

## EIGENVALUES VARIATION OF THE P-LAPLACIAN UNDER THE RICCI FLOW ON SM

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**Abstract.** Let  $(M, F)$  be a compact Finsler manifold. Studying the eigenvalues and eigenfunctions for the linear and nonlinear geometric operators is a known problem. In this paper we will consider the eigenvalue problem for the  $p$ -laplace operator for Sasakian metric acting on the space of functions on  $SM$ . We find the first variation formula for the eigenvalues of  $p$ -Laplacian on  $SM$  evolving by the Ricci flow on  $M$  and give some examples.

*Key words and Phrases:* Ricci flow, Finsler manifold,  $p$ -Laplace operator

**Abstrak.** Misalkan  $(M, F)$  adalah suatu manifold Finsler kompak. Sejauh ini telah dipelajari fungsi eigen dan nilai eigen untuk operator-operator geometri linier dan non linier. Dalam paper ini kami akan memperhatikan masalah nilai eigen untuk operator  $p$ -Laplace untuk metrik Sasakian yang berlaku pada ruang fungsi di  $SM$ . Kami memperoleh rumus variasi pertama dan memberikan beberapa contoh untuk nilai eigen dari  $p$ -Laplacian pada  $SM$  yang melibatkan aliran Ricci pada  $M$ .

*Kata kunci:* aliran Ricci, manifold Finsler, operator  $p$ -Laplace

### 1. INTRODUCTION

For a compact Finsler manifold  $(M, F)$ , studying the eigenvalues of geometric operators plays a powerful role in geometric analysis. In the classical theory of the Laplace or  $p$ -Laplace equation several main parts of mathematics are joined in a fruitful way: Calculus of Variation, Partial Differential Equation, Potential Theory,

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Analytic Function. Recently, there are many mathematicians who have investigated properties of the eigenvalues of  $p$ -Laplacian on Finsler manifolds and Riemannian manifolds to estimate the spectrum in terms of the other geometric quantities of the manifold. (see [3, 4, 9, 11, 18, 20]).

Also, geometric flows have been a topic of active research interest in mathematics and other sciences (see [5, 8, 10, 12, 13, 15, 16]). Hamilton's Ricci flow ([6]) is the best known example of a geometric evolution equation. The Ricci flow is related to dynamical systems in the infinite-dimensional space of all metrics on a given manifold. One of the aims of such flows is to obtain metrics with special properties. Special cases arise when the metric is invariant under a group of transformations and this property is preserved by the flow.

Let  $M$  be a manifold with a Finsler metric  $g_0$  (or  $F_0$ ), the family  $g(t)$  (or  $F_t$ ) of Finsler metrics on  $M$  is called an un-normalized Ricci flow when it satisfies the equations

$$\frac{\partial \log F}{\partial t} = -\mathcal{R}ic, \quad (1)$$

with the initial condition

$$F(0) = F_0$$

or equivalently satisfies the equations

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij}, \quad g(0) = g_0 \quad (2)$$

where  $Ric$  is the Ricci tensor of  $g(t)$ ,  $Ric_{ij} = (\frac{1}{2}F^2\mathcal{R}ic)_{y^i y^j}$ . In fact Ricci flow is a system of partial differential equations of parabolic type which was introduced by Hamilton on Riemannian manifolds for the first time in 1982 and Bao (see [2, 17]) studied Ricci flow equation in Finsler manifold. The Ricci flow has been proved to be a very useful tool to improve metrics in Finsler geometry, when  $M$  is compact. One often considers the normalized Ricci flow

$$\frac{\partial \log F}{\partial t} = -\mathcal{R}ic + \frac{1}{\text{vol}(SM)} \int_{SM} \mathcal{R}ic \, dv, \quad F(0) = F_0. \quad (3)$$

or

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij} + \frac{2}{\text{vol}(SM)} \int_{SM} Ric \, dv g_{ij}, \quad g(0) = g_0 \quad (4)$$

Under this normalized flow, the volume of the solution metrics remains constant in time. Short time existence and uniqueness for solution to the Ricci flow on  $[0, T)$  have been shown by Hamilton in [5] and by DeTurk in [7] for Riemannian manifolds and by the authors in [1] for important Berwald manifolds.

2. PRELIMINARIES

Let  $M$  be an  $n$ -dimensional  $C^\infty$  manifold. For a point  $x \in M$ , denote by  $T_x M$  the tangent space at  $x \in M$ , and by  $TM = \cup_{x \in M} T_x M$  the tangent bundle of  $M$ . Any element of  $TM$  has the form  $(x, y)$ , where  $x \in M$  and  $y \in T_x M$ .

**Definition 2.1.** A Finsler metric on a manifold  $M$  is a function  $F : TM_0 \rightarrow [0, \infty)$  which has the following properties:

- (i):  $F(x, \alpha y) = \alpha F(x, y), \forall \alpha > 0$ ;
- (ii):  $F(x, y)$  is  $C^\infty$  on  $TM_0$ ;
- (iii): For any non-zero tangent vector  $y \in T_x M$ , the associated quadratic form  $g_y : T_x M \times T_x M \rightarrow \mathbb{R}$  on  $TM$  is an inner product, where

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial r} [F^2(x, y + su + rv)] \Big|_{s=r=0}.$$

The pair  $(M, F)$  is called a Finsler manifold.

Let us denote by  $S_x M$  the set consisting of all rays  $[y] := \{\lambda y | \lambda > 0\}$ , where  $y \in T_x M_0$ . The Sphere bundle of  $M$ , i.e.  $SM$ , is the union of  $S_x M$ 's :

$$SM = \cup_x S_x M$$

$SM$  has a natural  $(2n - 1)$ -dimensional manifold structure. We denote the elements of  $SM$  by  $(x, [y])$  where  $y \in T_x M_0$ . If there is not any confusion we write  $(x, y)$  for  $(x, [y])$ . In a local coordinate system  $(x^i, y^i)$  we have  $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$  and  $(g^{ij}) := (g_{ij})^{-1}$ . The geodesics of  $F$  are characterized locally by

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$$

where

$$G^i = \frac{1}{4} g^{il} \left\{ 2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k. \tag{5}$$

**Definition 2.2.** The coefficients of the Riemann curvature  $R_y = R_k^i dx^i \otimes \frac{\partial}{\partial x^k}$  are given by

$$R_k^i := 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k} \tag{6}$$

and  $R_{jk}^i := \frac{1}{3} \left( \frac{\partial R_j^i}{\partial y^k} - \frac{\partial R_k^i}{\partial y^j} \right), R_{jkl}^i := \frac{1}{3} \left( \frac{\partial^2 R_k^i}{\partial y^j \partial y^l} - \frac{\partial^2 R_l^i}{\partial y^j \partial y^k} \right).$

The Ricci scalar function of  $F$  is given by  $\mathcal{R}ic := \frac{1}{F^2} R_i^i$ . A companion of the Ricci scalar is the Ricci tensor

$$Ric_{ij} := \left( \frac{1}{2} F^2 \mathcal{R}ic \right)_{y^i y^j}. \tag{7}$$

**Definition 2.3.** A Finsler metric is said to be an Einstein metric if the Ricci scalar function is a function of  $x$  alone, equivalently  $\text{Ric}_{ij} = \mathcal{R}(x)g_{ij}$  (see [14, 19]).

**Definition 2.4.** Let  $(M, F)$  be a Finsler manifold, the Sasakian metric  $\tilde{g}$  of  $g$  on  $TM_0$  is defined as

$$\tilde{g} = g_{ij}dx^i \otimes dx^j + g_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F} \quad (8)$$

then  $\tilde{g}$  is a Riemannian metric on  $TM_0$  and  $\{\frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i}\}$  is a coordinate base on  $TM_0$ , where  $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_j^i \frac{\partial}{\partial y^j}$  and  $\{dx^i, \frac{\delta y^i}{F}\}$  is the dual of  $\{\frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i}\}$  where  $\delta y^i = dy^i + G_j^i dx^j$ .

REMARK. The Levi-Civita connection  $\tilde{\nabla}$  on  $TM_0$  with respect to the Sasakian metric  $\tilde{g}$  is locally expressed as follows:

$$\begin{aligned} \tilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} &= -(C_{ij}^k + \frac{1}{2}R_{ij}^k) \frac{\partial}{\partial y^k} + F_{ij}^k \frac{\delta}{\delta x^k} \\ \tilde{\nabla}_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} &= C_{ij}^k \frac{\partial}{\partial y^k} - g_{ih}(F_{jk}^h - G_{jk}^h)g^{hk} \frac{\delta}{\delta x^k} \\ \tilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} &= F_{ij}^k \frac{\partial}{\partial y^k} + (C_{ij}^k + \frac{1}{2}g_{ih}R_{lj}^h g^{lk}) \frac{\delta}{\delta x^k} \\ &= \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} + G_{ij}^k \frac{\partial}{\partial y^k}, \end{aligned} \quad (9)$$

where

$$C_{ij}^k = \frac{1}{2}g^{kh} \frac{\partial g_{ij}}{\partial y^h}, \quad F_{ij}^k = \frac{1}{2}g^{kh} \left( \frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \right), \quad G_{ij}^k = \frac{\partial G_j^k}{\partial y^i},$$

and

$$R_{ij}^k = \frac{\delta G_i^k}{\delta x^j} - \frac{\delta G_j^k}{\delta x^i}, \quad \left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = R_{ij}^k \frac{\partial}{\partial y^k}, \quad \left[ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right] = G_{ij}^k \frac{\partial}{\partial y^k}.$$

**Lemma 2.5.** For a Sasakian metric  $\tilde{g}$  and any  $f : TM \rightarrow \mathbb{R}$ , there exists a unique vector field  $Y \in \mathcal{X}(TM)$  such that

$$\tilde{g}(Y, \tilde{X}) = df(\tilde{X}), \quad \forall \tilde{X} \in \mathcal{X}(TM) \quad (10)$$

where

$$\tilde{X} = X_1^i \frac{\delta}{\delta x^i} + X_2^i F \frac{\partial}{\partial y^i}$$

and  $X_1^i, X_2^i$  are  $C^\infty$  function on  $TM$ . Here we take  $Y = 0$  if  $df = 0$ .

Denote the vector field  $Y$  in (10) by  $\tilde{\nabla}f$ . We call  $\tilde{\nabla}f$  the gradient of  $f$  and define the divergence  $\text{div} \tilde{X}$  as follows:

$$\text{div} \tilde{X} = \text{tr} \tilde{\nabla} \tilde{X}$$

**Definition 2.6.** According to the above definition, the gradient of a function  $f$  is

$$\tilde{\nabla} f = g^{ij} \frac{\delta f}{\delta x^i} \frac{\delta}{\delta x^j} + F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial}{\partial y^j} \quad (11)$$

therefore, the norm of  $\tilde{\nabla} f$  with respect to the Riemannian metric  $\tilde{g}$  is given by

$$|\tilde{\nabla} f|^2 = \tilde{g}(\tilde{\nabla} f, \tilde{\nabla} f) = g^{ij} \frac{\delta f}{\delta x^i} \frac{\delta f}{\delta x^j} + F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}. \quad (12)$$

**Definition 2.7.** Let  $M$  be a compact Finsler manifold. The Laplace operator of  $f$  on  $TM$  is defined as follows:

$$\begin{aligned} \Delta f &= g^{ij} \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial G_j^r}{\partial x^i} \frac{\partial f}{\partial y^r} - G_j^r \frac{\partial^2 f}{\partial x^i \partial y^r} - G_i^s \frac{\partial^2 f}{\partial y^s \partial x^j} + G_j^r G_i^s \frac{\partial^2 f}{\partial y^s \partial y^r} \right) \\ &+ g^{ij} F^2 \frac{\partial^2 f}{\partial y^i \partial y^j} + g^{ij} (C_{ij}^k + \frac{1}{2} R_{ij}^k) \frac{\delta f}{\delta x^k} - F g^{ij} F_{ij}^k \frac{\delta f}{\delta x^k} \\ &- F g^{ij} C_{ij}^k \frac{\partial f}{\partial y^k} - F^2 g^{ij} g_{ih} (G_{jl}^h - F_{jl}^h) g^{lk} \frac{\delta f}{\delta y^k}. \end{aligned} \quad (13)$$

**Definition 2.8.** Let  $M$  be a compact Finsler manifold. The  $p$ -Laplace operator of  $f : SM \rightarrow \mathbb{R}$ ,  $f \in W^{1,p}(SM)$  for  $1 < p < \infty$  is defined as follows:

$$\begin{aligned} \Delta_p f &= \operatorname{div}(|\tilde{\nabla} f|^{p-2} \tilde{\nabla} f) \\ &= |\tilde{\nabla} f|^{p-2} \Delta f + (p-2) |\tilde{\nabla} f|^{p-4} (\operatorname{Hess} f)(\tilde{\nabla} f, \tilde{\nabla} f) \end{aligned} \quad (14)$$

where

$$(\operatorname{Hess} f)(X, Y) = \tilde{\nabla}(\tilde{\nabla} f)(X, Y) = Y.(X.f) - (\tilde{\nabla}_Y X).f, \quad X, Y \in \mathcal{X}(SM)$$

and in local coordinate, we have:

$$(\operatorname{Hess} f)(\partial_i, \partial_j) = \partial_i \partial_j f - \tilde{\Gamma}_{ij}^k \partial_k f.$$

NOTE. If  $f$  is a function of  $x$  alone, or suppose that is the lifting of  $f : M \rightarrow \mathbb{R}$  then

$$\Delta f = g^{ij} \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \tilde{\Gamma}_{ij}^k \frac{\partial f}{\partial x^k} \right) \quad (15)$$

where  $\tilde{\Gamma}_{ij}^k$  is christoffel symbol of  $\tilde{\nabla}$ .

## 2.1. Eigenvalues of the $p$ -Laplacian.

**Definition 2.9.** Let  $(M^n, F)$  be a compact Finsler manifold and  $f : SM \rightarrow \mathbb{R}$ . We say that  $\lambda$  is an eigenvalue of the  $p$ -Laplace operator whenever

$$\tilde{g} \Delta_p f + \lambda |f|^{p-2} f = 0 \quad (16)$$

then  $f$  is said to be the eigenfunction associated to  $\lambda$ , or equivalently they satisfy in

$$\int_{SM} |\tilde{\nabla} f|^{p-2} \langle \tilde{\nabla} f, \tilde{\nabla} \varphi \rangle dv = \lambda \int_{SM} |f|^{p-2} f \varphi dv \quad \forall \varphi \in W_0^{1,2}(SM) \quad (17)$$

where  $W_0^{1,p}(SM)$  is closure of  $C_0^\infty(SM)$  in Sobolev  $W^{1,p}(SM)$ .

By substitution  $\varphi = f$  in (17) we have:

$$\lambda = \frac{\int_{SM} |\tilde{\nabla} f|^p dv}{\int_{SM} |f|^p dv}. \quad (18)$$

Normalized eigenfunctions are defined as follows:

$$\int_{SM} f|f|^{p-2} dv = 0, \int_{SM} |f|^p dv = 1. \quad (19)$$

Suppose that  $(M^n, F_t)$  is a solution of the Ricci flow on the smooth manifold  $(M^n, F_0)$  in the interval  $[0, T)$  and

$$\lambda(t) = \int_{SM} |\tilde{\nabla} f(x, y)|^p dv_t \quad (20)$$

defines the evolution of an eigenvalue of  $P$ -Laplacian under the variation of  $F_t$  whose eigenfunction associated to  $\lambda(t)$  is normalized. Suppose that for any metric  $g(t)$  on  $M^n$

$$Spec_p(\tilde{g}) = \{0 = \lambda_0(g) \leq \lambda_1(g) \leq \lambda_2(g) \leq \dots \leq \lambda_k(g) \leq \dots\}$$

is the spectrum of  $\Delta_p = \tilde{g} \Delta_p$ . In what follows we assume the existence and  $C^1$ -differentiability of the elements  $\lambda(t)$  and  $f(t)$ , under a Ricci flow deformation  $g(t)$  of a given initial metric. We prove some propositions about the problem of the spectrum variation under a deformation of the metric given by a Ricci flow equation.

### 3. VARIATION OF $\lambda(t)$

In this part, we will give some useful evolution formulas for  $\lambda(t)$  under the Ricci flow. Let  $(M^n, F_t)$ ,  $t \in [0, T)$ , be a deformation of Finsler metric  $F_0$ . Assume that  $\lambda(t)$  is the eigenvalue of  $\Delta_p$ ,  $f = f(x, y, t)$  satisfies

$$\Delta_p f + \lambda |f|^{p-2} f = 0$$

and  $\int_{SM} |f|^p dv = 1$ , using (12), we have:

$$\begin{aligned} \frac{d}{dt} |\tilde{\nabla} f|^2 &= \frac{\partial}{\partial t} (g^{ij}) \frac{\delta f}{\delta x^i} \frac{\delta f}{\delta x^j} + g^{ij} \frac{\partial}{\partial t} \left( \frac{\delta f}{\delta x^i} \right) \frac{\delta f}{\delta x^j} \\ &+ g^{ij} \frac{\delta f}{\delta x^i} \frac{\partial}{\partial t} \left( \frac{\delta f}{\delta x^j} \right) + \frac{\partial (F^2)}{\partial t} g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \\ &+ F^2 \frac{\partial}{\partial t} (g^{ij}) \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} + 2F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j}. \end{aligned} \quad (21)$$

where

$$\frac{\partial}{\partial t} (g^{ij}) = -g^{il} g^{jk} \frac{\partial}{\partial t} (g_{lk}) \quad (22)$$

and

$$\begin{aligned}
\frac{\partial}{\partial t} \left( \frac{\delta f}{\delta x^i} \right) &= \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial x^i} - G_i^r \frac{\partial f}{\partial y^r} \right) \\
&= \frac{\partial f'}{\partial x^i} - G_i^r \frac{\partial f'}{\partial y^r} - \frac{\partial}{\partial t} (G_i^r) \frac{\partial f}{\partial y^r} \\
&= \frac{\delta f'}{\delta x^i} - \frac{\partial}{\partial t} (G_i^r) \frac{\partial f}{\partial y^r}
\end{aligned} \tag{23}$$

therefore, a substitution of (22) and (23) in (21), implies that:

*Proposition 3.1.* Let  $(M^n, F_t)$  be a deformation of Finsler manifold  $(M^n, F_0)$ , then

$$\begin{aligned}
\frac{d}{dt} |\tilde{\nabla} f|^p &= \frac{p}{2} |\tilde{\nabla} f|^{p-2} \left\{ -g^{il} g^{jk} \frac{\partial}{\partial t} (g_{lk}) \frac{\delta f}{\delta x^i} \frac{\delta f}{\delta x^j} + 2g^{ij} \frac{\delta f'}{\delta x^i} \frac{\delta f}{\delta x^j} \right. \\
&\quad \left. - 2g^{ij} \frac{\partial}{\partial t} (G_i^r) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} \right\} + \frac{p}{2} |\tilde{\nabla} f|^{p-2} \left\{ 2F \frac{\partial F}{\partial t} g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \right. \\
&\quad \left. - F^2 g^{il} g^{jk} \frac{\partial}{\partial t} (g_{lk}) \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} + 2F^2 g^{ij} \frac{\partial f'}{\partial y^i} \frac{\partial f}{\partial y^j} \right\}. \square
\end{aligned}$$

On the other hand we have

$$\frac{d}{dt} (dv) = \left\{ g^{ij} \frac{\partial}{\partial t} (g_{ij}) - n \frac{\partial}{\partial t} (\log F) \right\} dv. \tag{24}$$

Now, we get the following two integrability conditions:

$$0 = \frac{d}{dt} \int_{SM} |f|^{p-2} f dv = (p-1) \int_{SM} |f|^{p-2} f' dv + \int_{SM} |f|^{p-2} f \frac{d}{dt} dv$$

therefore

$$(p-1) \int_{SM} |f|^{p-2} f' dv = - \int_{SM} |f|^{p-2} f \left\{ g^{ij} \frac{\partial}{\partial t} (g_{ij}) - n \frac{\partial}{\partial t} (\log F) \right\} dv \tag{25}$$

and

$$0 = \frac{d}{dt} \int_{SM} |f|^p dv = p \int_{SM} f f' |f|^{p-2} dv + \int_{SM} |f|^p \frac{d}{dt} dv$$

which implies

$$p \int_{SM} f f' |f|^{p-2} dv = - \int_{SM} |f|^p \left\{ g^{ij} \frac{\partial}{\partial t} (g_{ij}) - n \frac{\partial}{\partial t} (\log F) \right\} dv. \tag{26}$$

Now, if we suppose that  $g(t)$  is a solution of the un-normalized Ricci flow (1) and (2), then we have:

$$\begin{aligned}
\frac{d\lambda}{dt} &= \int_{SM} \left( \frac{d}{dt} |\tilde{\nabla} f|^p \right) dv + \int_{SM} |\tilde{\nabla} f|^p \frac{d}{dt} (dv) \\
&= \frac{p}{2} \int_{SM} \left\{ -g^{il} g^{jk} (-2Ric_{lk}) \frac{\delta f}{\delta x^i} \frac{\delta f}{\delta x^j} + 2g^{ij} \frac{\delta f'}{\delta x^i} \frac{\delta f}{\delta x^j} \right. \\
&\quad \left. - 2g^{ij} \frac{\partial}{\partial t} (G_i^r) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} \right\} |\tilde{\nabla} f|^{p-2} dv + \frac{p}{2} \int_{SM} \left\{ 2F^2 (-Ric) g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \right. \\
&\quad \left. - F^2 g^{il} g^{jk} (-2Ric_{lk}) \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} + 2F^2 g^{ij} \frac{\partial f'}{\partial y^i} \frac{\partial f}{\partial y^j} \right\} |\tilde{\nabla} f|^{p-2} dv \\
&\quad + \int_{SM} |\tilde{\nabla} f|^p \{ g^{ij} (-2Ric_{ij}) - n(-Ric) \} dv \\
&= p \int_{SM} Ric(\tilde{\nabla} f, \tilde{\nabla} f) |\tilde{\nabla} f|^{p-2} dv + p \int_{SM} \tilde{g}(\tilde{\nabla} f', \tilde{\nabla} f) |\tilde{\nabla} f|^{p-2} dv \\
&\quad + \int_{SM} |\tilde{\nabla} f|^p \{ -2g^{ij} Ric_{ij} + nRic \} dv \\
&\quad - p \int_{SM} g^{ij} \frac{d}{dt} (G_i^r) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} |\tilde{\nabla} f|^{p-2} dv \\
&\quad - p \int_{SM} Ric F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^{p-2} dv
\end{aligned}$$

where  $\frac{d}{dt}(G_i^r)$  is obtained as follows:

$$G^r = \frac{1}{4} g^{rl} \left\{ 2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k, \quad G_i^r = \frac{\partial G^r}{\partial y^i}$$

Hence

$$\begin{aligned}
\frac{d}{dt}(G_i^r) &= -\frac{1}{4} g^{r\alpha} g^{a\beta} \frac{\partial}{\partial t} (g_{\alpha\beta}) g^{lb} \frac{\partial (g_{ab})}{\partial y^i} \left\{ 2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k \\
&\quad - \frac{1}{4} g^{l\alpha} g^{b\beta} \frac{\partial}{\partial t} (g_{\alpha\beta}) g^{ra} \frac{\partial (g_{ab})}{\partial y^i} \left\{ 2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k \\
&\quad - \frac{1}{4} g^{ra} g^{lb} \frac{\partial (g'_{ab})}{\partial y^i} \left\{ 2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k \\
&\quad - \frac{1}{4} g^{ra} g^{lb} \frac{\partial (g_{ab})}{\partial y^i} \left\{ 2 \frac{\partial g'_{jl}}{\partial x^k} - \frac{\partial g'_{jk}}{\partial x^l} \right\} y^j y^k \\
&\quad + \frac{1}{4} g^{r\alpha} g^{l\beta} \frac{\partial (g_{\alpha\beta})}{\partial t} \left\{ 2 \frac{\partial^2 g_{jl}}{\partial y^i \partial x^k} - \frac{\partial^2 g_{jk}}{\partial y^i \partial x^l} \right\} y^j y^k \\
&\quad + \frac{1}{4} g^{rl} \left\{ 2 \frac{\partial^2 g'_{jl}}{\partial y^i \partial x^k} - \frac{\partial^2 g'_{jk}}{\partial y^i \partial x^l} \right\} y^j y^k \\
&\quad + \frac{1}{4} g^{r\alpha} g^{l\beta} \frac{\partial (g_{\alpha\beta})}{\partial t} \left\{ 2 \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l} \right\} y^k + \frac{1}{4} g^{rl} \left\{ 2 \frac{\partial g'_{il}}{\partial x^k} - \frac{\partial g'_{ik}}{\partial x^l} \right\} y^k \\
&\quad + \frac{1}{4} g^{r\alpha} g^{l\beta} \frac{\partial (g_{\alpha\beta})}{\partial t} \left\{ 2 \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right\} y^j + \frac{1}{4} g^{rl} \left\{ 2 \frac{\partial g'_{jl}}{\partial x^i} - \frac{\partial g'_{ij}}{\partial x^l} \right\} y^j.
\end{aligned} \tag{27}$$



From the un-normalized Ricci flow, we can then write

$$\begin{aligned}
\frac{d}{dt}(G_i^r) &= \frac{1}{2}g^{r\alpha}g^{a\beta}Ric_{\alpha\beta}g^{lb}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k \\
&+ \frac{1}{2}g^{l\alpha}g^{b\beta}Ric_{\alpha\beta}g^{ra}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k \\
&+ \frac{1}{2}g^{ra}g^{lb}\frac{\partial(Ric_{ab})}{\partial y^i}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k \\
&+ \frac{1}{2}g^{ra}g^{lb}\frac{\partial(g_{ab})}{\partial y^i}\left\{2\frac{\partial Ric_{jl}}{\partial x^k}-\frac{\partial Ric_{jk}}{\partial x^l}\right\}y^jy^k \\
&- \frac{1}{2}g^{r\alpha}g^{l\beta}Ric_{\alpha\beta}\left\{2\frac{\partial^2 g_{jl}}{\partial y^i\partial x^k}-\frac{\partial^2 g_{jk}}{\partial y^i\partial x^l}\right\}y^jy^k \\
&- \frac{1}{2}g^{rl}\left\{2\frac{\partial^2 Ric_{jl}}{\partial y^i\partial x^k}-\frac{\partial^2 Ric_{jk}}{\partial y^i\partial x^l}\right\}y^jy^k \\
&- \frac{1}{2}g^{r\alpha}g^{l\beta}Ric_{\alpha\beta}\left\{2\frac{\partial g_{il}}{\partial x^k}-\frac{\partial g_{ik}}{\partial x^l}\right\}y^k \\
&- \frac{1}{2}g^{rl}\left\{2\frac{\partial Ric_{il}}{\partial x^k}-\frac{\partial Ric_{ik}}{\partial x^l}\right\}y^k \\
&- \frac{1}{2}g^{r\alpha}g^{l\beta}Ric_{\alpha\beta}\left\{2\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\right\}y^j \\
&- \frac{1}{2}g^{rl}\left\{2\frac{\partial Ric_{jl}}{\partial x^i}-\frac{\partial Ric_{ij}}{\partial x^l}\right\}y^j.
\end{aligned} \tag{28}$$

Using (26) we obtain

$$\begin{aligned}
p \int_{SM} \tilde{g}(\tilde{\nabla}f', \tilde{\nabla}f)|\tilde{\nabla}f|^{p-2}dv &= p\lambda \int_{SM} f'f|f|^{p-2}dv \\
&= -\lambda \int_{SM} |f|^p \{-2g^{ij}Ric_{ij} + nRic\} dv.
\end{aligned}$$

We have thus proved the following proposition:

*Proposition 3.2.* Let  $(M^n, F_t)$  be a solution of the un-normalized Ricci flow on the smooth Finsler manifold  $(M^n, F_0)$ . If  $\lambda(t)$  denotes the evolution of an eigenvalue under the Ricci flow, then

$$\begin{aligned}
\frac{d\lambda}{dt} &= p \int_{SM} Ric(\tilde{\nabla}f, \tilde{\nabla}f)|\tilde{\nabla}f|^{p-2}dv \\
&+ \int_{SM} (\lambda|f|^p - |\tilde{\nabla}f|^p) \{2g^{ij}Ric_{ij} - nRic\} dv
\end{aligned} \tag{29}$$

$$\begin{aligned}
&- p \int_{SM} g^{ij} \frac{\partial}{\partial t}(G_i^r) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} |\tilde{\nabla}f|^{p-2}dv \\
&- p \int_{SM} Ric F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla}f|^{p-2}dv
\end{aligned} \tag{30}$$

where  $f$  is the associated normalized evolving eigenfunction.  $\square$

NOTE. Let  $f : SM \rightarrow \mathbb{R}$  be a lifting of  $f : M \rightarrow \mathbb{R}$ . We have:

$$\begin{aligned} \frac{d\lambda}{dt} &= p \int_{SM} Ric(\tilde{\nabla}f, \tilde{\nabla}f)|\tilde{\nabla}f|^{p-2} dv \\ &\quad + \int_{SM} (\lambda|f|^p - |\tilde{\nabla}f|^p) \{-2g^{ij} Ric_{ij} + n\mathcal{R}ic\} dv \end{aligned}$$

and in this case, if  $-2g^{ij} Ric_{ij} + n\mathcal{R}ic$  is a constant, then

$$\frac{d\lambda}{dt} = p \int_{SM} Ric(\tilde{\nabla}f, \tilde{\nabla}f)|\tilde{\nabla}f|^{p-2} dv.$$

**Corollary 3.3.** *Let  $(M^n, F_t)$  be a solution of the un-normalized Ricci flow on the smooth Riemannian manifold  $(M^n, F_0)$ , i.e.  $F_t, F_0$  are Riemannian metric. If  $\lambda(t)$  denotes the evolution of an eigenvalue under Ricci flow, then:*

$$\begin{aligned} \frac{d\lambda}{dt} &= p \int_{SM} Ric(\tilde{\nabla}f, \tilde{\nabla}f)|\tilde{\nabla}f|^{p-2} dv + \int_{SM} R(\lambda|f|^p - |\tilde{\nabla}f|^p) dv \\ &\quad - p \int_{SM} g^{ij} \frac{\partial}{\partial t} (G_r^i) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} |\tilde{\nabla}f|^{p-2} dv \\ &\quad - \frac{p}{n} \int_{SM} RF^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla}f|^{p-2} dv \end{aligned} \tag{31}$$

where  $R$  is the scalar curvature of  $M$ .

**Proof:**

If  $F_t$  is the Riemannian metric, then

$$Ric = \frac{1}{n}R, \tag{32}$$

and

$$2g^{ij} Ric_{ij} - n\mathcal{R}ic = R, \tag{33}$$

therefore (31) is obtained by replacing (32) and (33) in (29).

□

**Corollary 3.4.** *Let  $(M^2, F_t)$  be a solution of the un-normalized Ricci flow on the smooth Riemannian surface  $(M^2, F_0)$ . If  $\lambda(t)$  denotes the evolution of an eigenvalue under the Ricci flow, then:*

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{p}{2} \int_{SM} R|\tilde{\nabla}f|^p dv + \int_{SM} R(\lambda|f|^p - |\tilde{\nabla}f|^p) dv \\ &\quad - p \int_{SM} g^{ij} \frac{\partial}{\partial t} (G_r^i) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} |\tilde{\nabla}f|^{p-2} dv \\ &\quad - \frac{p}{n} \int_{SM} RF^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla}f|^{p-2} dv \end{aligned}$$

where  $R$  is the scalar curvature of  $M$ .

**Proof:** In dimension  $n = 2$ , for a Riemannian manifold, we have:

$$Ric = \frac{1}{2}Rg, \tag{34}$$

hence the corollary is obtained by replacing (34) in (29).  $\square$

**Corollary 3.5.** *Let  $(M^n, F_t)$  be a solution of the un-normalized Ricci flow on the smooth homogenous Riemannian manifold  $(M^n, F_0)$ . If  $\lambda(t)$  denotes the evolution of an eigenvalue under the Ricci flow, then:*

$$\begin{aligned} \frac{d\lambda}{dt} &= p \int_{SM} Ric(\tilde{\nabla} f, \tilde{\nabla} f) |\tilde{\nabla} f|^{p-2} dv \\ &\quad - p \int_{SM} g^{ij} \frac{\partial}{\partial t} (G_r^i) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} |\tilde{\nabla} f|^{p-2} dv \\ &\quad - R \frac{p}{n} \int_{SM} F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^{p-2} dv \end{aligned}$$

where  $R$  is the scalar curvature of  $M$ .

**Proof:**

Since the evolving metric remains homogenous and a Riemannian homogenous manifold has constant scalar curvature, so the corollary is obtained by (29).  $\square$

Now, we give a variation of  $\lambda(t)$  under the normalized Ricci flow which is similar to the pervious proposition.

*Proposition 3.6.* Let  $(M^n, F_t)$  be a solution of the normalized Ricci flow on the smooth Finsler manifold  $(M^n, F_0)$ . If  $\lambda(t)$  denotes the evolution of an eigenvalue under Ricci flow, then:

$$\begin{aligned} \frac{d\lambda}{dt} &= (n-p)r\lambda + p \int_{SM} Ric(\tilde{\nabla} f, \tilde{\nabla} f) |\tilde{\nabla} f|^{p-2} dv \\ &\quad + \int_{SM} (\lambda |f|^p - |\tilde{\nabla} f|^p) \{2g^{ij} Ric_{ij} - n\mathcal{R}ic\} dv \quad (35) \\ &\quad - p \int_{SM} g^{ij} \frac{\partial}{\partial t} (G_i^s) \frac{\partial f}{\partial y^s} \frac{\delta f}{\delta x^j} |\tilde{\nabla} f|^{p-2} dv \\ &\quad - p \int_{SM} F^2 (\mathcal{R}ic - r) g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^{p-2} dv \end{aligned}$$

where  $f$  is the associated normalized evolving eigenfunction,  $r = \frac{\int_{SM} \mathcal{R}ic dv}{vol(SM)}$ .

**Proof:** In the normalized case, the integrability conditions read as follows

$$p \int_{SM} f' f |f|^{p-2} dv = \int_{SM} |f|^p \{2g^{ij} Ric_{ij} - nr - n\mathcal{R}ic\} dv. \quad (36)$$

Since

$$\frac{d}{dt}(dv) = \{-2g^{ij} Ric_{ij} + nr + n\mathcal{R}ic\} dv \quad (37)$$

using (24), (27) and the above equation, we can then write

$$\begin{aligned}
\frac{d\lambda}{dt} &= \int_{SM} \left( \frac{d}{dt} |\tilde{\nabla} f|^p \right) dv + \int_{SM} |\tilde{\nabla} f|^p \frac{d}{dt} (dv_t) \\
&= \frac{p}{2} \int_{SM} \left\{ -g^{il} g^{jk} (-2Ric_{lk} + 2rg_{lk}) \frac{\delta f}{\delta x^i} \frac{\delta f}{\delta x^j} + 2g^{ij} \frac{\delta f'}{\delta x^i} \frac{\delta f}{\delta x^j} \right\} |\tilde{\nabla} f|^{p-2} dv \\
&\quad + \frac{p}{2} \int_{SM} \left\{ 2F^2 (-\mathcal{R}ic + r) g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \right. \\
&\quad \left. - F^2 g^{il} g^{jk} (-2Ric_{lk} + 2rg_{lk}) \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \right\} |\tilde{\nabla} f|^{p-2} dv \\
&\quad + \frac{p}{2} \int_{SM} \left\{ -2g^{ij} \frac{\partial}{\partial t} (G_i^r) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} + 2F^2 g^{ij} \frac{\partial f'}{\partial y^i} \frac{\partial f}{\partial y^j} \right\} |\tilde{\nabla} f|^{p-2} dv \\
&\quad + \int_{SM} |\tilde{\nabla} f|^p \left\{ -2g^{ij} Ric_{ij} + nr + n\mathcal{R}ic \right\} dv \\
&= (n-p)r\lambda + p \int_{SM} Ric(\tilde{\nabla} f, \tilde{\nabla} f) |\tilde{\nabla} f|^{p-2} dv + p \int_{SM} \tilde{g}(\tilde{\nabla} f', \tilde{\nabla} f) |\tilde{\nabla} f|^{p-2} dv \\
&\quad + \int_{SM} |\tilde{\nabla} f|^p \left\{ -2g^{ij} Ric_{ij} + n\mathcal{R}ic \right\} dv \\
&\quad - p \int_{SM} g^{ij} \frac{d}{dt} (G_i^r) \frac{\partial f}{\partial y^r} \frac{\delta f}{\delta x^j} |\tilde{\nabla} f|^{p-2} dv \\
&\quad - p \int_{SM} F^2 (\mathcal{R}ic - r) g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^{p-2} dv
\end{aligned} \tag{38}$$

but

$$\begin{aligned}
p \int_{SM} \tilde{g}(\tilde{\nabla} f', \tilde{\nabla} f) |\tilde{\nabla} f|^{p-2} dv &= p\lambda \int_{SM} f' f |f|^{p-2} dv \\
&= \lambda \int_{SM} |f|^p \left\{ 2g^{ij} Ric_{ij} - nr - n\mathcal{R}ic \right\} dv \tag{39}
\end{aligned}$$

and  $\frac{\partial}{\partial t} (G_i^s)$  is obtained by replacing  $F'$  and  $g'_{ij}$  from (3) and (4), respectively, in (27). Thus the proposition is obtained by replacing (39) in (38).  $\square$

Similar to un-normalized case we have the following corollaries:

**Corollary 3.7.** *Let  $(M^n, F_t)$  be a solution of the normalized Ricci flow on the smooth Riemannian manifold  $(M^n, F_0)$ . If  $\lambda(t)$  denotes the evolution of an eigenvalue under Ricci flow, then:*

$$\begin{aligned}
\frac{d\lambda}{dt} &= (n-p)r\lambda + p \int_{SM} Ric(\tilde{\nabla} f, \tilde{\nabla} f) |\tilde{\nabla} f|^{p-2} dv \\
&\quad + \int_{SM} R(\lambda |f|^p - |\tilde{\nabla} f|^p) dv - p \int_{SM} g^{ij} \frac{\partial}{\partial t} (G_i^s) \frac{\partial f}{\partial y^s} \frac{\delta f}{\delta x^j} |\tilde{\nabla} f|^{p-2} dv \tag{40} \\
&\quad - p \int_{SM} \left( \frac{1}{n} R - r \right) F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^{p-2} dv
\end{aligned}$$

where  $R$  is the scalar curvature of  $M$ .

**Corollary 3.8.** *Let  $(M^2, F_t)$  be a solution of the normalized Ricci flow on the smooth Riemannian surface  $(M^2, F_0)$ . If  $\lambda(t)$  denotes the evolution of an eigenvalue*

under Ricci flow, then:

$$\begin{aligned} \frac{d\lambda}{dt} &= (2-p)r\lambda + \frac{p}{2} \int_{SM} R|\tilde{\nabla}f|^p dv \\ &+ \int_{SM} R(\lambda|f|^p - |\tilde{\nabla}f|^p)dv - p \int_{SM} g^{ij} \frac{\partial}{\partial t}(G_i^s) \frac{\partial f}{\partial y^s} \frac{\delta f}{\delta x^j} |\tilde{\nabla}f|^{p-2} dv \\ &- p \int_{SM} \left(\frac{R}{2} - r\right) F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla}f|^{p-2} dv \end{aligned}$$

where  $R$  is the scalar curvature of  $M$ .

**Corollary 3.9.** *Let  $(M^n, F_t)$  be a solution of the normalized Ricci flow on the smooth homogenous Riemannian manifold  $(M^n, F_0)$ . If  $\lambda(t)$  denotes the evolution of an eigenvalue under Ricci flow, then:*

$$\begin{aligned} \frac{d\lambda}{dt} &= (n-p)r\lambda + p \int_{SM} Ric(\tilde{\nabla}f, \tilde{\nabla}f) |\tilde{\nabla}f|^{p-2} dv \\ &- p \int_{SM} g^{ij} \frac{\partial}{\partial t}(G_i^s) \frac{\partial f}{\partial y^s} \frac{\delta f}{\delta x^j} |\tilde{\nabla}f|^{p-2} dv \\ &- p \int_{SM} \left(\frac{1}{n}R - r\right) F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla}f|^{p-2} dv \end{aligned}$$

where  $R$  is the scalar curvature of  $M$ .

#### 4. EXAMPLES

In this section, we will find the variational formula for some of Finsler manifolds.

**Example 4.1.** *Let  $(M^n, F_0)$  be an Einstein manifold i.e. there exists a constant  $a$  such that  $Ric(F_0) = aF_0^2$ . Therefore  $Ric_{ij}(g_0) = ag_{ij}(0)$ . Assume we have a solution to the Ricci flow which is of the form*

$$g(t) = u(t)g_0, \quad u(0) = 1$$

where  $u(t)$  is a positive function. Now (2) implies that

$$u(t) = -2at + 1,$$

so that we have

$$g(t) = (1 - 2at)g_0$$

which says that  $g(t)$  is an Einstein metric. On the other hand it is easily seen that

$$\begin{aligned} Ric(g(t)) &= Ric(g_0) = ag_0 = \frac{a}{1-2at}g(t), \\ Ric(g_t) &= \frac{1}{1-2at}Ric(g_0) = \frac{a}{1-2at}, \\ F_t^2 &= (1-2at)F_0^2 \end{aligned}$$

therefore

$$2g^{ij} Ric_{ij} - nRic = \frac{an}{1-2at}$$

and

$$Ric(\tilde{\nabla}f, \tilde{\nabla}f) = \frac{a}{1-2at} \tilde{g}(\tilde{\nabla}f, \tilde{\nabla}f) = \frac{a}{1-2at} |\tilde{\nabla}f|^2.$$

Also

$$\begin{aligned} G^i(t) &= \frac{1}{4} g^{il} \left\{ 2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k \\ &= \frac{1}{4} (g_0)^{il} \left\{ 2 \frac{\partial (g_0)_{jl}}{\partial x^k} - \frac{\partial (g_0)_{jk}}{\partial x^l} \right\} y^j y^k = G^i(0) \end{aligned}$$

therefore

$$\frac{\partial}{\partial t} (G_r^i) = 0.$$

Using the un-normalized Ricci flow equation (2) and (29), we obtain the following relation:

$$\begin{aligned} \frac{d\lambda}{dt} &= p \int_M \frac{a}{1-2at} |\tilde{\nabla}f|^p dv + \lambda \int_M |f|^p \frac{an}{1-2at} dv \\ &\quad - \int_M \frac{an}{1-2at} |\tilde{\nabla}f|^p dv - p \int_{SM} \frac{a}{1-2at} F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla}f|^{p-2} dv \\ &= \frac{pa}{1-2at} \left\{ \lambda - \int_{SM} F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla}f|^{p-2} dv \right\} \end{aligned}$$

Now, If we suppose that  $g_t = u(t)g_0$ ,  $u(0) = 1$  is a solution of the normalized Ricci flow and  $r = \frac{1}{Vol(SM)} \int_{SM} Ric dv$ , then from (3) we have

$$u'g_0 = -2ag_0 + 2rug_0.$$

It implies that

$$u(t) = \frac{a}{r} (1 - e^{2rt}) + e^{2rt}.$$

So that we have:

$$g(t) = \left\{ \frac{a}{r} (1 - e^{2rt}) + e^{2rt} \right\}$$

and

$$\begin{aligned} Ric(g(t)) &= a \left( \frac{a}{r} (1 - e^{2rt}) + e^{2rt} \right)^{-1} g_0 \\ Ric(g(t)) &= a \left( \frac{a}{r} (1 - e^{2rt}) + e^{2rt} \right)^{-1} \\ 2g^{ij} Ric_{ij} - Ric_{ij} &= an \left( \frac{a}{r} (1 - e^{2rt}) + e^{2rt} \right)^{-1}, \end{aligned}$$

also

$$G^i(t) = G^i(0),$$

therefore

$$\frac{\partial G_i^r}{\partial t} = 0.$$

Using (35), we obtain the following:

$$\begin{aligned} \frac{d\lambda}{dt} &= (n-p)r\lambda + pa\lambda \left( \frac{a}{r}(1-e^{2rt}) + e^{2rt} \right)^{-1} \\ &\quad - p \int_{SM} \left( a \left( \frac{a}{r}(1-e^{2rt}) + e^{2rt} \right)^{-1} - r \right) (y^i \frac{\partial f}{\partial y_i})^2 |\tilde{\nabla} f|^{p-2} dv \\ &= \left( (n-p)r + pa \left( \frac{a}{r}(1-e^{2rt}) + e^{2rt} \right)^{-1} \right) \lambda \\ &\quad - p \left( a \left( \frac{a}{r}(1-e^{2rt}) + e^{2rt} \right)^{-1} - r \right) \int_{SM} F^2 g^{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} |\tilde{\nabla} f|^{p-2} dv. \end{aligned}$$

REMARK. Let  $(M^n, F_0)$  be a Finsler manifold of dimension  $n \geq 3$ . Suppose that the flag curvature  $k = k(x)$  is isotropic and a function of  $x \in M$  alone then  $k = \text{constant}$  and therefore  $(M^n, F_0)$  is Einstein and the variation of its eigenvalues is similar to example (4.1).

**Definition 4.2.** A Finsler metric on an  $n$ -dimensional manifold is called a weak Einstein metric if

$$\text{Ric} = (n-1) \left\{ \frac{3\eta}{F} + \sigma \right\} F^2$$

where  $\eta$  is a 1-form and  $\sigma = \sigma(x)$  is scalar function.

**Example 4.3.** If we suppose that  $F_t = u(t)F_0$ ,  $u(0) = 1$  is a solution of the Ricci flow, then:

$$\text{Ric}(F_t) = \frac{\text{Ric}(F_t)}{F_t^2} = \frac{\text{Ric}(F_0)}{(u(t))^2 F_0^2} = \frac{(n-1) \left\{ \frac{3\eta_0}{F_0} + \sigma_0 \right\} F_0^2}{(u(t))^2 F_0^2} = \frac{(n-1) \left\{ \frac{3\eta_0}{F_0} + \sigma_0 \right\}}{(u(t))^2}$$

Now the Ricci flow (1) implies that

$$-\frac{(n-1) \left\{ \frac{3\eta_0}{F_0} + \sigma_0 \right\}}{(u(t))^2} = -\text{Ric} = \frac{\partial \log F}{\partial t} = \frac{F'_t}{F_t} = \frac{u'(t)}{u(t)}$$

or equivalently

$$uu' = (n-1) \left\{ \frac{3\eta_0}{F_0} + \sigma_0 \right\}$$

By integration we have:

$$u^2(t) = -2(n-1) \left\{ \frac{3\eta_0}{F_0} + \sigma_0 \right\} t + c$$

with condition  $u(0) = 1$  we have:

$$u^2(t) = 1 - 2(n-1) \left\{ \frac{3\eta_0}{F_0} + \sigma_0 \right\} t$$

therefore

$$F_t^2 = \left\{ 1 - 2(n-1) \left\{ \frac{3\eta_0}{F_0} + \sigma_0 \right\} t \right\} F_0^2 \quad (41)$$

and

$$\text{Ric}(\tilde{\nabla}f, \tilde{\nabla}f) = \frac{1}{2}(n-1)g^{il}g^{js} \{ (3\eta_0 F_0)_{y^t y^s} + 2\sigma_0(g_0)_{ls} \} \tilde{\nabla}_i f \tilde{\nabla}_j f. \quad (42)$$

Also we have

$$2g^{ij} \text{Ric}_{ij} - n\text{Ric} = (n-1)g^{ij} \{ (3\eta_0 F_0)_{y^i y^j} + 2\sigma_0(g_0)_{ij} \} - n \frac{(n-1) \{ \frac{3\eta_0}{F_0} + \sigma_0 \}}{1 - 2(n-1) \{ \frac{3\eta_0}{F_0} + \sigma_0 \} t}. \quad (43)$$

Note that

$$g_{ij}(t) = (1 - 2(n-1)\sigma_0 t)(g_0)_{ij} - 6(n-1)t \frac{\partial \eta}{\partial y^i} \frac{\partial F}{\partial y^j}. \quad (44)$$

By replacing (42), (43) and (44) in (29) we obtain the variation of an eigenvalue. Now, if we suppose that  $F_t = u(t)F_0$ ,  $u(0) = 1$  is a solution of the normalized Ricci flow and  $r = \frac{1}{\text{Vol}(SM)} \int_{SM} \text{Ric} dv$ , then from (3), we have:

$$r - \frac{(n-1) \frac{3\eta_0}{F_0} + \sigma_0}{u^2(t)} = r - \mathcal{R}ic = \frac{\partial \log F}{\partial t} = \frac{u'(t)}{u(t)}$$

. It implies that

$$uu' - u^2 r = -(n-1) \frac{3\eta_0}{F_0} + \sigma_0, \quad u(0) = 1$$

which is an ordinary differential equation and has a solution as follow:

$$u^2(t) = \frac{n-1}{r} \frac{3\eta_0}{F_0} + \sigma_0(1 - e^{2rt}) + e^{2rt}$$

therefore

$$F_t^2 = \left\{ \frac{n-1}{r} \frac{3\eta_0}{F_0} + \sigma_0(1 - e^{2rt}) + e^{2rt} \right\} F_0^2$$

and  $\frac{d\lambda}{dt}$  is obtained from (35).

**Example 4.4.** In this example we determine the behavior of the evolving spectrum on the Ricci solitons.

Let  $F_t$  is a solution of the Ricci flow  $\frac{\partial \log F}{\partial t} = -\mathcal{R}ic$ . If  $\varphi$  is a time-independent isometry such that

$$F(x, y) = \varphi^* F_t(x, y)$$

is a solution of the un-normalized Ricci flow, because of  $\frac{\partial}{\partial t} \log F_t(x, y) = -\mathcal{R}ic(x, y)$ ,  $F(0) = F_0$ ,

$$\frac{\partial}{\partial t} \log F(x, y) = \frac{\partial}{\partial t} \varphi^* \log F_t(x, y) = \frac{\partial}{\partial t} \log F_t(\varphi(x), \varphi_*(y))$$

since  $\varphi$  is a isometry we have

$$\frac{\partial}{\partial t} \log F(x, y) = \frac{\partial}{\partial t} \log F_t(x, y) = -\mathcal{R}ic(x, y) = -\mathcal{R}ic(\varphi(x), \varphi_*(y)) = -\mathcal{R}ic(F)$$

**Definition 4.5.** Let  $(M, F_t)$  is a solution of the Ricci flow and  $\varphi_t$  is a family of diffeomorphisms. We says  $F(t)$  is Ricci soliton, when satisfies in

$$F_t^2 = u(t)\varphi_t^* F_0^2. \quad (45)$$



Let  $(M, F)$  and  $(\overline{M}, \overline{F})$  be two closed Finsler manifolds and

$$\varphi : (M, g) \rightarrow (\overline{M}, \overline{F})$$

an isometry, then for  $p = 2$  we have

$$\tilde{g} \Delta_p \circ \varphi^* = \varphi^* \circ \tilde{g} \Delta_p.$$

Therefore given a diffeomorphism  $\varphi : M \rightarrow M$  we have that

$$\varphi : (SM, \varphi^* \tilde{g}) \rightarrow (SM, \tilde{g})$$

is an isometry, hence we conclude that  $(SM, \varphi^* \tilde{g})$ , and  $(SM, \tilde{g})$  have the same spectrum

$$Spec_p(\tilde{g}) = Spec_p(\varphi^* \tilde{g})$$

with eigenfunction  $f_k$  and  $\varphi^* f_k$  respectively. If  $g(t)$  is a Ricci soliton on  $(M^n, g_0)$  then

$$Spec_p(\tilde{g}(t)) = \frac{1}{u(t)} Spec_p(\tilde{g}_0)$$

so that  $\lambda(t)$  satisfies

$$\lambda(t) = \frac{1}{u(t)}, \quad \frac{d\lambda}{dt} = -\frac{u'(t)}{(u(t))^2}.$$

**Example 4.6.** Suppose that

$$\mathbb{R}_+^n = \{(x^1, x^2, \dots, x^n) \in \mathbb{R}^n \mid x^i > 0, i = 1, \dots, n\}$$

has the metric

$$g_{ij}(x, y) = \begin{cases} \frac{\phi_t(y)}{(x^i)^{\frac{2(p+1)}{p}}} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (46)$$

where it is a solution for the un-normalized Ricci flow and  $\phi_t(y)$  is a strictly positive  $C^\infty$  homogeneous function of degree zero and  $p > 0$ . We use the formula

$$g^{ij} = \begin{cases} \frac{(x^i)^{\frac{2(p+1)}{p}}}{\phi_t(y)} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (47)$$

for  $i = 1, \dots, n$ , we have

$$G^i = -\frac{p+1}{2p} \frac{(y^i)^2}{x^i}$$

and

$$F^2 Ric(g(t)) = 0$$

therefore  $Ric(g(t)) = 0$  and the un-normalized Ricci flow equation implies that

$$\frac{\partial g(t)}{\partial t} = 0$$

hence,  $g(t) = g_0$  and  $\lambda(t) = \lambda(0)$ .

**Example 4.7.** Suppose that  $\mathbb{R}^2$  has the metric

$$g_{ij}(x, y) = \phi_t(y) \begin{pmatrix} 4(x^1)^2 + 1 & -2x^2 \\ -2x^1 & 1 \end{pmatrix} \quad (48)$$

which is a solution for the the un-normalized Ricci flow, where  $\phi_t(y)$  is a strictly positive  $C^\infty$  homogeneous function of degree zero. We obtain

$$G^1 = 0, \quad G^2 = -(y^1)^2$$

and

$$F^2 \text{Ric}(g(t)) = 0$$

therefore  $\text{Ric}(g(t)) = 0$  and the un-normalized Ricci flow equation implies that

$$\frac{\partial g(t)}{\partial t} = 0$$

hence,  $g(t) = g_0$  and  $\lambda(t) = \lambda(0)$ .

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