

STRONGLY CLEAN ELEMENTS IN A CERTAIN BLOCK MATRIX RING

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Abstract. In this paper, we investigate the concepts of strongly clean, strongly π -regular and strongly J_n -clean related to the ring $A = \begin{pmatrix} R & {}_R M_S \\ 0 & S \end{pmatrix}$. Moreover, we give several equivalent conditions such that an element $\alpha \in \text{End}(A)$ is strongly clean.

Key words and Phrases: Strongly clean, strongly π -regular, strongly J_n -clean, endomorphism.

Abstrak. Dalam makalah ini, konsep dari *strongly clean*, *strongly π -regular* dan *strongly J_n -clean* yang berhubungan dengan ring $A = \begin{pmatrix} R & {}_R M_S \\ 0 & S \end{pmatrix}$ diselidiki. Selanjutnya, beberapa kondisi ekuivalen diberikan sedemikian sehingga sebuah element $\alpha \in \text{End}(A)$ adalah *strongly clean*.

Kata kunci: Strongly clean, strongly π -regular, strongly J_n -clean, endomorphism.

1. INTRODUCTION

An element in a ring R is strongly clean provided that it is the sum of an idempotent and a unit that commute with each other. This notion was firstly introduced by Nicholson in [?]. Let R, S be two rings, and let M be an $R - S$ -bimodule. This means that M is a left R -module and a right S -module such that $(rm)s = r(ms)$ for all $r \in R, m \in M$, and $s \in S$. Given such a bimodule M , we can form

$$A = \begin{pmatrix} R & {}_R M_S \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

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and define a multiplication on A by using formal matrix multiplication:

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rm' + ms' \\ 0 & ss' \end{pmatrix}.$$

In [?], a characterization of strongly J_n -clean rings by virtue of strongly π -regularity is given. The main purpose of this note is to study about strongly clean, strongly π -regular and strongly J_n -clean of the ring A . We give several equivalent conditions under which such element $\alpha \in \text{End}(A)$ is strongly clean.

2. STRONGLY CLEAN, STRONGLY π -REGULAR AND STRONGLY J_n -CLEAN

Let us define the following subsets of a ring R .

$$\begin{aligned} \mathcal{U}(R) &= \{r \in R \mid r \text{ is a unit of } R\}, \\ \mathcal{ID}(R) &= \{r \in R \mid r \text{ is an idempotent of } R\}. \end{aligned}$$

We begin with

Proposition 2.1. Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are commutative rings and ${}_R M_S$ is a bimodule. Then, A is a strongly clean ring if and only if R and S are strongly clean rings.

Proof. Suppose that $r \in R, s \in S$ and $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in A$. Then, $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix}$ such that $r_1 \in \mathcal{U}(R), s_1 \in \mathcal{U}(S), r_2 \in \mathcal{ID}(R), s_2 \in \mathcal{ID}(S), m_1, m_2 \in M$ and $r_2 m_2 + m_2 s_2 = m_2$. So, $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & m_1 + m_2 \\ 0 & s_1 + s_2 \end{pmatrix}$. Hence, $r = r_1 + r_2$ and $s = s_1 + s_2$.

For the converse, suppose that R and S are strongly clean rings and $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in A$. Then, $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & m \\ 0 & s_1 + s_2 \end{pmatrix}$ such that $r_1 \in \mathcal{U}(R), s_1 \in \mathcal{U}(S), r_2 \in \mathcal{ID}(R)$ and $s_2 \in \mathcal{ID}(S)$. So, $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix}$, where $m_1 + m_2 = m$ and $r_2 m_2 + m_2 s_2 = m_2$. Hence, $\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} \in \mathcal{U}(A)$ and $\begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix} \in \mathcal{ID}(A)$. \square

Definition 2.2. An element in a ring R is called strongly π -regular if for every $a \in R$, the chain $aR \supseteq a^2R \supseteq \dots$ terminates (or equivalently, the chain $Ra \supseteq Ra^2 \supseteq \dots$ terminates, see [?, ?]).

As defined by artinian ring, if $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is artinian, then the chain

$$A \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \supseteq A \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}^2 \supseteq \dots$$

terminates. So, A is strongly π -regular.

REMARK 1. [?] Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are rings and ${}_R M_S$ is a bimodule. Then, A is artinian if and only if R and S are artinian.

Proposition 2.3. Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are artinian rings and M as a $R - S$ -module is artinian. Then, A is strongly π -regular.

Proof. The verification is straightforward. \square

Definition 2.4. We say that $x \in R$ is strongly J_n -clean provided that there exists an idempotent $e \in R$ such that $x - e \in \mathcal{U}(R)$, $ex = xe$ and $(ex)^n \in J(R)$, where $J(R)$ is the Jacobson radical of R .

Proposition 2.5. Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are commutative local rings and ${}_R M_S$ is a bimodule. Then, $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in A$ is strongly J_n -clean if there exists an idempotent $\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} \in A$ such that $r_1 m + m_1 s = r m_1 + m s_1$ and $(r_1 r)^n = (s_1 s)^n = 0$.

Proof. By $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} = \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ and $J(A) = \begin{pmatrix} J(R) & M \\ 0 & J(S) \end{pmatrix} = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$, the proof is clear. \square

3. ENDOMORPHISMS AND STRONGLY CLEAN

In this section, we study the endomorphisms of the ring A and also necessary and sufficient conditions under which an endomorphism of the ring A is strongly clean.

Lemma 3.1. Let $A = \begin{pmatrix} R & {}_R M_S \\ 0 & S \end{pmatrix}$, $\alpha \in \text{End}(A)$ and let $\pi^2 = \pi \in \text{End}(A)$. Then,

- (1) $\pi(A)$ is α -invariant if and only if $\pi\alpha = \pi\alpha\pi$.
- (2) Both $\pi(A)$ and $(1 - \pi)(A)$ are α -invariant if and only if $\pi\alpha = \alpha\pi$.

Proof. The proof is a routine verification. \square

Proposition 3.2. Let $A = \begin{pmatrix} R & {}_R M_S \\ 0 & S \end{pmatrix}$, $\alpha \in \text{End}(A)$ and let

$$A = \underbrace{\begin{pmatrix} R & {}_R P_S \\ 0 & S \end{pmatrix}}_{A'} \oplus \underbrace{\begin{pmatrix} R & {}_R Q_S \\ 0 & S \end{pmatrix}}_{A''},$$

where A' and A'' are both α -invariant. The following are equivalent.

- (1) $\alpha|_{A'}$ is an isomorphism.
- (2) $\ker(\alpha) \subseteq A''$ and $A' \subseteq \alpha(A)$.

Proof. (1) \Rightarrow (2). If $F \in \ker(\alpha)$ write $F = G + H$. Then, $0 = \alpha(F) = \alpha(G + H) = \alpha(G) + \alpha(H) \in A' \oplus A''$, so $0 = \alpha(G) = \alpha|_{A'}(G)$. Thus, we have $G = 0$ by (1), so $F = H \in A''$. Hence, $\ker(\alpha) \subseteq A''$. Finally, $A' = \alpha|_{A'}(A') = \alpha(A') \subseteq \alpha(A)$.

(2) \Rightarrow (1). $\alpha|_{A'}$ is one-to-one because $\ker(\alpha|_{A'}) = \ker(\alpha) \cap A' \subseteq A'' \cap A' = 0$. Let $\pi^2 = \pi \in \text{End}(A)$ satisfy $\ker(\pi) = A''$ and $\pi(A) = A'$. Then, $\alpha\pi = \pi\alpha$ by ??, so $A' = \pi(A') \subseteq \pi\alpha(A) = \alpha\pi(A) = \alpha(A')$ using (2). Hence, $\alpha|_{A'}$ is onto. \square

REMARK 2. In Proposition ??, it is important the existence of a ring R with the property $R \cong R \oplus R$. Here we mention a well known fact. If V is an infinite dimensional vector space over a field F , then $R = \text{End}(V_F)$ is a ring with the property that $R \cong R \oplus R$.

Theorem 3.3. Let $A = \begin{pmatrix} R & {}_R M_S \\ 0 & S \end{pmatrix}$ and $E = \text{End}(A)$. The following are equivalent for $\alpha \in E$.

- (1) α is strongly clean in E .
- (2) There exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi$ is a unit in $\pi E\pi$ and $(1 - \alpha)(1 - \pi)$ is a unit in $(1 - \pi)E(1 - \pi)$.
- (3) $A = \underbrace{\begin{pmatrix} R & {}_R P_S \\ 0 & S \end{pmatrix}}_{A'} \oplus \underbrace{\begin{pmatrix} R & {}_R Q_S \\ 0 & S \end{pmatrix}}_{A''}$, where A' and A'' are α -invariant, and $\alpha|_{A'}$ and $(1 - \alpha)|_{A''}$ are isomorphisms.
- (4) $A = A' \oplus A''$, where A' and A'' are α -invariant, $\ker(\alpha) \subseteq A'' \subseteq (1 - \alpha)(A)$ and $\ker(1 - \alpha) \subseteq A' \subseteq \alpha(A)$.
- (5) $A = A_1 \oplus A_2 \oplus \dots \oplus A_n$ for some $n \geq 1$ where A_i is α -invariant and $\alpha|_{A_i}$ is strongly clean in $\text{End}(A_i)$ for each i .

Proof. (1) \Rightarrow (2). Let $\alpha = (1 - \pi) + \sigma$ where $\pi\sigma = \sigma\pi$, $\pi^2 = \pi$, and $\sigma \in \mathcal{U}(E)$. Note that α, π , and σ all commute. Now $\alpha - \sigma = 1 - \pi$ so $\alpha\pi = \sigma\pi$. Since $\sigma^{-1}\alpha \in \pi E\pi$ this gives $(\alpha\pi)(\sigma^{-1}\pi) = (\sigma\pi)(\sigma^{-1}\pi) = \pi$. Similarly, $(\sigma^{-1}\pi)(\alpha\pi) = \pi$ so $\alpha\pi$ is a unit in $\pi E\pi$. Finally, observe that $1 - \alpha = \pi + (-\sigma)$ is strongly clean too, so an analog of the above argument shows that $(1 - \alpha)(1 - \pi)$ is a unit in $(1 - \pi)E(1 - \pi)$.

(2) \Rightarrow (3). Given π as in (2) let $A' = \pi(A)$ and $A'' = (1 - \pi)(A)$. Then, $A = A' \oplus A''$ and $\alpha(A') = \alpha\pi(A) = \pi\alpha(A) \subseteq \pi(A) = A'$, so A' is α -invariant. Similarly, $\alpha(A'')$. If $(\alpha\pi)^{-1} = \pi\gamma\pi$ in $\pi E\pi$, let $\gamma_0 = (\pi\gamma\pi)|_{A'}$. Then, $\gamma_0 \in$

$End(A')$ so, if $F \in A'$ we have $(\gamma_0(\alpha|_{A'}))(F) = (\gamma_0\alpha)(\pi(F)) = \gamma_0(\alpha\pi)(F) = (\gamma_0\pi\alpha)(F) = \pi(F) = F$. Thus $\gamma_0\alpha|_{A'} = 1_{A'}$. Similarly $\alpha|_{A'}\gamma_0 = 1_{A'}$, because $\alpha|_{A'}\gamma_0(F) = (\alpha|_{A'}(\pi\gamma\pi))(F) = (\alpha\pi\gamma\pi)(F) = \pi(F) = F$. Thus $\alpha|_{A'}$ is a unit in $End(A')$. A similar argument shows that $(1-\alpha)|_{A''}$ is a unit in $End(A'')$.

(3) \Rightarrow (4). This follows from Proposition ??.

(4) \Rightarrow (5). Suppose that $A = A' \oplus A''$ as in (4). Then, $\alpha|_{A'}$ is strongly clean in $End(A')$ because it is a unit by Proposition ??. Similarly A' and A'' are $(1-\alpha)$ -invariant so $(1-\alpha)|_{A''}$ is a unit in $End(A'')$, again by Proposition ??. This gives (5) with $n = 2$, $A_1 = A'$ and $A_2 = A''$.

(5) \Rightarrow (1). Given the situation in (5), extend maps $\lambda_i \in End(A_i)$ to $\widehat{\lambda}_i \in End(A)$ by defining

$$\widehat{\lambda}_i(F_1 + F_2 + \dots + F_n) = \lambda_i(F_i).$$

Then, $\widehat{\lambda}_i\widehat{\lambda}_j = 0$ if $i \neq j$ while $\widehat{\lambda}_i\widehat{\mu}_i = \widehat{\lambda}_i\mu_i$ for all $\mu_i \in End(A_i)$. Now let $\alpha|_{A_i} = \pi_i + \sigma_i \in End(A_i)$ where $\pi_i^2 = \pi_i$, $\sigma_i \in \mathcal{U}(End(A_i))$ and $\pi_i\sigma_i = \sigma_i\pi_i$. If $\pi = \sum_i \widehat{\pi}_i$ and $\sigma = \sum_i \widehat{\sigma}_i$, then $\pi^2 = \sum_i \widehat{\pi}_i^2 = \pi$, $\pi\sigma = \sum_i \widehat{\pi}_i\widehat{\sigma}_i = \sum_i \widehat{\sigma}_i\widehat{\pi}_i = \sigma\pi$, and $\sigma \in \mathcal{U}(End(A))$ because $\sigma^{-1} = \sum_i \widehat{\sigma}_i^{-1}$. Since $\alpha = \sum_i \widehat{\alpha}|_{A_i} = \sum_i (\widehat{\pi} + \widehat{\sigma}_i) = \pi + \sigma$, the proof of (1) is complete. \square

A ring is called uniquely clean [?] if every element is uniquely the sum of an idempotent and a unit.

Definition 3.4. An element a of a ring R is called uniquely strongly clean (or USC for short) if a has a unique strongly clean expression in R as stated above. The ring R is called uniquely strongly clean (or USC for short) if every element of R is uniquely strongly clean.

Proposition 3.5. Let $A = \begin{pmatrix} R & {}_R M_S \\ 0 & S \end{pmatrix}$ and $\alpha \in End(A)$. The following are equivalent:

- (1) α is USC in $End(A)$.
- (2) There exists a unique decomposition $A = A' \oplus A''$ where A' and A'' are α -invariant, and $\alpha|_{A'}$ and $(1-\alpha)|_{A''}$ are isomorphisms.

Conclusion. A ring is said to be clean if every element of A can be written as a sum of an idempotent and a unit. Till now, many authors considered clean rings and obtained many results in this respect. In this work, we investigated the ring $A = \begin{pmatrix} R & {}_R M_S \\ 0 & S \end{pmatrix}$. In particular, we studied the endomorphisms of the ring A and gave necessary and sufficient conditions under which an endomorphism of A is strongly clean.

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