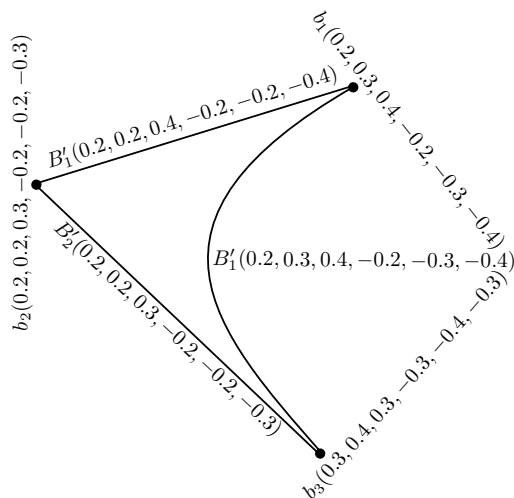


FIGURE 3. A BSVN induced subgraph-structure

$$\begin{aligned}
 T'^P(b) &= T^P(b), I'^P(b) = I^P(b), F'^P(b) = F^P(b), \\
 T'^N(b) &= T^N(b), I'^N(b) = I^N(b), F'^N(b) = F^N(b), \\
 T'_k{}^P(b, d) &= T_k^P(b, d), I'_k{}^P(b, d) = I_k^P(b, d), F'_k{}^P(b, d) = F_k^P(b, d), \\
 T'_k{}^N(b, d) &= T_k^N(b, d), I'_k{}^N(b, d) = I_k^N(b, d), F'_k{}^N(b, d) = F_k^N(b, d), \\
 &\forall b, d \in Q, k = 1, 2, \dots, m.
 \end{aligned}$$

**Example 2.10.** A BSVNGS depicted in Fig.3 is a BSVN induced subgraph-structure of BSVNGS represented in Fig. 1.

**Definition 2.11.** A BSVNGS  $\check{H}_{bn} = (B', B'_1, B'_2, \dots, B'_m)$  is called *BSVN spanning subgraph-structure* of BSVNGS  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$  if  $B' = B$  and

$$\begin{aligned}
 T'_k{}^P(b, d) &\leq T_k^P(b, d), I'_k{}^P(b, d) \leq I_k^P(b, d), F'_k{}^P(b, d) \geq F_k^P(b, d), \\
 T'_k{}^N(b, d) &\geq T_k^N(b, d), I'_k{}^N(b, d) \geq I_k^N(b, d), F'_k{}^N(b, d) \leq F_k^N(b, d), k = 1, 2, \dots, m.
 \end{aligned}$$

**Example 2.12.** A BSVNGS represented in Fig. 4 is a BSVN spanning subgraph-structure of BSVNGS represented in Fig. 1.

**Definition 2.13.** Let  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$  be a BSVNGS. Then  $bd \in B_k$  is called a *BSVN  $B_k$ -edge* or shortly  *$B_k$ -edge*, if

$$T_k^P(b, d) > 0 \text{ or } I_k^P(b, d) > 0 \text{ or } F_k^P(b, d) > 0 \text{ or } T_k^N(b, d) < 0 \text{ or } I_k^N(b, d) < 0 \text{ or } F_k^N(b, d) < 0 \text{ or all these conditions are satisfied.}$$

Consequently support of  $B_k$  is defined as:

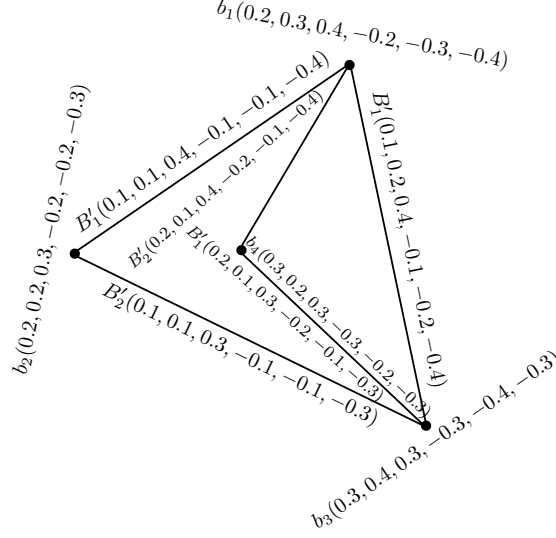


FIGURE 4. A BSVN spanning subgraph-structure

$$\text{supp}(B_k) = \{bd \in B_k : T_k^P(b, d) > 0\} \cup \{bd \in B_k : I_k^P(b, d) > 0\} \cup \{bd \in B_k : F_k^P(b, d) > 0\} \cup \{bd \in B_k : T_k^N(b, d) < 0\} \cup \{bd \in B_k : I_k^N(b, d) < 0\} \cup \{bd \in B_k : F_k^N(b, d) < 0\}, k = 1, 2, \dots, m.$$

**Definition 2.14.**  $B_k$ -path in BSVNGS  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$  is a sequence  $b_1, b_2, \dots, b_m$  of distinct nodes(vertices) (except  $b_m = b_1$ ) in  $V$  such that  $b_{k-1}b_k$  is a BSVN  $B_k$ -edge for all  $k = 2, \dots, m$ .

**Definition 2.15.** A BSVNGS  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$  is  $B_k$ -strong for any  $k \in \{1, 2, \dots, m\}$  if

$$\begin{aligned} T_k^P(b, d) &= \min\{T^P(b), T^P(d)\}, I_k^P(b, d) = \min\{I^P(b), I^P(d)\}, \\ F_k^P(b, d) &= \max\{F^P(b), F^P(d)\}, T_k^N(b, d) = \max\{T^N(b), T^N(d)\}, \\ I_k^N(b, d) &= \max\{I^N(b), I^N(d)\}, F_k^N(b, d) = \min\{F^N(b), F^N(d)\}, \end{aligned}$$

$\forall bd \in \text{supp}(B_k)$ . If  $\check{G}_{bn}$  is  $B_k$ -strong for all  $k \in \{1, 2, \dots, m\}$ , then  $\check{G}_{bn}$  is called strong BSVNGS.

**Example 2.16.** Consider BSVNGS  $\check{G}_{bn} = (B, B_1, B_2, B_3)$  as depicted in Fig. 5. Then  $\check{G}_{bn}$  is strong BSVNGS, since it is  $B_1$ -,  $B_2$ - and  $B_3$ - strong.

**Definition 2.17.** A BSVNGS  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$  is called complete BSVNGS if

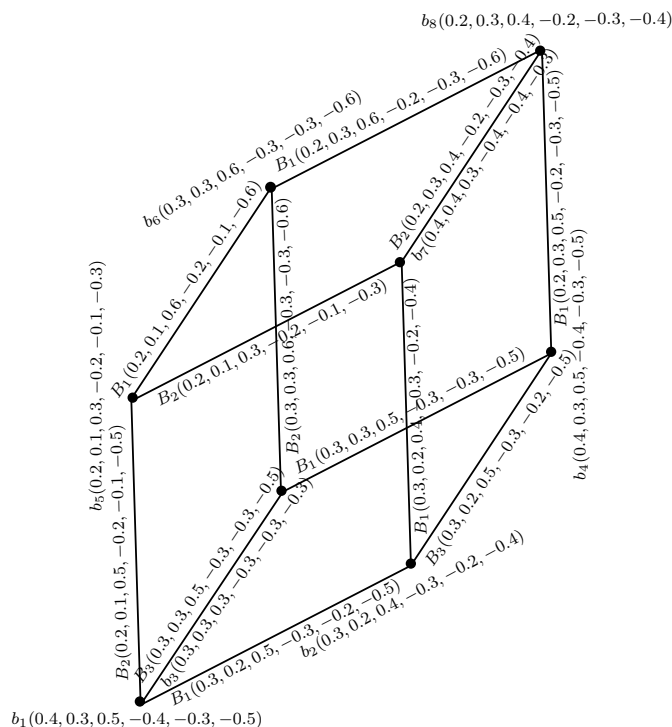


FIGURE 5. A Strong BSVNGS

- (1)  $\check{G}_{bn}$  is strong BSVNGS.
- (2)  $supp(B_k) \neq \emptyset$ , for all  $k = 1, 2, \dots, m$ .
- (3) For all  $b, d \in V$ ,  $bd$  is a  $B_k$  - edge for some  $k$ .

**Example 2.18.** Let  $\check{G}_{bn} = (B, B_1, B_2)$  be BSVNGS of GS  $\check{G} = (V, V_1, V_2)$ , such that  $V = \{b_1, b_2, b_3, b_4\}$ ,  $V_1 = \{b_1b_2, b_3b_4\}$ ,  $V_2 = \{b_1b_3, b_2b_3, b_1b_4, b_2b_4\}$ . Through direct calculations it is easily shown that  $\check{G}_{bn}$  is strong BSVNGS. Moreover,  $supp(B_1) \neq \emptyset$ ,  $supp(B_2) \neq \emptyset$  and each pair  $b_k b_l$  of nodes in  $V$  is either a  $B_1$ -edge or  $B_2$ -edge. Hence  $\check{G}_{bn}$  is complete BSVNGS, that is,  $B_1 - B_2$ -complete BSVNGS.

**Definition 2.19.** Let  $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \dots, B_{1m})$  and  $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \dots, B_{2m})$  be two BSVNGSs. *Lexicographic product* of  $\check{G}_{b1}$  and  $\check{G}_{b2}$ , denoted by

$$\check{G}_{b1} \bullet \check{G}_{b2} = (B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \dots, B_{1m} \bullet B_{2m}),$$

is defined as:

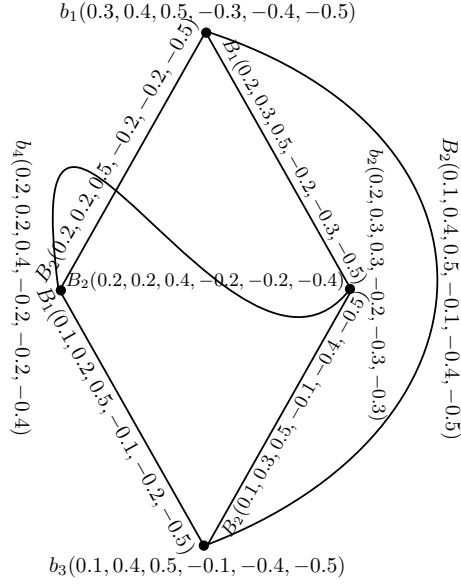


FIGURE 6. A complete BSVNGS

$$\begin{aligned}
 & \text{(i) } \begin{cases} T_{(B_1 \bullet B_2)}^P(bd) = (T_{B_1}^P \bullet T_{B_2}^P)(bd) = T_{B_1}^P(b) \wedge T_{B_2}^P(d) \\ I_{(B_1 \bullet B_2)}^P(bd) = (I_{B_1}^P \bullet I_{B_2}^P)(bd) = I_{B_1}^P(b) \wedge I_{B_2}^P(d) \\ F_{(B_1 \bullet B_2)}^P(bd) = (F_{B_1}^P \bullet F_{B_2}^P)(bd) = F_{B_1}^P(b) \vee F_{B_2}^P(d) \end{cases} \\
 & \text{(ii) } \begin{cases} T_{(B_1 \bullet B_2)}^N(bd) = (T_{B_1}^N \bullet T_{B_2}^N)(bd) = T_{B_1}^N(b) \vee T_{B_2}^N(d) \\ I_{(B_1 \bullet B_2)}^N(bd) = (I_{B_1}^N \bullet I_{B_2}^N)(bd) = I_{B_1}^N(b) \vee I_{B_2}^N(d) \\ F_{(B_1 \bullet B_2)}^N(bd) = (F_{B_1}^N \bullet F_{B_2}^N)(bd) = F_{B_1}^N(b) \wedge F_{B_2}^N(d) \end{cases} \\
 & \text{for all } (bd) \in V_1 \times V_2, \\
 & \text{(iii) } \begin{cases} T_{(B_{1k} \bullet B_{2k})}^P(bd_1)(bd_2) = (T_{B_{1k}}^P \bullet T_{B_{2k}}^P)(bd_1)(bd_2) = T_{B_{1k}}^P(b) \wedge T_{B_{2k}}^P(d_1 d_2) \\ I_{(B_{1k} \bullet B_{2k})}^P(bd_1)(bd_2) = (I_{B_{1k}}^P \bullet I_{B_{2k}}^P)(bd_1)(bd_2) = I_{B_{1k}}^P(b) \wedge I_{B_{2k}}^P(d_1 d_2) \\ F_{(B_{1k} \bullet B_{2k})}^P(bd_1)(bd_2) = (F_{B_{1k}}^P \bullet F_{B_{2k}}^P)(bd_1)(bd_2) = F_{B_{1k}}^P(b) \vee F_{B_{2k}}^P(d_1 d_2) \end{cases} \\
 & \text{(iv) } \begin{cases} T_{(B_{1k} \bullet B_{2k})}^N(bd_1)(bd_2) = (T_{B_{1k}}^N \bullet T_{B_{2k}}^N)(bd_1)(bd_2) = T_{B_{1k}}^N(b) \vee T_{B_{2k}}^N(d_1 d_2) \\ I_{(B_{1k} \bullet B_{2k})}^N(bd_1)(bd_2) = (I_{B_{1k}}^N \bullet I_{B_{2k}}^N)(bd_1)(bd_2) = I_{B_{1k}}^N(b) \vee I_{B_{2k}}^N(d_1 d_2) \\ F_{(B_{1k} \bullet B_{2k})}^N(bd_1)(bd_2) = (F_{B_{1k}}^N \bullet F_{B_{2k}}^N)(bd_1)(bd_2) = F_{B_{1k}}^N(b) \wedge F_{B_{2k}}^N(d_1 d_2) \end{cases} \\
 & \text{for all } b \in V_1, (d_1 d_2) \in V_{2k}, \\
 & \text{(v) } \begin{cases} T_{(B_{1k} \bullet B_{2k})}^P(b_1 d_1)(b_2 d_2) = (T_{B_{1k}}^P \bullet T_{B_{2k}}^P)(b_1 d_1)(b_2 d_2) = T_{B_{1k}}^P(b_1 b_2) \wedge T_{B_{2k}}^P(d_1 d_2) \\ I_{(B_{1k} \bullet B_{2k})}^P(b_1 d_1)(b_2 d_2) = (I_{B_{1k}}^P \bullet I_{B_{2k}}^P)(b_1 d_1)(b_2 d_2) = I_{B_{1k}}^P(b_1 b_2) \wedge I_{B_{2k}}^P(d_1 d_2) \\ F_{(B_{1k} \bullet B_{2k})}^P(b_1 d_1)(b_2 d_2) = (F_{B_{1k}}^P \bullet F_{B_{2k}}^P)(b_1 d_1)(b_2 d_2) = F_{B_{1k}}^P(b_1 b_2) \vee F_{B_{2k}}^P(d_1 d_2) \end{cases}
 \end{aligned}$$



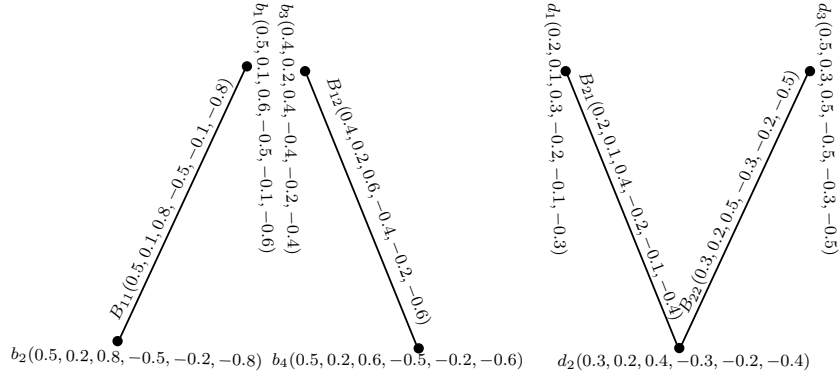
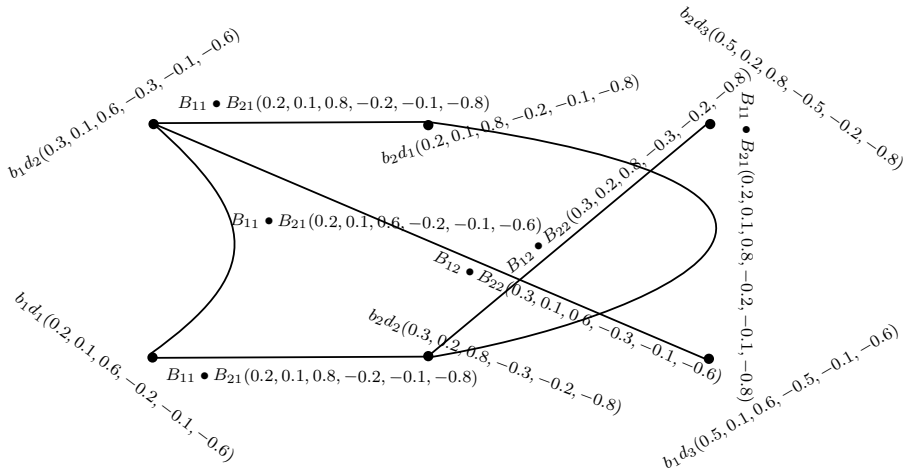


FIGURE 7. Two BSVNGSs  $\check{G}_{b_1}$  and  $\check{G}_{b_2}$

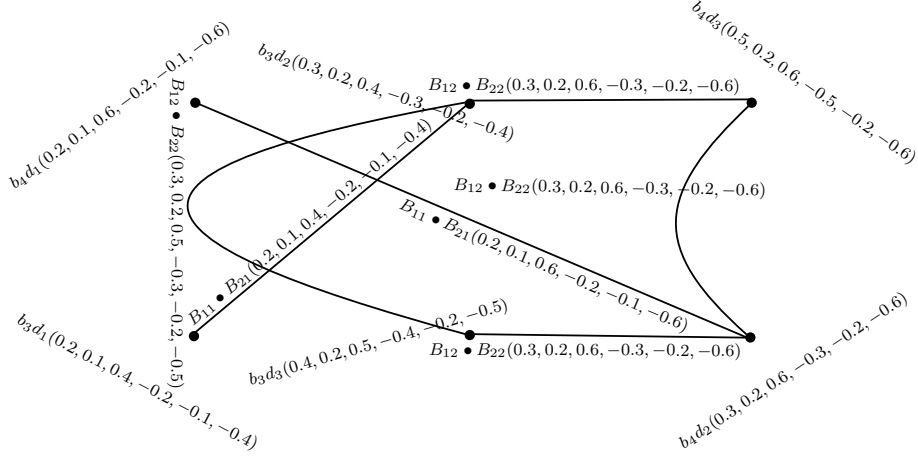


$$(vi) \begin{cases} T_{(B_{1k} \bullet B_{2k})}^N(b_1d_1)(b_2d_2) = (T_{B_{1k}}^N \bullet T_{B_{2k}}^N)(b_1d_1)(b_2d_2) = T_{B_{1k}}^N(b_1b_2) \vee T_{B_{2k}}^N(d_1d_2) \\ I_{(B_{1k} \bullet B_{2k})}^N(b_1d_1)(b_2d_2) = (I_{B_{1k}}^N \bullet I_{B_{2k}}^N)(b_1d_1)(b_2d_2) = I_{B_{1k}}^N(b_1b_2) \vee I_{B_{2k}}^N(d_1d_2) \\ F_{(B_{1k} \bullet B_{2k})}^N(b_1d_1)(b_2d_2) = (F_{B_{1k}}^N \bullet F_{B_{2k}}^N)(b_1d_1)(b_2d_2) = F_{B_{1k}}^N(b_1b_2) \wedge F_{B_{2k}}^N(d_1d_2) \end{cases}$$

for all  $(b_1b_2) \in V_{1k}$ ,  $(d_1d_2) \in V_{2k}$ .

**Example 2.20.** Consider  $\check{G}_{b_1} = (B_1, B_{11}, B_{12})$  and  $\check{G}_{b_2} = (B_2, B_{21}, B_{22})$  are two BSVNGSs of GSs  $\check{G}_{s_1} = (V_1, V_{11}, V_{12})$  and  $\check{G}_{s_2} = (V_2, V_{21}, V_{22})$ , respectively, as depicted in Fig. 7, where  $V_{11} = \{b_1b_2\}$ ,  $V_{12} = \{b_3b_4\}$ ,  $V_{21} = \{d_1d_2\}$ ,  $V_{22} = \{d_2d_3\}$ .

Lexicographic product of BSVNGSs  $\check{G}_{b_1}$  and  $\check{G}_{b_2}$  shown in Fig. 7 is defined as:  $\check{G}_{b_1} \bullet \check{G}_{b_2} = \{B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}\}$  and is depicted in Fig. 8.

FIGURE 8.  $\check{G}_{b1} \bullet \check{G}_{b2}$ 

**Theorem 2.21.** Lexicographic product  $\check{G}_{b1} \bullet \check{G}_{b2} = (B_1 \bullet B_2, B_{11} \bullet B_{21}, B_{12} \bullet B_{22}, \dots, B_{1m} \bullet B_{2m})$  of two BSVNSGSs of GSs  $\check{G}_{s1}$  and  $\check{G}_{s2}$  is a BSVNSGS of  $\check{G}_{s1} \bullet \check{G}_{s2}$ .

PROOF. Consider two cases:

**Case 1.:** For  $b \in V_1$ ,  $d_1 d_2 \in V_{2k}$

$$\begin{aligned}
 T_{(B_{1k} \bullet B_{2k})}^P((bd_1)(bd_2)) &= T_{B_1}^P(b) \wedge T_{B_{2k}}^P(d_1 d_2) \\
 &\leq T_{B_1}^P(b) \wedge [T_{B_2}^P(d_1) \wedge T_{B_2}^P(d_2)] \\
 &= [T_{B_1}^P(b) \wedge T_{B_2}^P(d_1)] \wedge [T_{B_1}^P(b) \wedge T_{B_2}^P(d_2)] \\
 &= T_{(B_1 \bullet B_2)}^P(bd_1) \wedge T_{(B_1 \bullet B_2)}^P(bd_2),
 \end{aligned}$$

$$\begin{aligned}
 T_{(B_{1k} \bullet B_{2k})}^N((bd_1)(bd_2)) &= T_{B_1}^N(b) \vee T_{B_{2k}}^N(d_1 d_2) \\
 &\geq T_{B_1}^N(b) \vee [T_{B_2}^N(d_1) \vee T_{B_2}^N(d_2)] \\
 &= [T_{B_1}^N(b) \vee T_{B_2}^N(d_1)] \vee [T_{B_1}^N(b) \vee T_{B_2}^N(d_2)] \\
 &= T_{(B_1 \bullet B_2)}^N(bd_1) \vee T_{(B_1 \bullet B_2)}^N(bd_2),
 \end{aligned}$$

$$\begin{aligned}
 I_{(B_{1k} \bullet B_{2k})}^P((bd_1)(bd_2)) &= I_{B_1}^P(b) \wedge I_{B_{2k}}^P(d_1 d_2) \\
 &\leq I_{B_1}^P(b) \wedge [I_{B_2}^P(d_1) \wedge I_{B_2}^P(d_2)] \\
 &= [I_{B_1}^P(b) \wedge I_{B_2}^P(d_1)] \wedge [I_{B_1}^P(b) \wedge I_{B_2}^P(d_2)] \\
 &= I_{(B_1 \bullet B_2)}^P(bd_1) \wedge I_{(B_1 \bullet B_2)}^P(bd_2),
 \end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \bullet B_{2k})}^N((bd_1)(bd_2)) &= I_{B_1}^N(b) \vee I_{B_{2k}}^N(d_1 d_2) \\
&\geq I_{B_1}^N(b) \vee [I_{B_2}^N(d_1) \vee I_{B_2}^N(d_2)] \\
&= [I_{B_1}^N(b) \vee I_{B_2}^N(d_1)] \vee [I_{B_1}^N(b) \vee I_{B_2}^N(d_2)] \\
&= I_{(B_1 \bullet B_2)}^N(bd_1) \vee I_{(B_1 \bullet B_2)}^N(bd_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \bullet B_{2k})}^P((bd_1)(bd_2)) &= F_{B_1}^P(b) \vee F_{B_{2k}}^P(d_1 d_2) \\
&\leq F_{B_1}^P(b) \vee [F_{B_2}^P(d_1) \vee F_{B_2}^P(d_2)] \\
&= [F_{B_1}^P(b) \vee F_{B_2}^P(d_1)] \vee [F_{B_1}^P(b) \vee F_{B_2}^P(d_2)] \\
&= F_{(B_1 \bullet B_2)}^P(bd_1) \vee F_{(B_1 \bullet B_2)}^P(bd_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \bullet B_{2k})}^N((bd_1)(bd_2)) &= F_{B_1}^N(b) \wedge F_{B_{2k}}^N(d_1 d_2) \\
&\geq F_{B_1}^N(b) \wedge [F_{B_2}^N(d_1) \wedge F_{B_2}^N(d_2)] \\
&= [F_{B_1}^N(b) \wedge F_{B_2}^N(d_1)] \wedge [F_{B_1}^N(b) \wedge F_{B_2}^N(d_2)] \\
&= F_{(B_1 \bullet B_2)}^N(bd_1) \wedge F_{(B_1 \bullet B_2)}^N(bd_2),
\end{aligned}$$

for  $bd_1, bd_2 \in V_1 \bullet V_2$ .

**Case 2.:** For  $b_1 b_2 \in V_{1k}, d_1 d_2 \in V_{2k}$

$$\begin{aligned}
T_{(B_{1k} \bullet B_{2k})}^P((b_1 d_1)(b_2 d_2)) &= T_{B_{1k}}^P(b_1 b_2) \wedge T_{B_{2k}}^P(d_1 d_2) \\
&\leq [T_{B_1}^P(b_1) \wedge T_{B_1}^P(b_2)] \wedge [T_{B_2}^P(d_1) \wedge T_{B_2}^P(d_2)] \\
&= [T_{B_1}^P(b_1) \wedge T_{B_2}^P(d_1)] \wedge [T_{B_1}^P(b_2) \wedge T_{B_2}^P(d_2)] \\
&= T_{(B_1 \bullet B_2)}^P(b_1 d_1) \wedge T_{(B_1 \bullet B_2)}^P(b_2 d_2),
\end{aligned}$$

$$\begin{aligned}
T_{(B_{1k} \bullet B_{2k})}^N((b_1 d_1)(b_2 d_2)) &= T_{B_{1k}}^N(b_1 b_2) \vee T_{B_{2k}}^N(d_1 d_2) \\
&\geq [T_{B_1}^N(b_1) \vee T_{B_1}^N(b_2)] \vee [T_{B_2}^N(d_1) \vee T_{B_2}^N(d_2)] \\
&= [T_{B_1}^N(b_1) \vee T_{B_2}^N(d_1)] \vee [T_{B_1}^N(b_2) \vee T_{B_2}^N(d_2)] \\
&= T_{(B_1 \bullet B_2)}^N(b_1 d_1) \vee T_{(B_1 \bullet B_2)}^N(b_2 d_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \bullet B_{2k})}^P((b_1 d_1)(b_2 d_2)) &= I_{B_{1k}}^P(b_1 b_2) \wedge I_{B_{2k}}^P(d_1 d_2) \\
&\leq [I_{B_1}^P(b_1) \wedge I_{B_1}^P(b_2)] \wedge [I_{B_2}^P(d_1) \wedge I_{B_2}^P(d_2)] \\
&= [I_{B_1}^P(b_1) \wedge I_{B_2}^P(d_1)] \wedge [I_{B_1}^P(b_2) \wedge I_{B_2}^P(d_2)] \\
&= I_{(B_1 \bullet B_2)}^P(b_1 d_1) \wedge I_{(B_1 \bullet B_2)}^P(b_2 d_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \bullet B_{2k})}^N((b_1 d_1)(b_2 d_2)) &= I_{B_{1k}}^N(b_1 b_2) \vee I_{B_{2k}}^N(d_1 d_2) \\
&\geq [I_{B_1}^N(b_1) \vee I_{B_1}^N(b_2) \vee [I_{B_2}^N(d_1) \vee I_{B_2}^N(d_2)]] \\
&= [I_{B_1}^N(b_1) \vee I_{B_2}^N(d_1)] \vee [I_{B_1}^N(b_2) \vee I_{B_2}^N(d_2)] \\
&= I_{(B_1 \bullet B_2)}^N(b_1 d_1) \vee I_{(B_1 \bullet B_2)}^N(b_2 d_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \bullet B_{2k})}^P((b_1 d_1)(b_2 d_2)) &= F_{B_{1k}}^P(b_1 b_2) \vee F_{B_{2k}}^P(d_1 d_2) \\
&\leq [F_{B_1}^P(b_1) \vee F_{B_1}^P(b_2) \vee [F_{B_2}^P(d_1) \vee F_{B_2}^P(d_2)]] \\
&= [F_{B_1}^P(b_1) \vee F_{B_2}^P(d_1)] \vee [F_{B_1}^P(b_2) \vee F_{B_2}^P(d_2)] \\
&= F_{(B_1 \bullet B_2)}^P(b_1 d_1) \vee F_{(B_1 \bullet B_2)}^P(b_2 d_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \bullet B_{2k})}^N((b_1 d_1)(b_2 d_2)) &= F_{B_{1k}}^N(b_1 b_2) \wedge F_{B_{2k}}^N(d_1 d_2) \\
&\geq [F_{B_1}^N(b_1) \wedge F_{B_1}^N(b_2) \wedge [F_{B_2}^N(d_1) \wedge F_{B_2}^N(d_2)]] \\
&= [F_{B_1}^N(b_1) \wedge F_{B_2}^N(d_1)] \wedge [F_{B_1}^N(b_2) \wedge F_{B_2}^N(d_2)] \\
&= F_{(B_1 \bullet B_2)}^N(b_1 d_1) \wedge F_{(B_1 \bullet B_2)}^N(b_2 d_2),
\end{aligned}$$

$b_1 d_1, b_2 d_2 \in V_1 \bullet V_2$  and  $h \in \{1, 2, \dots, m\}$ . This completes the proof.

**Definition 2.22.** Let  $\check{G}_{b_1} = (B_1, B_{11}, B_{12}, \dots, B_{1m})$  and  $\check{G}_{b_2} = (B_2, B_{21}, B_{22}, \dots, B_{2m})$  be two BSVNGSs. *Strong product* of  $\check{G}_{b_1}$  and  $\check{G}_{b_2}$ , denoted by

$$\check{G}_{b_1} \boxtimes \check{G}_{b_2} = (B_1 \boxtimes B_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \dots, B_{1m} \boxtimes B_{2m}),$$

is defined as:

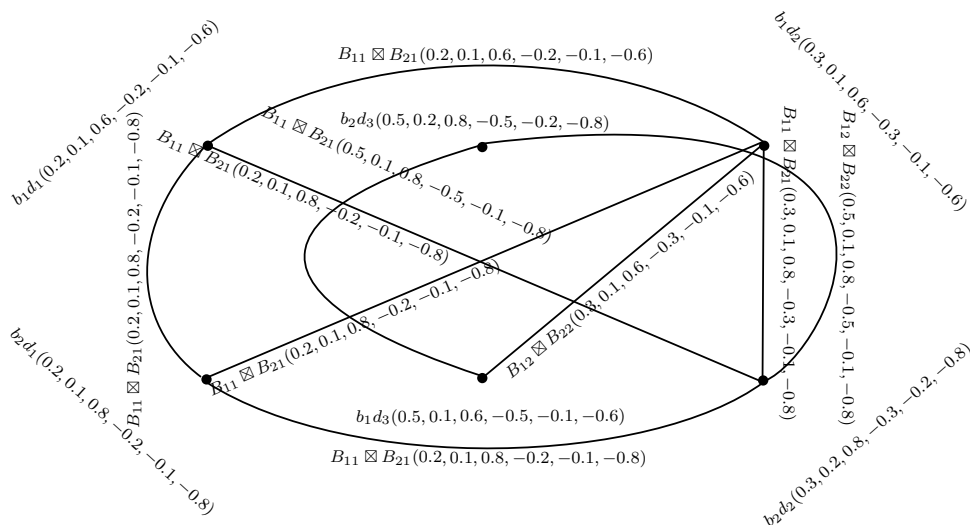
$$\begin{aligned}
\text{(i)} \quad &\begin{cases} T_{(B_1 \boxtimes B_2)}^P(bd) = (T_{B_1}^P \boxtimes T_{B_2}^P)(bd) = T_{B_1}^P(b) \wedge T_{B_2}^P(d) \\ I_{(B_1 \boxtimes B_2)}^P(bd) = (I_{B_1}^P \boxtimes I_{B_2}^P)(bd) = I_{B_1}^P(b) \wedge I_{B_2}^P(d) \\ F_{(B_1 \boxtimes B_2)}^P(bd) = (F_{B_1}^P \boxtimes F_{B_2}^P)(bd) = F_{B_1}^P(b) \vee F_{B_2}^P(d) \end{cases} \\
\text{(ii)} \quad &\begin{cases} T_{(B_1 \boxtimes B_2)}^N(bd) = (T_{B_1}^N \boxtimes T_{B_2}^N)(bd) = T_{B_1}^N(b) \vee T_{B_2}^N(d) \\ I_{(B_1 \boxtimes B_2)}^N(bd) = (I_{B_1}^N \boxtimes I_{B_2}^N)(bd) = I_{B_1}^N(b) \vee I_{B_2}^N(d) \\ F_{(B_1 \boxtimes B_2)}^N(bd) = (F_{B_1}^N \boxtimes F_{B_2}^N)(bd) = F_{B_1}^N(b) \wedge F_{B_2}^N(d) \end{cases}
\end{aligned}$$

for all  $(bd) \in V_1 \times V_2$ ,

$$\begin{aligned}
\text{(iii)} \quad &\begin{cases} T_{(B_{1k} \boxtimes B_{2k})}^P(bd_1)(bd_2) = (T_{B_{1k}}^P \boxtimes T_{B_{2k}}^P)(bd_1)(bd_2) = T_{B_{1k}}^P(b) \wedge T_{B_{2k}}^P(d_1 d_2) \\ I_{(B_{1k} \boxtimes B_{2k})}^P(bd_1)(bd_2) = (I_{B_{1k}}^P \boxtimes I_{B_{2k}}^P)(bd_1)(bd_2) = I_{B_{1k}}^P(b) \wedge I_{B_{2k}}^P(d_1 d_2) \\ F_{(B_{1k} \boxtimes B_{2k})}^P(bd_1)(bd_2) = (F_{B_{1k}}^P \boxtimes F_{B_{2k}}^P)(bd_1)(bd_2) = F_{B_{1k}}^P(b) \vee F_{B_{2k}}^P(d_1 d_2) \end{cases} \\
\text{(iv)} \quad &\begin{cases} T_{(B_{1k} \boxtimes B_{2k})}^N(bd_1)(bd_2) = (T_{B_{1k}}^N \boxtimes T_{B_{2k}}^N)(bd_1)(bd_2) = T_{B_{1k}}^N(b) \vee T_{B_{2k}}^N(d_1 d_2) \\ I_{(B_{1k} \boxtimes B_{2k})}^N(bd_1)(bd_2) = (I_{B_{1k}}^N \boxtimes I_{B_{2k}}^N)(bd_1)(bd_2) = I_{B_{1k}}^N(b) \vee I_{B_{2k}}^N(d_1 d_2) \\ F_{(B_{1k} \boxtimes B_{2k})}^N(bd_1)(bd_2) = (F_{B_{1k}}^N \boxtimes F_{B_{2k}}^N)(bd_1)(bd_2) = F_{B_{1k}}^N(b) \wedge F_{B_{2k}}^N(d_1 d_2) \end{cases}
\end{aligned}$$

for all  $b \in V_1$ ,  $(d_1 d_2) \in V_{2k}$ ,

$$\begin{aligned}
\text{(v)} \quad &\begin{cases} T_{(B_{1k} \boxtimes B_{2k})}^P(b_1 d)(b_2 d) = (T_{B_{1k}}^P \boxtimes T_{B_{2k}}^P)(b_1 d)(b_2 d) = T_{B_2}^P(d) \wedge T_{B_{1k}}^P(b_1 b_2) \\ I_{(B_{1k} \boxtimes B_{2k})}^P(b_1 d)(b_2 d) = (I_{B_{1k}}^P \boxtimes I_{B_{2k}}^P)(b_1 d)(b_2 d) = I_{B_2}^P(d) \wedge I_{B_{2k}}^P(b_1 b_2) \\ F_{(B_{1k} \boxtimes B_{2k})}^P(b_1 d)(b_2 d) = (F_{B_{1k}}^P \boxtimes F_{B_{2k}}^P)(b_1 d)(b_2 d) = F_{B_2}^P(d) \vee F_{B_{2k}}^P(b_1 b_2) \end{cases}
\end{aligned}$$

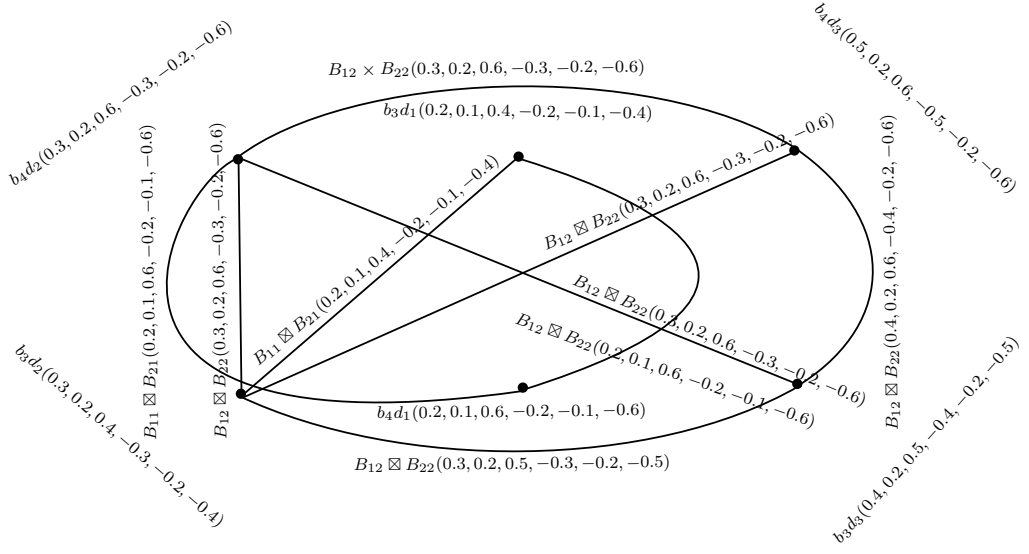
FIGURE 9.  $\check{G}_{b_1} \boxtimes \check{G}_{b_2}$ 

$$\begin{aligned}
 \text{(vi)} \quad & \left\{ \begin{aligned} T_{(B_{1k} \boxtimes B_{2k})}^N(b_1 d)(b_2 d) &= (T_{B_{1k}}^N \boxtimes T_{B_{2k}}^N)(b_1 d)(b_2 d) = T_{B_2}^N(d) \vee T_{B_{1k}}^N(b_1 b_2) \\ I_{(B_{1k} \boxtimes B_{2k})}^N(b_1 d)(b_2 d) &= (I_{B_{1k}}^N \boxtimes I_{B_{2k}}^N)(b_1 d)(b_2 d) = I_{B_2}^N(d) \vee I_{B_{2k}}^N(b_1 b_2) \\ F_{(B_{1k} \boxtimes B_{2k})}^N(b_1 d)(b_2 d) &= (F_{B_{1k}}^N \boxtimes F_{B_{2k}}^N)(b_1 d)(b_2 d) = F_{B_2}^N(d) \wedge F_{B_{2k}}^N(b_1 b_2) \end{aligned} \right. \\
 & \text{for all } d \in V_2, (b_1 b_2) \in V_{1k}. \\
 \text{(vii)} \quad & \left\{ \begin{aligned} T_{(B_{1k} \boxtimes B_{2k})}^P(b_1 d_1)(b_2 d_2) &= (T_{B_{1k}}^P \boxtimes T_{B_{2k}}^P)(b_1 d_1)(b_2 d_2) = T_{B_{1k}}^P(b_1 b_2) \wedge T_{B_{2k}}^P(d_1 d_2) \\ I_{(B_{1k} \boxtimes B_{2k})}^P(b_1 d_1)(b_2 d_2) &= (I_{B_{1k}}^P \boxtimes I_{B_{2k}}^P)(b_1 d_1)(b_2 d_2) = I_{B_{1k}}^P(b_1 b_2) \wedge I_{B_{2k}}^P(d_1 d_2) \\ F_{(B_{1k} \boxtimes B_{2k})}^P(b_1 d_1)(b_2 d_2) &= (F_{B_{1k}}^P \boxtimes F_{B_{2k}}^P)(b_1 d_1)(b_2 d_2) = F_{B_{1k}}^P(b_1 b_2) \vee F_{B_{2k}}^P(d_1 d_2) \end{aligned} \right. \\
 \text{(viii)} \quad & \left\{ \begin{aligned} T_{(B_{1k} \boxtimes B_{2k})}^N(b_1 d_1)(b_2 d_2) &= (T_{B_{1k}}^N \boxtimes T_{B_{2k}}^N)(b_1 d_1)(b_2 d_2) = T_{B_{1k}}^N(b_1 b_2) \vee T_{B_{2k}}^N(d_1 d_2) \\ I_{(B_{1k} \boxtimes B_{2k})}^N(b_1 d_1)(b_2 d_2) &= (I_{B_{1k}}^N \boxtimes I_{B_{2k}}^N)(b_1 d_1)(b_2 d_2) = I_{B_{1k}}^N(b_1 b_2) \vee I_{B_{2k}}^N(d_1 d_2) \\ F_{(B_{1k} \boxtimes B_{2k})}^N(b_1 d_1)(b_2 d_2) &= (F_{B_{1k}}^N \boxtimes F_{B_{2k}}^N)(b_1 d_1)(b_2 d_2) = F_{B_{1k}}^N(b_1 b_2) \wedge F_{B_{2k}}^N(d_1 d_2) \end{aligned} \right. \\
 & \text{for all } (b_1 b_2) \in V_{1k}, (d_1 d_2) \in V_{2k}.
 \end{aligned}$$

**Example 2.23.** Strong product of BSVNGSs  $\check{G}_{b_1}$  and  $\check{G}_{b_2}$  shown in Fig. 7 is defined as  $\check{G}_{b_1} \boxtimes \check{G}_{b_2} = \{B_1 \boxtimes B_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}\}$  and is depicted in Fig. 9 and 10.

**Theorem 2.24.** Strong product  $\check{G}_{b_1} \boxtimes \check{G}_{b_2} = (B_1 \boxtimes B_2, B_{11} \boxtimes B_{21}, B_{12} \boxtimes B_{22}, \dots, B_{1m} \boxtimes B_{2m})$  of two BSVNGSs of GSs  $\check{G}_{s_1}$  and  $\check{G}_{s_2}$  is a BSVNGS of  $\check{G}_{s_1} \boxtimes \check{G}_{s_2}$ .

PROOF. Consider three cases:

FIGURE 10.  $\check{G}_{b_1} \boxtimes \check{G}_{b_2}$ 

**Case 1.:** For  $b \in V_1$ ,  $d_1 d_2 \in V_{2k}$

$$\begin{aligned}
 T_{(B_{1k} \boxtimes B_{2k})}^P((bd_1)(bd_2)) &= T_{B_1}^P(b) \wedge T_{B_{2k}}^P(d_1 d_2) \\
 &\leq T_{B_1}^P(b) \wedge [T_{B_2}^P(d_1) \wedge T_{B_2}^P(d_2)] \\
 &= [T_{B_1}^P(b) \wedge T_{B_2}^P(d_1)] \wedge [T_{B_1}^P(b) \wedge T_{B_2}^P(d_2)] \\
 &= T_{(B_1 \boxtimes B_2)}^P(bd_1) \wedge T_{(B_1 \boxtimes B_2)}^P(bd_2),
 \end{aligned}$$

$$\begin{aligned}
 T_{(B_{1k} \boxtimes B_{2k})}^N((bd_1)(bd_2)) &= T_{B_1}^N(b) \vee T_{B_{2k}}^N(d_1 d_2) \\
 &\geq T_{B_1}^N(b) \vee [T_{B_2}^N(d_1) \vee T_{B_2}^N(d_2)] \\
 &= [T_{B_1}^N(b) \vee T_{B_2}^N(d_1)] \vee [T_{B_1}^N(b) \vee T_{B_2}^N(d_2)] \\
 &= T_{(B_1 \boxtimes B_2)}^N(bd_1) \vee T_{(B_1 \boxtimes B_2)}^N(bd_2),
 \end{aligned}$$

$$\begin{aligned}
 I_{(B_{1k} \boxtimes B_{2k})}^P((bd_1)(bd_2)) &= I_{B_1}^P(b) \wedge I_{B_{2k}}^P(d_1 d_2) \\
 &\leq I_{B_1}^P(b) \wedge [I_{B_2}^P(d_1) \wedge I_{B_2}^P(d_2)] \\
 &= [I_{B_1}^P(b) \wedge I_{B_2}^P(d_1)] \wedge [I_{B_1}^P(b) \wedge I_{B_2}^P(d_2)] \\
 &= I_{(B_1 \boxtimes B_2)}^P(bd_1) \wedge I_{(B_1 \boxtimes B_2)}^P(bd_2),
 \end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \boxtimes B_{2k})}^N((bd_1)(bd_2)) &= I_{B_1}^N(b) \vee I_{B_{2k}}^N(d_1 d_2) \\
&\geq I_{B_1}^N(b) \vee [I_{B_2}^N(d_1) \vee I_{B_2}^N(d_2)] \\
&= [I_{B_1}^N(b) \vee I_{B_2}^N(d_1)] \vee [I_{B_1}^N(b) \vee I_{B_2}^N(d_2)] \\
&= I_{(B_1 \boxtimes B_2)}^N(bd_1) \vee I_{(B_1 \boxtimes B_2)}^N(bd_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \boxtimes B_{2k})}^P((bd_1)(bd_2)) &= F_{B_1}^P(b) \vee F_{B_{2k}}^P(d_1 d_2) \\
&\leq F_{B_1}^P(b) \vee [F_{B_2}^P(d_1) \vee F_{B_2}^P(d_2)] \\
&= [F_{B_1}^P(b) \vee F_{B_2}^P(d_1)] \vee [F_{B_1}^P(b) \vee F_{B_2}^P(d_2)] \\
&= F_{(B_1 \boxtimes B_2)}^P(bd_1) \vee F_{(B_1 \boxtimes B_2)}^P(bd_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \boxtimes B_{2k})}^N((bd_1)(bd_2)) &= F_{B_1}^N(b) \wedge F_{B_{2k}}^N(d_1 d_2) \\
&\geq F_{B_1}^N(b) \wedge [F_{B_2}^N(d_1) \wedge F_{B_2}^N(d_2)] \\
&= [F_{B_1}^N(b) \wedge F_{B_2}^N(d_1)] \wedge [F_{B_1}^N(b) \wedge F_{B_2}^N(d_2)] \\
&= F_{(B_1 \boxtimes B_2)}^N(bd_1) \wedge F_{(B_1 \boxtimes B_2)}^N(bd_2),
\end{aligned}$$

for  $bd_1, bd_2 \in V_1 \boxtimes V_2$ .

**Case 2.:** For  $b \in V_2, d_1 d_2 \in V_{1k}$

$$\begin{aligned}
T_{(B_{1k} \boxtimes B_{2k})}^P((d_1 b)(d_2 b)) &= T_{B_2}^P(b) \wedge T_{B_{1k}}^P(d_1 d_2) \\
&\leq T_{B_2}^P(b) \wedge [T_{B_1}^P(d_1) \wedge T_{B_1}^P(d_2)] \\
&= [T_{B_2}^P(b) \wedge T_{B_1}^P(d_1)] \wedge [T_{B_2}^P(b) \wedge T_{B_1}^P(d_2)] \\
&= T_{(B_1 \boxtimes B_2)}^P(d_1 b) \wedge T_{(B_1 \boxtimes B_2)}^P(d_2 b),
\end{aligned}$$

$$\begin{aligned}
T_{(B_{1k} \boxtimes B_{2k})}^N((d_1 b)(d_2 b)) &= T_{B_2}^N(b) \vee T_{B_{1k}}^N(d_1 d_2) \\
&\geq T_{B_2}^N(b) \vee [T_{B_1}^N(d_1) \vee T_{B_1}^N(d_2)] \\
&= [T_{B_2}^N(b) \vee T_{B_1}^N(d_1)] \vee [T_{B_2}^N(b) \vee T_{B_1}^N(d_2)] \\
&= T_{(B_1 \boxtimes B_2)}^N(d_1 b) \vee T_{(B_1 \boxtimes B_2)}^N(d_2 b),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \boxtimes B_{2k})}^P((d_1 b)(d_2 b)) &= I_{B_2}^P(b) \wedge I_{B_{1k}}^P(d_1 d_2) \\
&\leq I_{B_2}^P(b) \wedge [I_{B_1}^P(d_1) \wedge I_{B_1}^P(d_2)] \\
&= [I_{B_2}^P(b) \wedge I_{B_1}^P(d_1)] \wedge [I_{B_2}^P(b) \wedge I_{B_1}^P(d_2)] \\
&= I_{(B_1 \boxtimes B_2)}^P(d_1 b) \wedge I_{(B_1 \boxtimes B_2)}^P(d_2 b),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \boxtimes B_{2k})}^N((d_1b)(d_2b)) &= I_{B_2}^N(b) \vee I_{B_{1k}}^N(d_1d_2) \\
&\geq I_{B_2}^N(b) \vee [I_{B_1}^N(d_1) \vee I_{B_1}^N(d_2)] \\
&= [I_{B_2}^N(b) \vee I_{B_1}^N(d_1)] \vee [I_{B_2}^N(b) \vee I_{B_1}^N(d_2)] \\
&= I_{(B_1 \boxtimes B_2)}^N(d_1b) \vee I_{(B_1 \boxtimes B_2)}^N(d_2b),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \boxtimes B_{2k})}^P((d_1b)(d_2b)) &= F_{B_2}^P(b) \vee F_{B_{1k}}^P(d_1d_2) \\
&\leq F_{B_2}^P(b) \vee [F_{B_1}^P(d_1) \vee F_{B_1}^P(d_2)] \\
&= [F_{B_2}^P(b) \vee F_{B_1}^P(d_1)] \vee [F_{B_2}^P(b) \vee F_{B_1}^P(d_2)] \\
&= F_{(B_1 \boxtimes B_2)}^P(d_1b) \vee F_{(B_1 \boxtimes B_2)}^P(d_2b),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \boxtimes B_{2k})}^N((d_1b)(d_2b)) &= F_{B_2}^N(b) \wedge F_{B_{1k}}^N(d_1d_2) \\
&\geq F_{B_2}^N(b) \wedge [F_{B_1}^N(d_1) \wedge F_{B_1}^N(d_2)] \\
&= [F_{B_2}^N(b) \wedge F_{B_1}^N(d_1)] \wedge [F_{B_2}^N(b) \wedge F_{B_1}^N(d_2)] \\
&= F_{(B_1 \boxtimes B_2)}^N(d_1b) \wedge F_{(B_1 \boxtimes B_2)}^N(d_2b),
\end{aligned}$$

for  $d_1b, d_2b \in V_1 \boxtimes V_2$ .

**Case 3.:** For  $b_1b_2 \in V_{1k}, d_1d_2 \in V_{2k}$

$$\begin{aligned}
T_{(B_{1k} \boxtimes B_{2k})}^P((b_1d_1)(b_2d_2)) &= T_{B_{1k}}^P(b_1b_2) \wedge T_{B_{2k}}^P(d_1d_2) \\
&\leq [T_{B_1}^P(b_1) \wedge T_{B_1}^P(b_2)] \wedge [T_{B_2}^P(d_1) \wedge T_{B_2}^P(d_2)] \\
&= [T_{B_1}^P(b_1) \wedge T_{B_2}^P(d_1)] \wedge [T_{B_1}^P(b_2) \wedge T_{B_2}^P(d_2)] \\
&= T_{(B_1 \boxtimes B_2)}^P(b_1d_1) \wedge T_{(B_1 \boxtimes B_2)}^P(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
T_{(B_{1k} \boxtimes B_{2k})}^N((b_1d_1)(b_2d_2)) &= T_{B_{1k}}^N(b_1b_2) \vee T_{B_{2k}}^N(d_1d_2) \\
&\geq [T_{B_1}^N(b_1) \vee T_{B_1}^N(b_2)] \vee [T_{B_2}^N(d_1) \vee T_{B_2}^N(d_2)] \\
&= [T_{B_1}^N(b_1) \vee T_{B_2}^N(d_1)] \vee [T_{B_1}^N(b_2) \vee T_{B_2}^N(d_2)] \\
&= T_{(B_1 \boxtimes B_2)}^N(b_1d_1) \vee T_{(B_1 \boxtimes B_2)}^N(b_2d_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \boxtimes B_{2k})}^P((b_1d_1)(b_2d_2)) &= I_{B_{1k}}^P(b_1b_2) \wedge I_{B_{2k}}^P(d_1d_2) \\
&\leq [I_{B_1}^P(b_1) \wedge I_{B_1}^P(b_2)] \wedge [I_{B_2}^P(d_1) \wedge I_{B_2}^P(d_2)] \\
&= [I_{B_1}^P(b_1) \wedge I_{B_2}^P(d_1)] \wedge [I_{B_1}^P(b_2) \wedge I_{B_2}^P(d_2)] \\
&= I_{(B_1 \boxtimes B_2)}^P(b_1d_1) \wedge I_{(B_1 \boxtimes B_2)}^P(b_2d_2),
\end{aligned}$$



$$\begin{aligned}
I_{(B_{1k} \boxtimes B_{2k})}^N((b_1 d_1)(b_2 d_2)) &= I_{B_{1k}}^N(b_1 b_2) \vee I_{B_{2k}}^N(d_1 d_2) \\
&\geq [I_{B_1}^N(b_1) \vee I_{B_1}^N(b_2)] \vee [I_{B_2}^N(d_1) \vee I_{B_2}^N(d_2)] \\
&= [I_{B_1}^N(b_1) \vee I_{B_2}^N(d_1)] \vee [I_{B_1}^N(b_2) \vee I_{B_2}^N(d_2)] \\
&= I_{(B_1 \boxtimes B_2)}^N(b_1 d_1) \vee I_{(B_1 \boxtimes B_2)}^N(b_2 d_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \boxtimes B_{2k})}^P((b_1 d_1)(b_2 d_2)) &= F_{B_{1k}}^P(b_1 b_2) \vee F_{B_{2k}}^P(d_1 d_2) \\
&\leq [F_{B_1}^P(b_1) \vee F_{B_1}^P(b_2)] \vee [F_{B_2}^P(d_1) \vee F_{B_2}^P(d_2)] \\
&= [F_{B_1}^P(b_1) \vee F_{B_2}^P(d_1)] \vee [F_{B_1}^P(b_2) \vee F_{B_2}^P(d_2)] \\
&= F_{(B_1 \boxtimes B_2)}^P(b_1 d_1) \vee F_{(B_1 \boxtimes B_2)}^P(b_2 d_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \boxtimes B_{2k})}^N((b_1 d_1)(b_2 d_2)) &= F_{B_{1k}}^N(b_1 b_2) \wedge F_{B_{2k}}^N(d_1 d_2) \\
&\geq [F_{B_1}^N(b_1) \wedge F_{B_1}^N(b_2)] \wedge [F_{B_2}^N(d_1) \wedge F_{B_2}^N(d_2)] \\
&= [F_{B_1}^N(b_1) \wedge F_{B_2}^N(d_1)] \wedge [F_{B_1}^N(b_2) \wedge F_{B_2}^N(d_2)] \\
&= F_{(B_1 \boxtimes B_2)}^N(b_1 d_1) \wedge F_{(B_1 \boxtimes B_2)}^N(b_2 d_2),
\end{aligned}$$

$$b_1 d_1, b_2 d_2 \in V_1 \boxtimes V_2.$$

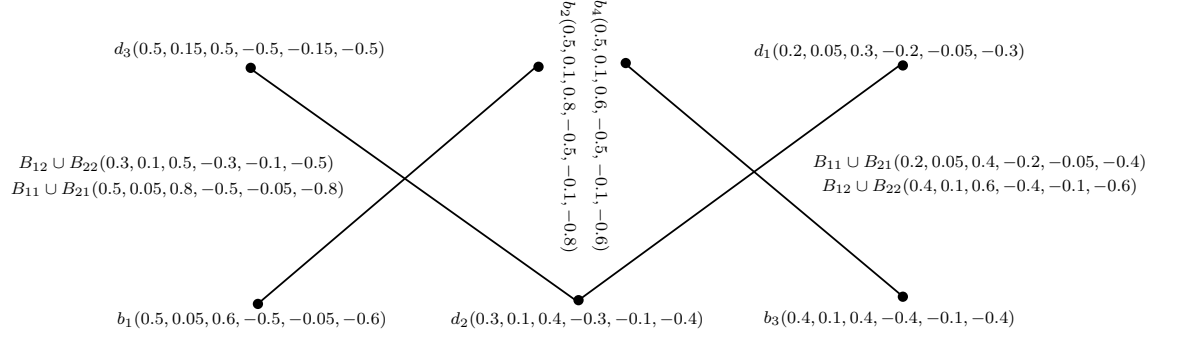
All cases hold  $\forall k \in \{1, 2, \dots, m\}$ .

**Definition 2.25.** Let  $\check{G}_{b_1} = (B_1, B_{11}, B_{12}, \dots, B_{1m})$  and  $\check{G}_{b_2} = (B_2, B_{21}, B_{22}, \dots, B_{2m})$  be BSVNGSs. Union of  $\check{G}_{b_1}$  and  $\check{G}_{b_2}$ , denoted by

$$\check{G}_{b_1} \cup \check{G}_{b_2} = (B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \dots, B_{1m} \cup B_{2m}),$$

is defined as:

$$\begin{aligned}
\text{(i)} \quad &\begin{cases} T_{(B_1 \cup B_2)}^P(b) = (T_{B_1}^P \cup T_{B_2}^P)(b) = T_{B_1}^P(b) \vee T_{B_2}^P(b) \\ I_{(B_1 \cup B_2)}^P(b) = (I_{B_1}^P \cup I_{B_2}^P)(b) = (I_{B_1}^P(b) + I_{B_2}^P(b))/2 \\ F_{(B_1 \cup B_2)}^P(b) = (F_{B_1}^P \cup F_{B_2}^P)(b) = F_{B_1}^P(b) \wedge F_{B_2}^P(b) \end{cases} \\
\text{(ii)} \quad &\begin{cases} T_{(B_1 \cup B_2)}^N(b) = (T_{B_1}^N \cup T_{B_2}^N)(b) = T_{B_1}^N(b) \wedge T_{B_2}^N(b) \\ I_{(B_1 \cup B_2)}^N(b) = (I_{B_1}^N \cup I_{B_2}^N)(b) = (I_{B_1}^N(b) + I_{B_2}^N(b))/2 \\ F_{(B_1 \cup B_2)}^N(b) = (F_{B_1}^N \cup F_{B_2}^N)(b) = F_{B_1}^N(b) \vee F_{B_2}^N(b) \end{cases} \\
&\text{for all } b \in V_1 \cup V_2, \\
\text{(iii)} \quad &\begin{cases} T_{(B_{1k} \cup B_{2k})}^P(bd) = (T_{B_{1k}}^P \cup T_{B_{2k}}^P)(bd) = T_{B_{1k}}^P(bd) \vee T_{B_{2k}}^P(bd) \\ I_{(B_{1k} \cup B_{2k})}^P(bd) = (I_{B_{1k}}^P \cup I_{B_{2k}}^P)(bd) = (I_{B_{1k}}^P(bd) + I_{B_{2k}}^P(bd))/2 \\ F_{(B_{1k} \cup B_{2k})}^P(bd) = (F_{B_{1k}}^P \cup F_{B_{2k}}^P)(bd) = F_{B_{1k}}^P(bd) \wedge F_{B_{2k}}^P(bd) \end{cases} \\
\text{(iv)} \quad &\begin{cases} T_{(B_{1k} \cup B_{2k})}^N(bd) = (T_{B_{1k}}^N \cup T_{B_{2k}}^N)(bd) = T_{B_{1k}}^N(bd) \wedge T_{B_{2k}}^N(bd) \\ I_{(B_{1k} \cup B_{2k})}^N(bd) = (I_{B_{1k}}^N \cup I_{B_{2k}}^N)(bd) = (I_{B_{1k}}^N(bd) + I_{B_{2k}}^N(bd))/2 \\ F_{(B_{1k} \cup B_{2k})}^N(bd) = (F_{B_{1k}}^N \cup F_{B_{2k}}^N)(bd) = F_{B_{1k}}^N(bd) \vee F_{B_{2k}}^N(bd) \end{cases} \\
&\text{for all } (bd) \in V_{1k} \cup V_{2k}.
\end{aligned}$$

FIGURE 11.  $\check{G}_{b_1} \cup \check{G}_{b_2}$ 

**Example 2.26.** Union of two BSVNGSs  $\check{G}_{b_1}$  and  $\check{G}_{b_2}$  shown in Fig. 7 is defined as  $\check{G}_{b_1} \cup \check{G}_{b_2} = \{B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}\}$  and is depicted in Fig. 11.

**Theorem 2.27.** Union  $\check{G}_{b_1} \cup \check{G}_{b_2} = (B_1 \cup B_2, B_{11} \cup B_{21}, B_{12} \cup B_{22}, \dots, B_{1m} \cup B_{2m})$  of two BSVNGSs of the GSs  $\check{G}_1$  and  $\check{G}_2$  is BSVNGS of  $\check{G}_1 \cup \check{G}_2$ .

PROOF. Let  $b_1 b_2 \in V_{1k} \cup V_{2k}$ . Two cases arise:

**Case 1.:** For  $b_1, b_2 \in V_1$ , by definition 2.25

$$\begin{aligned} T_{B_2}^P(b_1) &= T_{B_2}^P(b_2) = T_{B_{2k}}^P(b_1 b_2) = 0, \quad I_{B_2}^P(b_1) = I_{B_2}^P(b_2) = I_{B_{2k}}^P(b_1 b_2) = 0, \\ F_{B_2}^P(b_1) &= F_{B_2}^P(b_2) = F_{B_{2k}}^P(b_1 b_2) = 1, \\ T_{B_2}^N(b_1) &= T_{B_2}^N(b_2) = T_{B_{2k}}^N(b_1 b_2) = 0, \quad I_{B_2}^N(b_1) = I_{B_2}^N(b_2) = I_{B_{2k}}^N(b_1 b_2) = 0, \\ F_{B_2}^N(b_1) &= F_{B_2}^N(b_2) = F_{B_{2k}}^N(b_1 b_2) = -1, \text{ so} \end{aligned}$$

$$\begin{aligned} T_{(B_{1k} \cup B_{2k})}^P(b_1 b_2) &= T_{B_{1k}}^P(b_1 b_2) \vee T_{B_{2k}}^P(b_1 b_2) \\ &= T_{B_{1k}}^P(b_1 b_2) \vee 0 \\ &\leq [T_{B_1}^P(b_1) \wedge T_{B_1}^P(b_2)] \vee 0 \\ &= [T_{B_1}^P(b_1) \vee 0] \wedge [T_{B_1}^P(b_2) \vee 0] \\ &= [T_{B_1}^P(b_1) \vee T_{B_2}^P(b_1)] \wedge [T_{B_1}^P(b_2) \vee T_{B_2}^P(b_2)] \\ &= T_{(B_1 \cup B_2)}^P(b_1) \wedge T_{(B_1 \cup B_2)}^P(b_2), \end{aligned}$$

$$\begin{aligned} T_{(B_{1k} \cup B_{2k})}^N(b_1 b_2) &= T_{B_{1k}}^N(b_1 b_2) \wedge T_{B_{2k}}^N(b_1 b_2) \\ &= T_{B_{1k}}^N(b_1 b_2) \wedge 0 \\ &\geq [T_{B_1}^N(b_1) \vee T_{B_1}^N(b_2)] \wedge 0 \\ &= [T_{B_1}^N(b_1) \wedge 0] \vee [T_{B_1}^N(b_2) \wedge 0] \\ &= [T_{B_1}^N(b_1) \wedge T_{B_2}^N(b_1)] \vee [T_{B_1}^N(b_2) \wedge T_{B_2}^N(b_2)] \\ &= T_{(B_1 \cup B_2)}^N(b_1) \vee T_{(B_1 \cup B_2)}^N(b_2), \end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \cup B_{2k})}^P(b_1 b_2) &= F_{B_{1k}}^P(b_1 b_2) \wedge F_{B_{2k}}^P(b_1 b_2) \\
&= F_{B_{1k}}^P(b_1 b_2) \wedge 1 \\
&\leq [F_{B_1}^P(b_1) \vee F_{B_1}^P(b_2)] \wedge 1 \\
&= [F_{B_1}^P(b_1) \wedge 1] \vee [F_{B_1}^P(b_2) \wedge 1] \\
&= [F_{B_1}^P(b_1) \wedge F_{B_2}^P(b_1)] \vee [F_{B_1}^P(b_2) \wedge F_{B_2}^P(b_2)] \\
&= F_{(B_1 \cup B_2)}^P(b_1) \vee F_{(B_1 \cup B_2)}^P(b_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \cup B_{2k})}^N(b_1 b_2) &= F_{B_{1k}}^N(b_1 b_2) \vee F_{B_{2k}}^N(b_1 b_2) \\
&= F_{B_{1k}}^N(b_1 b_2) \vee -1 \\
&\geq [F_{B_1}^N(b_1) \wedge F_{B_1}^N(b_2)] \vee -1 \\
&= [F_{B_1}^N(b_1) \vee -1] \wedge [F_{B_1}^N(b_2) \vee -1] \\
&= [F_{B_1}^N(b_1) \vee F_{B_2}^N(b_1)] \wedge [F_{B_1}^N(b_2) \vee F_{B_2}^N(b_2)] \\
&= F_{(B_1 \cup B_2)}^N(b_1) \wedge F_{(B_1 \cup B_2)}^N(b_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \cup B_{2k})}^P(b_1 b_2) &= \frac{I_{B_{1k}}^P(b_1 b_2) + I_{B_{2k}}^P(b_1 b_2)}{2} \\
&= \frac{I_{B_{1k}}^P(b_1 b_2) + 0}{2} \\
&\leq \frac{[I_{B_1}^P(b_1) \wedge I_{B_1}^P(b_2)] + 0}{2} \\
&= \left[ \frac{I_{B_1}^P(b_1)}{2} + 0 \right] \wedge \left[ \frac{I_{B_1}^P(b_2)}{2} + 0 \right] \\
&= \frac{[I_{B_1}^P(b_1) + I_{B_2}^P(b_1)]}{2} \wedge \frac{[I_{B_1}^P(b_2) + I_{B_2}^P(b_2)]}{2} \\
&= I_{(B_1 \cup B_2)}^P(b_1) \wedge I_{(B_1 \cup B_2)}^P(b_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \cup B_{2k})}^N(b_1 b_2) &= \frac{I_{B_{1k}}^N(b_1 b_2) + I_{B_{2k}}^N(b_1 b_2)}{2} \\
&= \frac{I_{B_{1k}}^N(b_1 b_2) + 0}{2} \\
&\geq \frac{[I_{B_1}^N(b_1) \vee I_{B_1}^N(b_2)] + 0}{2} \\
&= \left[ \frac{I_{B_1}^N(b_1)}{2} + 0 \right] \vee \left[ \frac{I_{B_1}^N(b_2)}{2} + 0 \right] \\
&= \frac{[I_{B_1}^N(b_1) + I_{B_2}^N(b_1)]}{2} \vee \frac{[I_{B_1}^N(b_2) + I_{B_2}^N(b_2)]}{2} \\
&= I_{(B_1 \cup B_2)}^N(b_1) \vee I_{(B_1 \cup B_2)}^N(b_2),
\end{aligned}$$

for  $b_1, b_2 \in V_1 \cup V_2$ .

**Case 2.:** For  $b_1, b_2 \in V_2$ , by definition 2.25

$$\begin{aligned}
T_{B_1}^P(b_1) &= T_{B_1}^P(b_2) = T_{B_{1k}}^P(b_1 b_2) = 0, \quad I_{B_1}^P(b_1) = I_{B_1}^P(b_2) = I_{B_{1k}}^P(b_1 b_2) = 0, \\
F_{B_1}^P(b_1) &= F_{B_2}^P(b_2) = F_{B_{1k}}^P(b_1 b_2) = 1, \\
T_{B_1}^N(b_1) &= T_{B_1}^N(b_2) = T_{B_{1k}}^N(b_1 b_2) = 0, \quad I_{B_1}^N(b_1) = I_{B_1}^N(b_2) = I_{B_{1k}}^N(b_1 b_2) = 0, \\
F_{B_1}^N(b_1) &= F_{B_2}^N(b_2) = F_{B_{1k}}^N(b_1 b_2) = -1, \text{ so}
\end{aligned}$$

$$\begin{aligned}
T_{(B_{1k} \cup B_{2k})}^P(b_1 b_2) &= T_{B_{1k}}^P(b_1 b_2) \vee T_{B_{2k}}^P(b_1 b_2) \\
&= T_{B_{2k}}^P(b_1 b_2) \vee 0 \\
&\leq [T_{B_2}^P(b_1) \wedge T_{B_2}^P(b_2)] \vee 0 \\
&= [T_{B_2}^P(b_1) \vee 0] \wedge [T_{B_2}^P(b_2) \vee 0] \\
&= [T_{B_2}^P(b_1) \vee T_{B_1}^P(b_1)] \wedge [T_{B_2}^P(b_2) \vee T_{B_1}^P(b_2)] \\
&= T_{(B_1 \cup B_2)}^P(b_1) \wedge T_{(B_1 \cup B_2)}^P(b_2),
\end{aligned}$$

$$\begin{aligned}
T_{(B_{1k} \cup B_{2k})}^N(b_1 b_2) &= T_{B_{1k}}^N(b_1 b_2) \wedge T_{B_{2k}}^N(b_1 b_2) \\
&= T_{B_{2k}}^N(b_1 b_2) \wedge 0 \\
&\geq [T_{B_2}^N(b_1) \vee T_{B_2}^N(b_2)] \wedge 0 \\
&= [T_{B_2}^N(b_1) \wedge 0] \vee [T_{B_2}^N(b_2) \wedge 0] \\
&= [T_{B_2}^N(b_1) \wedge T_{B_1}^N(b_1)] \vee [T_{B_2}^N(b_2) \wedge T_{B_1}^N(b_2)] \\
&= T_{(B_1 \cup B_2)}^N(b_1) \vee T_{(B_1 \cup B_2)}^N(b_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \cup B_{2k})}^P(b_1 b_2) &= F_{B_{1k}}^P(b_1 b_2) \wedge F_{B_{2k}}^P(b_1 b_2) \\
&= F_{B_{2k}}^P(b_1 b_2) \wedge (1) \\
&\leq [F_{B_2}^P(b_1) \vee F_{B_2}^P(b_2)] \wedge (1) \\
&= [F_{B_2}^P(b_1) \wedge (1)] \vee [F_{B_2}^P(b_2) \wedge (1)] \\
&= [F_{B_2}^P(b_1) \wedge F_{B_1}^P(b_1)] \vee [F_{B_2}^P(b_2) \wedge F_{B_1}^P(b_2)] \\
&= F_{(B_1 \cup B_2)}^P(b_1) \vee F_{(B_1 \cup B_2)}^P(b_2),
\end{aligned}$$

$$\begin{aligned}
F_{(B_{1k} \cup B_{2k})}^N(b_1 b_2) &= F_{B_{1k}}^N(b_1 b_2) \vee F_{B_{2k}}^N(b_1 b_2) \\
&= F_{B_{2k}}^N(b_1 b_2) \vee (-1) \\
&\geq [F_{B_2}^N(b_1) \wedge F_{B_2}^N(b_2)] \vee (-1) \\
&= [F_{B_2}^N(b_1) \vee (-1)] \wedge [F_{B_2}^N(b_2) \vee (-1)] \\
&= [F_{B_2}^N(b_1) \vee F_{B_1}^N(b_1)] \wedge [F_{B_2}^N(b_2) \vee F_{B_1}^N(b_2)] \\
&= F_{(B_1 \cup B_2)}^N(b_1) \wedge F_{(B_1 \cup B_2)}^N(b_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \cup B_{2k})}^P(b_1 b_2) &= \frac{I_{B_{1k}}^P(b_1 b_2) + I_{B_{2k}}^P(b_1 b_2)}{2} \\
&= \frac{I_{B_{2k}}^P(b_1 b_2) + 0}{2} \\
&\leq \frac{[I_{B_2}^P(b_1) \wedge I_{B_2}^P(b_2)] + 0}{2} \\
&= \left[ \frac{I_{B_2}^P(b_1)}{2} + 0 \right] \wedge \left[ \frac{I_{B_2}^P(b_2)}{2} + 0 \right] \\
&= \frac{[I_{B_2}^P(b_1) + I_{B_1}^P(b_1)]}{2} \wedge \frac{[I_{B_2}^P(b_2) + I_{B_1}^P(b_2)]}{2} \\
&= I_{(B_1 \cup B_2)}^P(b_1) \wedge I_{(B_1 \cup B_2)}^P(b_2),
\end{aligned}$$

$$\begin{aligned}
I_{(B_{1k} \cup B_{2k})}^N(b_1 b_2) &= \frac{I_{B_{1k}}^N(b_1 b_2) + I_{B_{2k}}^N(b_1 b_2)}{2} \\
&= \frac{I_{B_{2k}}^N(b_1 b_2) + 0}{2} \\
&\geq \frac{[I_{B_2}^N(b_1) \vee I_{B_2}^N(b_2)] + 0}{2} \\
&= \left[ \frac{I_{B_2}^N(b_1)}{2} + 0 \right] \vee \left[ \frac{I_{B_2}^N(b_2)}{2} + 0 \right] \\
&= \frac{[I_{B_2}^N(b_1) + I_{B_1}^N(b_1)]}{2} \vee \frac{[I_{B_2}^N(b_2) + I_{B_1}^N(b_2)]}{2} \\
&= I_{(B_1 \cup B_2)}^N(b_1) \vee I_{(B_1 \cup B_2)}^N(b_2),
\end{aligned}$$

for  $b_1, b_2 \in V_1 \cup V_2$ .

Both cases hold  $\forall k \in \{1, 2, \dots, m\}$ . This completes the proof.

**Theorem 2.28.** Let  $\check{G}_s = (V_1 \cup V_2, V_{11} \cup V_{21}, V_{12} \cup V_{22}, \dots, V_{1m} \cup V_{2m})$  be union of GSs  $\check{G}_{s1} = (V_1, V_{11}, V_{12}, \dots, V_{1m})$  and  $\check{G}_{s2} = (V_2, V_{21}, V_{22}, \dots, V_{2m})$ . Then every BSVNGS  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$  of  $\check{G}_s$  is union of two BSVNGSs  $\check{G}_{b1}$  and  $\check{G}_{b2}$  of GSs  $\check{G}_{s1}$  and  $\check{G}_{s2}$ , respectively.

PROOF Firstly, we define  $B_1, B_2, B_{1k}$  and  $B_{2k}$  for  $k \in \{1, 2, \dots, m\}$  as:

$$\begin{aligned}
T_{B_1}^P(b) &= T_B^P(b), I_{B_1}^P(b) = I_B^P(b), F_{B_1}^P(b) = F_B^P(b), \\
T_{B_1}^N(b) &= T_B^N(b), I_{B_1}^N(b) = I_B^N(b), F_{B_1}^N(b) = F_B^N(b), \text{ if } b \in V_1.
\end{aligned}$$

$$\begin{aligned}
T_{B_2}^P(b) &= T_B^P(b), I_{B_2}^P(b) = I_B^P(b), F_{B_2}^P(b) = F_B^P(b), \\
T_{B_2}^N(b) &= T_B^N(b), I_{B_2}^N(b) = I_B^N(b), F_{B_2}^N(b) = F_B^N(b), \text{ if } b \in V_2.
\end{aligned}$$

$$\begin{aligned}
T_{B_{1k}}^P(b_1 b_2) &= T_{B_k}^P(b_1 b_2), I_{B_{1k}}^P(b_1 b_2) = I_{B_k}^P(b_1 b_2), F_{B_{1k}}^P(b_1 b_2) = F_{B_k}^P(b_1 b_2), T_{B_{1k}}^N(b_1 b_2) = \\
&T_{B_k}^N(b_1 b_2), I_{B_{1k}}^N(b_1 b_2) = I_{B_k}^N(b_1 b_2), F_{B_{1k}}^N(b_1 b_2) = F_{B_k}^N(b_1 b_2), \text{ if } b_1 b_2 \in V_{1k}. \\
T_{B_{2k}}^P(b_1 b_2) &= T_{B_k}^P(b_1 b_2), I_{B_{2k}}^P(b_1 b_2) = I_{B_k}^P(b_1 b_2), F_{B_{2k}}^P(b_1 b_2) = F_{B_k}^P(b_1 b_2), T_{B_{2k}}^N(b_1 b_2) = \\
&T_{B_k}^N(b_1 b_2), I_{B_{2k}}^N(b_1 b_2) = I_{B_k}^N(b_1 b_2), F_{B_{2k}}^N(b_1 b_2) = F_{B_k}^N(b_1 b_2), \text{ if } b_1 b_2 \in V_{2k}.
\end{aligned}$$

Then  $B = B_1 \cup B_2$  and  $B_k = B_{1k} \cup B_{2k}$ ,  $k \in \{1, 2, \dots, m\}$ . Now for  $b_1 b_2 \in V_{tk}$ ,  $t = 1, 2$ ,  $k \in \{1, 2, \dots, m\}$ :

$$\begin{aligned}
T_{B_{tk}}^P(b_1 b_2) &= T_{B_k}^P(b_1 b_2) \leq T_B^P(b_1) \wedge T_B^P(b_2) = T_{B_t}^P(b_1) \wedge T_{B_t}^P(b_2), I_{B_{tk}}^P(b_1 b_2) = \\
&I_{B_k}^P(b_1 b_2) \leq I_B^P(b_1) \wedge I_B^P(b_2) = I_{B_t}^P(b_1) \wedge I_{B_t}^P(b_2), F_{B_{tk}}^P(b_1 b_2) = F_{B_k}^P(b_1 b_2) \leq F_B^P(b_1) \vee \\
&F_B^P(b_2) = F_{B_t}^P(b_1) \vee F_{B_t}^P(b_2),
\end{aligned}$$

$$\begin{aligned}
T_{B_{tk}}^N(b_1 b_2) &= T_{B_k}^N(b_1 b_2) \geq T_B^N(b_1) \vee T_B^N(b_2) = T_{B_t}^N(b_1) \vee T_{B_t}^N(b_2), I_{B_{tk}}^N(b_1 b_2) = \\
&I_{B_k}^N(b_1 b_2) \geq I_B^N(b_1) \vee I_B^N(b_2) = I_{B_t}^N(b_1) \vee I_{B_t}^N(b_2), F_{B_{tk}}^N(b_1 b_2) = F_{B_k}^N(b_1 b_2) \geq \\
&F_B^N(b_1) \wedge F_B^N(b_2) = F_{B_t}^N(b_1) \wedge F_{B_t}^N(b_2), \text{ i.e.,}
\end{aligned}$$

$\check{G}_{bl} = (B_l, B_{l1}, B_{l2}, \dots, B_{lm})$  is a BSVNGS of  $\check{G}_t$ ,  $t = 1, 2$ . Thus  $\check{G}_{bn} = (B, B_1, B_2, \dots, B_m)$ , a BSVNGS of  $\check{G}_s = \check{G}_{s1} \cup \check{G}_{s2}$ , is the union of two BSVNGSs  $\check{G}_{b1}$  and  $\check{G}_{b2}$ .

**Definition 2.29.** Let  $\check{G}_{b1} = (B_1, B_{11}, B_{12}, \dots, B_{1m})$  and  $\check{G}_{b2} = (B_2, B_{21}, B_{22}, \dots, B_{2m})$  be BSVNGSs and let  $V_1 \cap V_2 = \emptyset$ . Join of  $\check{G}_{b1}$  and  $\check{G}_{b2}$ , denoted by

$$\check{G}_{b1} + \check{G}_{b2} = (B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}, \dots, B_{1m} + B_{2m}),$$

is defined as:

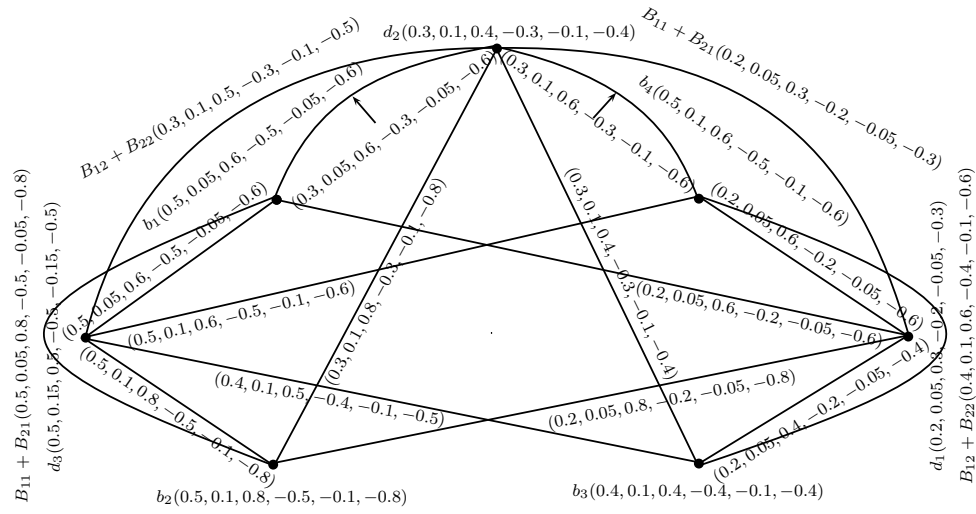
$$\begin{aligned} & \text{(i) } \begin{cases} T_{(B_1+B_2)}^P(b) = T_{(B_1 \cup B_2)}^P(b) \\ I_{(B_1+B_2)}^P(b) = I_{(B_1 \cup B_2)}^P(b) \\ F_{(B_1+B_2)}^P(b) = F_{(B_1 \cup B_2)}^P(b) \end{cases} \\ & \text{(ii) } \begin{cases} T_{(B_1+B_2)}^N(b) = T_{(B_1 \cup B_2)}^N(b) \\ I_{(B_1+B_2)}^N(b) = I_{(B_1 \cup B_2)}^N(b) \\ F_{(B_1+B_2)}^N(b) = F_{(B_1 \cup B_2)}^N(b) \end{cases} \\ & \text{for all } b \in V_1 \cup V_2, \\ & \text{(iii) } \begin{cases} T_{(B_{1k}+B_{2k})}^P(bd) = T_{(B_{1k} \cup B_{2k})}^P(bd) \\ I_{(B_{1k}+B_{2k})}^P(bd) = I_{(B_{1k} \cup B_{2k})}^P(bd) \\ F_{(B_{1k}+B_{2k})}^P(bd) = F_{(B_{1k} \cup B_{2k})}^P(bd) \end{cases} \\ & \text{(iv) } \begin{cases} T_{(B_{1k}+B_{2k})}^N(bd) = T_{(B_{1k} \cup B_{2k})}^N(bd) \\ I_{(B_{1k}+B_{2k})}^N(bd) = I_{(B_{1k} \cup B_{2k})}^N(bd) \\ F_{(B_{1k}+B_{2k})}^N(bd) = F_{(B_{1k} \cup B_{2k})}^N(bd) \end{cases} \\ & \text{for all } (bd) \in V_{1k} \cup V_{2k}, \\ & \text{(v) } \begin{cases} T_{(B_{1k}+B_{2k})}^P(bd) = (T_{B_{1k}}^P + T_{B_{2k}}^P)(bd) = T_{B_1}^P(b) \wedge T_{B_2}^P(d) \\ I_{(B_{1k}+B_{2k})}^P(bd) = (I_{B_{1k}}^P + I_{B_{2k}}^P)(bd) = I_{B_1}^P(b) \wedge I_{B_2}^P(d) \\ F_{(B_{1k}+B_{2k})}^P(bd) = (F_{B_{1k}}^P + F_{B_{2k}}^P)(bd) = F_{B_1}^P(b) \vee F_{B_2}^P(d) \end{cases} \\ & \text{(vi) } \begin{cases} T_{(B_{1k}+B_{2k})}^N(bd) = (T_{B_{1k}}^N + T_{B_{2k}}^N)(bd) = T_{B_1}^N(b) \vee T_{B_2}^N(d) \\ I_{(B_{1k}+B_{2k})}^N(bd) = (I_{B_{1k}}^N + I_{B_{2k}}^N)(bd) = I_{B_1}^N(b) \vee I_{B_2}^N(d) \\ F_{(B_{1k}+B_{2k})}^N(bd) = (F_{B_{1k}}^N + F_{B_{2k}}^N)(bd) = F_{B_1}^N(b) \wedge F_{B_2}^N(d) \end{cases} \\ & \text{for all } b \in V_1, d \in V_2. \end{aligned}$$

**Example 2.30.** Join of two BSVNGSs  $\check{G}_{b1}$  and  $\check{G}_{b2}$  shown in Fig. 7 is defined as  $\check{G}_{b1} + \check{G}_{b2} = \{B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}\}$  and is depicted in Fig. 12.

**Theorem 2.31.** Join  $\check{G}_{b1} + \check{G}_{b2} = (B_1 + B_2, B_{11} + B_{21}, B_{12} + B_{22}, \dots, B_{1m} + B_{2m})$  of two BSVNGSs of the GSs  $\check{G}_1$  and  $\check{G}_2$  is BSVNGS of  $\check{G}_1 + \check{G}_2$ .

### 3. CONCLUDING REMARKS

Bipolar fuzzy graph theory has numerous applications in various fields of science and technology including, artificial intelligence, operations research and decision making. A bipolar neutrosophic graph constitutes a generalization of the notion bipolar fuzzy graph. In this research paper, We have introduced the idea of bipolar single-valued neutrosophic graph structure and discussed many relevant notions. We also discussed a worthwhile application of bipolar single-valued neutrosophic graph structure in decision-making. In future, we aim to generalize our notions to (1) BSVN hypergraph structures, (2) BSVN vague hypergraph structures,

FIGURE 12.  $\check{G}_{b_1} + \check{G}_{b_2}$ 

(3) BSVN interval-valued hypergraph structures, and(4) BSVN rough hypergraph structures.

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