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CHARACTERIZATION OF NAKAYAMA m-CLUSTER TILTED ALGEBRAS OF TYPE A_n

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Abstract. For any natural natural number m, the m-cluster tilted algebras are generalization of cluster tilted algebras. These class algebras are defined as the endomorphism of certain object in m-cluster category called m-cluster tilting object. Finding such object in the m-cluster category has become a combinatorial problem. In this article we characterize Nakayama m-cluster tilted algebras of type A_n by geometric description given by Baur and Marsh.

Key words and Phrases: Cluster tilted algebras, cluster category, tilting object, Nakayama algebra

Abstrak. Untuk setiap bilangan asli m, aljabar teralih m-kluster adalah generalisasi dari aljabar teralih kluster. Kelas aljabar ini didefinisikan sebagai endomorfisma objek tertentu di kategori m-kluster yang disebut objek pengalih m-kluster. Mencari objek tersebut dalam kategori m-kluster dapat menjadi masalah kombinatorial. Dalam artikel ini dikarakterisasi aljabar Nakayama yang merupakan aljabar teralih m-kluster jenis A_n berdasarkan deskripsi geometris yang diberikan oleh Baur dan Marsh.

Kata kunci: aljabar teralih kluster, kategori kluster, objek pengalih, aljabar Nakayama.

1. Introduction

Let K be an algebraically closed field, and Q a finite acyclic quiver with n vertices. Let $\mathcal{D}^b(H)$ be a bounded derived category of mod H where H is a basic, finite dimensional hereditary algebra over K. We can assume H as a path algebra KQ of some quiver Q. The m-cluster category is the orbit category $C_H^m = \mathcal{D}^b(H)/\tau^{-1}[m]$ where τ is the Auslander-Reiten translation of $\mathcal{D}^b(H)$ and $[m]$ denotes m-th power of shift [1] in the derived category $\mathcal{D}^b(H)$. The m-cluster category is triangulated

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[5] and it is a Krull-Schmidt category [2]. These categories are generalization of cluster categories defined in [2] and independently [3] for the Dynkin type A_n case.

In m-cluster category we consider a class of objects called m-cluster tilting objects. These objects have nice combinatorial properties. By definition, an object T is an m-cluster tilting object if for any object X, we have $X \in \text{add } T$ if only if $\text{Ext}^i_{C^m_H}(T, X) = 0$ for all $i \in \{1, 2, \ldots, m\}$. The objects T always have exactly n indecomposable direct summands [7]. The endomorphism algebra $\text{End}^{op}_{C^m_H}(T)$ is called m-cluster tilted algebra.

In this paper we investigate m-Cluster Tilted Algebras(m-CTA) of type A_n which are Nakayama algebras. Nakayama algebra itself by its quiver is divided into two types, namely type A_n and cyclic. In this paper we focus on m-CTAs which are Nakayama algebras of type A_n and all possible relations as from [6] we have known all m-CTAs which are Nakayama algebras of type cyclic, see also [4]. In order to do this we use the geometric description of m-cluster category type A_n in [1]. We will divide into three cases in the search of m-CTAs of type A_n. We divide these two cases based on the relationship between m and n. The first case is when $m \geq n - 2$, the second case is $m < n - 2$.

This article is organized as follows. In Section 2 we describe the geometric description and the relations of Nakayama m-CTAs; in Section 3 we give a characterization of Nakayama m-CTA of cyclic type; in Section 4 we give a characterization of Nakayama m-CTA of acyclic type which will be divided into two cases.

2. GEOMETRIC DESCRIPTION AND RELATIONS IN NAKAYAMA m-CTAS

The geometric description of m-cluster category type A_n in [1] briefly representing indecomposable objects and arrows of the AR-quiver of m-cluster category in a regular gon. The indecomposable object is described as a diagonal of a regular gon while an arrow between two indecomposable objects described as two diagonals that have a common endpoint. From this geometric description we can also see the relations of quivers of the m-CTAs of type A_n.

Let $\mathcal{P}_{m(n+1)+2}$ be $(m(n+1)+2)$-regular gon, $m, n \in \mathbb{N}$, where its corner points are numbered clockwise from 1 to $m(n+1)+2$. A diagonal D of $\mathcal{P}_{m(n+1)+2}$ can be denoted as a pair (i, j). Consequently, the diagonal (i, j) is the diagonal (j, i). We said a diagonal D of $\mathcal{P}_{m(n+1)+2}$ is an m**-diagonal** if D divide $\mathcal{P}_{m(n+1)+2}$ into two parts that is $(mj+2)$-gon and $(m(n-j)+2)$-gon where $j = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$. For $i \neq j$, an arc D_{ij} of $\mathcal{P}_{m(n+1)+2}$ is a part of boundary that connect i to j clockwise. Note that if j is a clockwise direct neighbor of i then arc D_{ij} is an edge ij of $\mathcal{P}_{m(n+1)+2}$. We always have two arcs D_{ij}, D_{ji}. Let $\Gamma^m_{A_n}$ be a quiver with the vertices are all m-diagonals of polygon $\mathcal{P}_{m(n+1)+2}$ while arrows obtained in the following way: suppose $D = (i, j)$ and $D' = (i, j')$ are m-diagonals which have a common vertex i in $\mathcal{P}_{m(n+1)+2}$ then there is an arrow from D to D' if D, D'

together with arc from j to j' form $(m+2)$-gon in $\mathcal{P}_{m(n+1)+2}$ and D can be rotated clockwise to D' about the common endpoint i.

Using this regular gon we can easily make a quiver of an m-CTA. The set of indecomposable objects of a tilting object of m-cluster category of type A_n can be identified as the set of maximal m-diagonals in $\mathcal{P}_{m(n+1)+2}$ and the number of direct summands of this object is always n. Such a set is called an $(m+2)-$angulation of $\mathcal{P}_{m(n+1)+2}$. By definition, we can conclude that if X and Y are m-diagonals of a tilting object T that has a common endpoint then there is a path from T_X and T_Y in the Auslander-Reiten(AR) quiver of m-cluster category where T_X and T_Y are indecomposable objects associated to X and Y. It is clear that the composition of the arrows in this path is not zero. If there is no m-diagonal between X and Y in $\mathcal{P}_{m(n+1)+2}$ then the composition of irreducible maps from T_X to T_Y does not pass through another indecomposable object which is a direct summand of a tilting object T. It means that there is an arrow from the point corresponding to X and Y in the quiver of m-CTA $\mathrm{End}^{op}(T)$.

By the above argument we can define a quiver of an m-CTA independently from $(m+2)$-angulation of $\mathcal{P}_{m(n+1)+2}$. Let $T = \{T_1, T_2, \ldots, T_n\}$ be an $(m+2)$-angulation. Define a quiver Q_T as follows: The vertices of Q_T are the numbers $1, 2, \ldots, n$ which are in bijective correspondence with the m-diagonals $T_1, T_2, \ldots, T_n$. Given two vertices a, b of Q_T, there is an arrow from a to b if

(i) T_a and T_b have a common point in $\mathcal{P}_{m(n+1)+2}$,
(ii) there is no m-diagonal of T between T_a and T_b and
(iii) T_a can be rotated clockwise to T_b at the common endpoint.

Our first lemma characterize the possible forms of two m-diagonals in polygon $\mathcal{P}_{m(n+1)+2}$, correspond to a path of length two in the quiver of an m-CTA. We have the following easy lemma.

Lemma 2.1. *Let H =End$^{op}(T)$ be an m-CTA with T is an m-cluster tilting object of $\mathcal{C}^m_{A_n}$. If $x \to y \to z$ is a path of length two in Q_H and T_x, T_y, T_z respectively are m-diagonals correspond to points x, y, z then*

(1) $T_x = (x_1, x_2), T_y = (x_2, x_3), T_z = (x_3, x_4)$ *with* x_4 *in arc* $D_{x_3x_1}$

or

(2) $T_x = (x_1, x_2), T_y = (x_2, x_3), T_z = (x_2, x_4)$ *with* x_4 *in arc* $D_{x_3x_2}$,

where $x_i \neq x_j$ *if* $i \neq j$.

Proof. Let $T_x = (x_1, x_2)$. Since there is an arrow from x to y then T_x and T_y have a common endpoint. Without loss of generality, suppose $T_y = (x_2, x_3)$. Since there is an arrow from y to z then T_y and T_z have a common endpoint. If x_3 is a common endpoint of T_y and T_z then $T_z = (x_3, x_4)$ where x_4 in arc $D_{x_1x_3}$, otherwise T_z will cross T_x. If x_2 is a common endpoint of T_y and T_z then $T_z = (x_2, x_4)$ where x_4 in arc $D_{x_3x_2}$. □

Let Q be a finite quiver without cycle and $H = KQ/\mathcal{I}$ where $\mathcal{I}$ is an admissible ideal of KQ. If Q is not connected then the algebra H is not connected. Indeed let $\mathcal{Q}$ be the collection of maximal connected subquivers of Q. It can be shown that $H = \prod_{Q' \in \mathcal{Q}} KQ'/\mathcal{I}'$ where $\mathcal{I}'$ is an ideal of Q', but then H is a finite direct product of some algebras. Hence, H is not connected.

In order to know the condition of an $(m+2)$-angulation such that the quiver of m-cluster tilted algebra is connected, we have the following easy lemma.

Lemma 2.2. *Let T be an $(m+2)$-angulation of $\mathcal{P}_{m(n+1)+2}$. The graph generated by the diagonals in T is connected if only if the quiver Q_T is connected.*

Let $X = (x_1, x_2)$ be a diagonal of $\mathcal{P}_{m(n+1)+2}$. We may assume $x_2 > x_1$. Define the length of diagonal X to be the $\min\{x_2 - x_1, m(n+1) + 2 + x_1 - x_2\}$. Thus, the length of X is equal to the minimum of the number of sides between arc $D_{x_1 x_2}$ and $D_{x_2 x_1}$. An m-diagonal X of $\mathcal{P}_{m(n+1)+2}$ is said to be **short** if its length

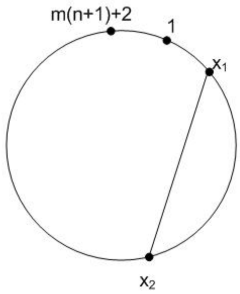


FIGURE 1. short m-diagonal

is minimal, that is of length $m+1$. An m-diagonal X is short if only if there is no m-diagonal whose endpoints are in smaller polygon divided by X.

Lemma 2.3. *Let T be an $(m+2)$-angulation of $\mathcal{P}_{m(n+1)+2}$ with $n \geq 3$. If Q_T is cyclic then all m-diagonals in T are short.*

Proof. Let X be an m-diagonal of T which is not short . Without loss of generality, let $X = (1, x_1)$ and X has length which is minimal among the diagonals in T which are not short . First, assume that $x_1 \leq \dfrac{m(n+1)+2}{2}$. The diagonal X will divide $\mathcal{P}_{m(n+1)+2}$ into two smaller polygons P_1 and P_2 with P_1 is the smallest polygon (see Figure 2). Since X is not short and T is maximal, there exists an m-diagonal of T whose endpoints in arc $D_{x_1 x_2}$. By the same argument we also have another m-diagonal of T which divides the polygon P_2. We then have that all m-diagonals in P_1 are short by the minimality of X. Since Q_T is connected there exists a short m-diagonal X_1 of T in P_1 that adjacent to X. We may assume that $X_1 = (1, b)$. Now there exists a short m-diagonal that adjacent to X_1, namely X_2. By the same argument we have a collection of short m-diagonals $X_1 = (1, a_1), X_2 = (a_1, a_2) \ldots, X_k = (a_{k-1}, a_k)$ where all of these are in P_1 and maximal with respect

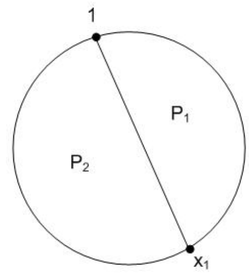


FIGURE 2. m-diagonal X

to this property. It follows that $x_k = x_1$, otherwise there is no arrow which target is X_k in Q_T. We describe this situation in the following figure

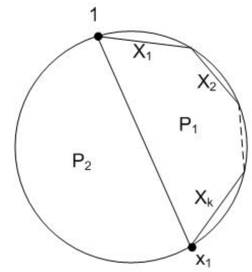


FIGURE 3. m-diagonals in P_1

But now we have a path $X_1 \to X \to X_k$ in Q_T. So there can be no further m-diagonals adjacent to X, which is a contradiction.

If $x_1 > \dfrac{m(n+1)+2}{2}$ we get similar proof for P_2 since in this case P_2 becomes the smallest polygon divided by X. $\square$

Lemma 2.3 gives us a characterization of m-cluster tilting object such that the corresponding m-CTA is a Nakayama algebra of cyclic type. We will find all m-cluster tilting objects in this form in the next section. Now we look at the configuration of an $(m+2)$-angulation T which Q_T is of A_n type.

Lemma 2.4. *Let T be an $(m+2)$-angulation of $\mathcal{P}_{m(n+1)+2}$ with $n \geq 3$. If Q_T is of A_n type then*

$$T = T_C \cup T_{\alpha_1} \cup T_{\alpha_2} \cup \cdots \cup T_{\alpha_{r-1}}$$

for some $r \geq 2$ where (up to rotation) $T_C = \{(1, x_1), (x_1, x_2), \ldots, (x_{r-1}, x_r)\}$ and all m-diagonals in T_C are short,

$$T_{\alpha_1} = \{(x_1, y_{11}), (x_1, y_{12}), \ldots, (x_1, y_{1j_1})\},\ j_1 \geq 0$$
$$T_{\alpha_2} = \{(x_2, y_{21}), (x_1, y_{22}), \ldots, (x_1, y_{2j_2})\},\ j_2 \geq 0$$
$$\vdots$$
$$T_{\alpha_{r-1}} = \{(x_{r-1}, y_{r-1,1}), (x_1, y_{r-1,2}), \ldots, (x_{r-1}, y_{r-1,j_{r-1}})\},\ j_{r-1} \geq 0$$

with $y_{11} < y_{12} < \cdots < y_{1j_1} < y_{21} < \cdots < y_{2j_2} < \cdots < y_{n-1,j_{n-1}}$.

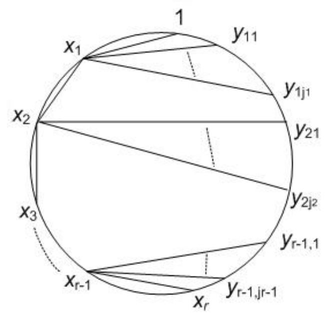


FIGURE 4. $(m+2)$-angulation of T with $Q_T = A_n$

Proof. Let $(1, x_1)$ be an m-diagonal of $\mathcal{P}_{m(n+1)+2}$ correspond to a source in Q_T. We claim that $(1, x_1)$ is short. If $(1, x_1)$ is not short then either there is an m-diagonal (x_1, t) with $t > x_1$ or there is an m diagonal $(1, u)$ with $u > x_1$ (see Figure 5). Consider the first case , if there is an m-diagonal (x_1, t), we chose t maximal

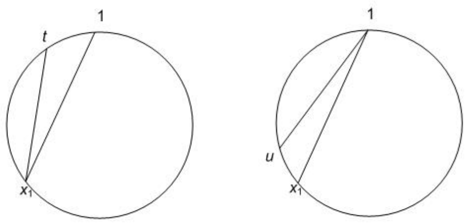


FIGURE 5. m-diagonals (x_1, t) and $(1, u)$

such that $t > x_1$. Then we have an arrow $(x_1, t) \to (1, x_1)$, but it contradicts that $(1, x_1)$ is a source. Second case, if there is an m-diagonal $(1, u)$ we chose u minimal such that $u > x_1$. Since $(1, x_1)$ is not short, there is either an m-diagonal (x_1, a)

with $1 < a < x_1$ or an m-diagonal $(1, b)$ with $1 < b < x_1$. We may assume that a is minimal and b maximal. If there is a diagonal (x_1, a) then there is an arrow $(1, b) \to (x_1, a)$. It contradicts the fact that there is also an arrow $(1, x_1) \to (1, u)$. So we can assume that there is a diagonal $(1, b)$. It follows that there is an arrow $(1, b) \to (1, x_1)$. This is a contradiction since $(1, x_1)$ is a source. Therefore $(1, x_1)$ is short, this proves our claim.

Let $(1, x_1) \to (x_1, z)$ be the arrow starting in $(1, x_1)$ then $z > 1$. Now there are two cases, either (x_1, z) is short or (x_1, z) is not short.

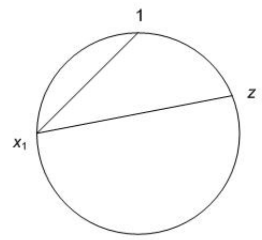
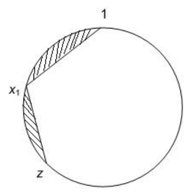


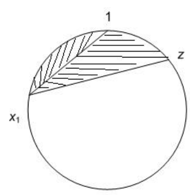
FIGURE 6. m-diagonal (x_1, z)

(1) (x_1, z) **is short**.
If $T_\alpha = (x_1, z)$ is short then arc D_{zx_1} together with T_α is a smaller polygon divided by (x_1, z). Hence, there is no m-diagonal with endpoints in arc D_{zx_1}. We also have that there is no m-diagonal (x_1, y) with $1 < y < z$ since otherwise the arrow $(1, x_1) \to (x_1, z)$ will not exist.

(2) (x_1, z) **is not short**.
If (x_1, z) is not short then there is no m-diagonal (z, v) with $1 < v < z$. Indeed, assume to the contrary that there is an m-diagonal (z, v) with $1 < v < z$. It follows that there is no m-diagonal (x_1, u) for $z < u < x_1$ since otherwise there is also an arrow $(x_1, z) \to (x_1, u)$. If there is an m-diagonal (z, l) for $z < l < x_1$, and choose z maximal, then there is an arrow $(z, l) \to (x_1, z)$, a contradiction. Therefore there is no m-diagonal with endpoints in arc D_{zx_1}. This is a contradiction since (x_1, z) is not short. Hence there is no diagonal (z, v). Therefore arc D_{1z} together with $(1, x_1)$ and (x_1, z) forms an $(m + 2)$-gon.

We describe condition 1 and 2 respectively as follows





where the shaded polygons are $m+2$-gons and hence there is no m-diagonal in these polygons. Now we perform same analysis by consider the arrow starting at (x_1, z).

Indeed, in case (x_1, z) is short then the arrow starting at (x_1, z) is $(x_1, z) \to (z, w)$ with $1 < w < z$. In case (x_1, z) is not short then the arrow starting at (x_1, z) is $(x_1, z) \to (x_1, w)$ with $1 < w < x$. We have similar case for the third m-diagonal from the source which adjacent to (x_1, z). There are again two cases to consider, that is either this m-diagonal is short or not short. These two cases will be similar to the condition 1 and 2 above. We complete the proof by induction using the fact that the the next m-diagonal adjacent to the previous have two possibilities like condition 1 and 2. □

Two cases in Lemma 2.1 hold for any path of length two in the quiver of m-CTAs of type A_n. For both cases the picture is as follows

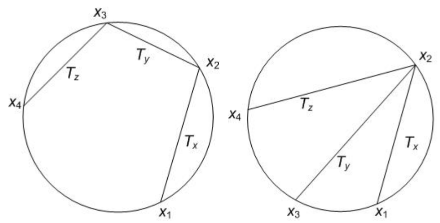


FIGURE 7. m-diagonals correspond a path of length two

Using the above lemma we can conclude that each path of length two in the quiver of m-CTAs of type A_n is one of these two cases.

Now we will see the composition of paths of length two in End $(T) \cong KQ/\mathcal{I}$ for both cases. We have the following facts.

Lemma 2.5. *Let $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ be an m-cluster tilting object of $\mathcal{C}^m_{A_n}$ and Q be a quiver of m-CTA End$^{op}(T)$. Suppose $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ is a path of length two in Q corresponding to the m-diagonals T_i, T_j, T_k in $\mathcal{P}_{m(n+1)+2}$.*

(1) If $T_i = (x_1, x_2), T_j = (x_2, x_3), T_k = (x_3, x_4)$ with x_4 in arc $D_{x_3x_1}$ then the composition $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ in End$^{op}(T)$ is zero.

(2) If $T_x = (x_1, x_2), T_y = (x_2, x_3), T_z = (x_2, x_4)$ with x_4 in arc $D_{x_3x_2}$ then the composition $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ in End$^{op}(T)$ is not zero.

Proof. See [4]. □

Now we can identify the relation of connected Nakayama m-cluster tilted algebras using Lemma 2.3, 2.4 and 2.5.

Theorem 2.6. *Let $H = KQ/\mathcal{I}$ be a connected Nakayama m-cluster tilted algebra of $\mathcal{C}^m_{A_n}$. An ideal $\mathcal{I}$ of H is generated by a relation of paths of length two.*

Proof. If Q is cyclic then by Lemma 2.3, $Q = Q_T$ where T is an $(m+2)$-angulation such that all m-diagonals in T are short. Therefore, every path of length two in

Q_T is in case 1 of Lemma 2.1. By Lemma 2.5 all paths of length two is zero. If Q is of type A_n then by Lemma 2.4 every path of length two is either case one or case two of Lemma 2.5. It remains to prove that every path $\mathbb{P} = \alpha_1\alpha_2\ldots\alpha_\ell$ with $\ell \geq 3$ is not zero in H if every subpath of $\mathbb{P}$ is not zero in H. It follows that every subpath of length two in $\mathbb{P}$ is case two of Lemma 2.5. We may assume that $T_{\alpha_1} = (1, mr+2)$ with $1 \leq r < n$ whose common endpoint with T_{α_2} and T_{α_3} is 1. Hence, $T_{\alpha_j} = (1, mr_j + 2)$ for every $j \geq 2$ with $r < r_i < r_{i+1}$ for all i. We have that $T_{\alpha_1}, T_{\alpha_2}, \ldots, T_{\alpha_\ell}$ will be in the subquiver of $\Gamma^m_{A_n}$ as in Figure 8. Since the

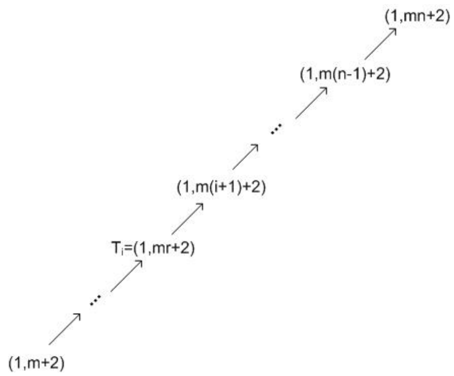


FIGURE 8. subquiver of $\Gamma^m_{A_n}$

composition of irreducible morphism $T_{\alpha_1} \to T_{\alpha_2} \to \cdots \to_{\alpha_\ell}$ is not zero in m-cluster category, we conclude that $\alpha_1\alpha_2\ldots\alpha_\ell$ not zero in H. This finishes the proof. $\square$

3. m-CTAs which are Nakayama Algebra of Cyclic Type

In this section we will show that m-CTAs which are Nakayama algebras of cyclic type only occur if $m = n-2$. It means that there is no m-CTA whose quiver is cyclic when $m \neq n-2$. In addition, in m-CTA there is only one possibility relation that is relations of paths of length two. More generally, m-CTAs which have cyclic quivers have been stated by Murphy in [6]. However, in this section we explain how to characterize m-CTAs which quivers are cyclic by using geometric description in [1]. The results in this section have been proved in [4]. We state again here with more structured proofs.

We show that if $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ then T is a m-cluster tilting object for $m \geq n+2$ where T_i's are m-diagonals described in Proposition 3.1. The quivers of m-CTAs $\mathrm{End}^{op}(T)$ have different forms for each case $m = n-2$ and $m > n-2$. Indeed, for $1 \leq i \leq n-1$ diagonals T_i and T_{i+1} have a common endpoint in $\mathcal{P}_{m(n+1)+2}$ for $m \geq n-2$. It means that for every i, we have an arrow $i \to i+1$ in the quiver of $\mathrm{End}^{op}(T)$. Now consider m-diagonals $T_n = (3m-(n-5), 2m-(n-4))$ and $T_1 = (1, m+2)$. If $m = n-2$ then $T_n = (2m+3, m+2)$. Hence, T_n and T_1 have a common endpoint $(m+2)$ in $\mathcal{P}_{m(n+1)+2}$. Therefore there exists an arrow $n \to 1$ in quiver of $\mathrm{End}^{op}(T)$. Thus, for $m = n-2$ the quiver of m-cluster tilted algebra $\mathrm{End}^{op}(T)$ is Figure 9.

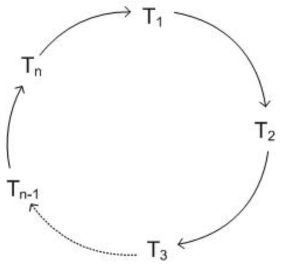


FIGURE 9. Quiver of $\mathrm{End}^{op}(T)$ for $m = n-2$

Proposition 3.1. *Let* $\mathcal{C}^m_{A_n} = \mathcal{D}^b(KA_n)/F_m$*, where* $F_m = \tau^{-1}[m]$ *and* $m = n-2$. *Suppose that* $T_1 = (1, m+2), T_2 = (1, nm+2)$ *and for* $3 \le i \le n$,

$$T_i = ((n-(i-2))m-(i-4), (n-(i-3))m-(i-5))$$

then

(1) $T_1, T_2, \ldots, T_n$ *are* m*-diagonals of* $\mathcal{P}_{m(n+1)+2}$.
(2) $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ *is an* m*-cluster tilting object.*
(3) m*-cluster tilted algebra* $\mathrm{End}^{op}(T)$ *is isomorphic to* $KQ/\mathcal{I}$ *where* Q *is cyclic with* n *vertices and* $\mathcal{I}$ *is an ideal generated by all paths of length two.*

Proof. It is clear that if $T_1 = (1, m+2), T_2 = (1, nm+2)$ and for $3 \le i \le n$,

$$T_i = ((n-(i-2))m-(i-4), (n-(i-3))m-(i-5))$$

then $T_1, T_2, \ldots, T_n$ are m-diagonals of $\mathcal{P}_{m(n+1)+2}$. For $i = n$ we have that $T_n = ((n-(n-2))m-(n-4), (n-(n-3))m-(n-5)) = (3m-(n-5), 2m-(n-4))$. Consider m-diagonals $T_1, T_2, \ldots, T_n$ in $\mathcal{P}_{m(n+1)+2}$, see Figure 10. Because $T_1, T_2 \ldots, T_n$ are not crossing each other then T is an m-cluster tilting object. Let Q be a quiver of m-cluster tilted algebra $\mathrm{End}^{op}(T)$, then there is only one arrow $i \to i+1$ for every $1 \le i \le n-1$. Since $m = n-2$, we obtain that $T_n = (2m+3, m+2)$ and $T_1 = (1, m+2)$ have a common endpoint. Consequently, there is exactly one arrow $n \to 1$ in Q. It means that Q is a cyclic quiver with n vertices. By Lemma 2.5 the composition of all paths of length two is zero. □

Next we show that the m-CTA of type A_n whose quiver is cyclic is the algebra stated in Proposition 3.1.

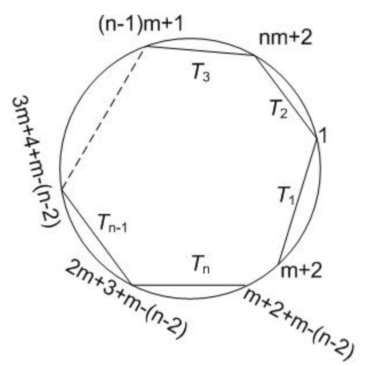


FIGURE 10. m-diagonals $T_1, T_2, \ldots, T_n$

Proposition 3.2. *If T is an m-cluster tilting object of m-cluster category $\mathcal{C}^m_{A_n}$ such that the quiver of m-cluster tilted algebra $End^{op}(T)$ is connected and cyclic, then $m = n-2$. Moreover, $End^{op}(T) = KQ/\mathcal{I}$ with $\mathcal{I}$ an ideal generated by all paths of length two.*

Proof. Let Q be a quiver of m-cluster tilted algebra $\mathrm{End}^{op}(T)$. Suppose $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$,we may assume $\{T_1, T_2, \ldots, T_n\}$ is a set of maximal non-crossing m-diagonals in $(m(n+1)+2)$-gon $\mathcal{P}_{m(n+1)+2}$. Assume that $Q_0 = \{T_1, T_2, \ldots, T_n\}$ the set of vertices of Q, and the set of arrows $Q_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n\}$ with $\alpha_i : T_i \to T_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$ and $\alpha_n : T_n \to T_1$. Consider any path of length two $T_p \to T_q \to T_r$ in Q. By Lemma 2.3 T_q, T_r, T_s are short. It follows that $T_q = (x_1, x_2), T_r = (x_2, x_3), T_s = (x_3, x_4)$ can be described as in Figure 11. By applying the above argument, the picture of m-diagonals $T_1, T_2, \ldots, T_n$ in

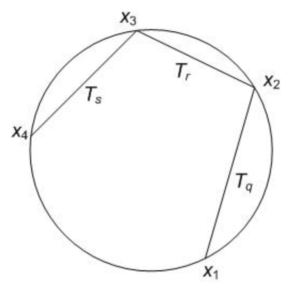


FIGURE 11. m-diagonals correspond to T_q, T_r and T_s

$\mathcal{P}_{m(n+1)+2}$ is Figure 12.

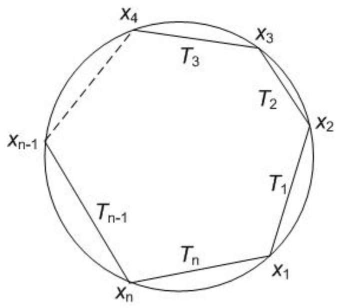


FIGURE 12. m-diagonals $T_1, T_2, \ldots, T_n$ for $m = n - 2$

Since all T_i are short then the length of T_i is $m + 1$. Consequently, we have the equation

$$\underbrace{(m+1) + (m+1) + \cdots + (m+1)}_{n} = m(n+1) + 2.$$

Therefore,

$$(m+1)n = m(n+1) + 2 \Leftrightarrow n = m + 2$$

For the last statement we apply Lemma 3.1. □

Example 3.3. *Let* $m = 4$ *and* $n = 6$ *then* $m(n+1)+2 = 4(6+1)+2 = 30$. *Consider 30-gon* $\mathcal{P}_{30}$, *let* $T_1 = (1, 6), T_2 = (1, 26), T_3 = (26, 21), T_4 = (21, 16), T_5 = (16, 11)$ *and* $T_6 = (11, 6)$ *then* $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6$ *is a 4-cluster tilting object. The picture of* $\mathcal{P}_{30}$ *together with the six* m*-diagonals is*

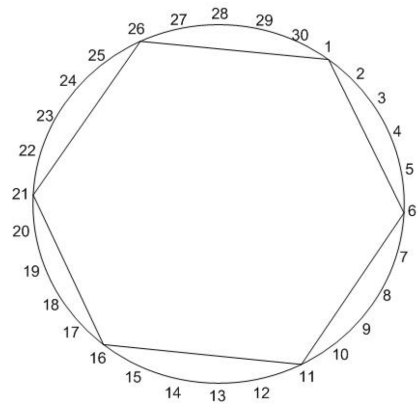


FIGURE 13. $(m + 2)$-angulation T for $m = 4$ and $n = 6$

4. m-CTAs which are Nakayama Algebras with Acyclic Quivers

In this section we will characterize m-CTA which are Nakayama algebras whose quivers are connected acyclic. In other words, we find m-cluster tilting objects $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ such that $\mathrm{End}^{op}(T) \cong KQ/\mathcal{I}$ where Q is

$$T_1 \xrightarrow{\alpha_1} T_2 \xrightarrow{\alpha_2} T_3 \to \cdots \to T_{n-1} \xrightarrow{\alpha_{n-1}} T_n.$$

Throughout, Q is assumed to be the above quiver, unless otherwise specified.

We will also observe the relation in this type of m-CTA. To do this we divide into two cases correspond to m and n. These three cases are $m \geq n-2$ and $m < n-2$.

The following is the list of m-diagonals in $\mathcal{P}_{m(n+1)+2}$.

Table 1. m-diagonals

$(1,-)$	$(nm+2,-)$	$((n-1)m+1,-)$	$((n-2)m,-)$	$((n-3)m-1,-)$
$m+2$	1	$nm+2$	$(n-1)m+1$	$(n-2)m$
$2m+2$	$m+1$	$(n+1)m+2$	$nm+1$	$(n-1)m$
$3m+2$	$2m+1$	m	$(n+1)m+1$	nm
$4m+2$	$3m+1$	$2m$	$m-1$	$(n+1)m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$nm+2$	$(n-1)m+1$	$(n-2)m$	$(n-3)m-1$	$(n-4)m-2$

$((n-4)m-2,-)$	$((n-5)m-3,-)$	$\dots$	$((n-i)m-(i-2),-)$	$((n-(i+1))m-(i-1),-)$
$(n-3)m-1$	$(n-4)m-2$	$\dots$	$(n-(i-1))m-(i-3)$	$(n-i)m-(i-2)$
$(n-2)m-1$	$(n-3)m-2$	$\dots$	$(n-(i-2))m-(i-3)$	$(n-(i-1))m-(i-2)$
$(n-1)m-1$	$(n-2)m-2$	$\dots$	$(n-(i-3))m-(i-3)$	$(n-(i-2))m-(i-2)$
$nm-1$	$(n-1)m-2$	$\dots$	$\vdots$	$\vdots$
$(n+1)m-1$	$nm-2$	$\dots$	$nm-(i-3)$	$(n-1)m-(i-2)$
$m-3$	$(n+1)m-2$	$\dots$	$(n+1)m-(i-3)$	$nm-(i-2)$
$2m-3$	$m-4$	$\dots$	$m-(i-1)$	$(n+1)m-(i-2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-5)m-3$	$(n-6)m-5$	$\dots$	$(n-(i+1))m-(i-1)$	$(n-(i+2))m-i$

From Table 1 we take m-diagonals which will be used as a direct summand of an m-cluster tilting object such that the quiver of m-CTA is A_n. The following table lists some m-diagonals which will be used for our m-cluster tilting object.

TABLE 2. m-diagonals of m-cluster tilting objects

$X_{1,1} = (1, 2m + 2)$	$X_{1,2} = (nm + 2, 2m + 1)$
$X_{2,1} = (1, 3m + 2)$	$X_{2,2} = (nm + 2, 3m + 1)$
$X_{3,1} = (1, 4m + 2)$	$X_{3,2} = (nm + 2, 4m + 1)$
$\vdots$	$\vdots$
$X_{n-2,1} = (1, (n - 1)m + 2)$	$X_{n-2,2} = (nm + 2, (n - 1)m + 1)$

$X_{1,3} = ((n - 1)m + 1, 2m)$	$\dots$	$X_{1,i} = ((n - (i - 2))m - (i - 4), 2m - (i - 3))$
$X_{2,3} = ((n - 1)m + 1, 3m)$	$\dots$	$X_{2,i} = ((n - (i - 2))m - (i - 4), 3m - (i - 3))$
$X_{3,3} = ((n - 1)m + 1, 4m)$	$\dots$	$X_{3,i} = ((n - (i - 2))m - (i - 4), 4m - (i - 3))$
$\vdots$	$\vdots$	$\vdots$
$X_{n-3,3} = ((n - 1)m + 1, (n - 2)m)$	$\dots$	$X_{n-i,i} = ((n - (i - 2))m - (i - 4), (n - i + 1)m - (i - 3))$

Throughout, for every $1 \le i \le n$, T_i is assumed to be the m-diagonal described in Proposition 3.1.

4.1. Case $m \ge n - 2$.

Recall that $T_1 = (1, m + 2), T_2 = (1, nm + 2)$ and for $3 \le i \le n - t$ we have

$$T_i = ((n - (i - 2))m - (i - 4), (n - (i - 3))m - (i - 5)).$$

We have that all m-diagonals in the set $T = \{T_1, T_2, \dots T_{n-1}, T_n\}$ are short. In the case $m = n - 2$ the quiver of Q_T is a cyclic quiver and every path of length of two is a relation in the corresponding m-CTA. We will prove that there is no m-CTA whose quiver is A_n and every path of length two is zero in the case $m = n-2$. But in the case $m > n - 2$ the quiver Q_T is a path and every path of length of two is a relation in the corresponding m-CTA.

Lemma 4.1. *Suppose that $\mathcal{C}^m_{A_n} = D^b(KA_n)/F_m$, where $F_m = \tau^{-1}[m]$ with $m > n - 2$.*

(1) $T_1, T_2, \dots, T_n$ are m-diagonals of $\mathcal{P}_{m(n+1)+2}$.
(2) $T = T_1 \oplus T_2 \oplus \dots \oplus T_n$ is an m-cluster tilting object.
(3) The m-cluster tilted algebra End$^{op}(T)$ is isomorphic to $KQ/\mathcal{I}$ where Q is

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \dots \to (n - 1) \xrightarrow{\alpha_{n-1}} n.$$

and $\mathcal{I}$ is an ideal generated by all paths of length two.

Proof. It is clear that $T_1, T_2, \ldots, T_n$ are m-diagonals of $\mathcal{P}_{m(n+1)+2}$, where if $i = n$ then $T_n = ((n-(n-2))m-(n-4), (n-(n-3))m-(n-5)) = (3m-(n-5), 2m-(n-4))$. Observe that the picture of m-diagonals $T_1, T_2, \ldots, T_n$ in $\mathcal{P}_{m(n+1)+2}$ is Figure 14. Since $T_1, T_2 \ldots, T_n$ are not crossing each other then T is an m-cluster

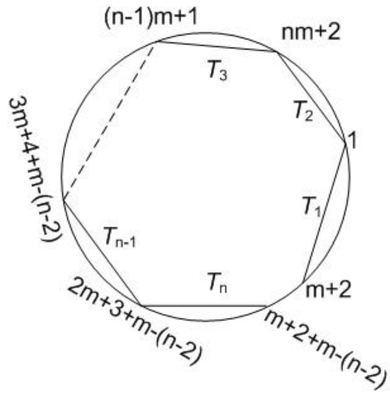


FIGURE 14. m-diagonals of T

tilting object. Let Q be the quiver of m-cluster tilted algebra $\mathrm{End}^{op}(T)$, then there exists exactly one arrow $T_i \to T_{i+1}$ for every $1 \leq i \leq n-1$. If $m > n-2$ then $m-(n-2) > 0$ and consequently $m+2+m-(n-2) > m+2$. Hence, T_n and T_1 don't have common endpoint. In other words there is no arrow from T_n to T_1. We conclude Q is the quiver in the proposition. Finally, by Lemma 2.5 the composition of all paths of length two is zero. $\square$

Lemma 4.2. *Let $m \geq n-2$ and $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{n-1} \oplus X_{1,i}$ with $1 \leq i \leq n-2$ then*

(1) T is an m-cluster tilting object in $\mathcal{C}^m_{A_n}$.
(2) If Q is a quiver of $End^{op}(T)$ then Q is

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to n-1 \xrightarrow{\alpha_{n-1}} n.$$

(3) If $\rho_j = \alpha_j\alpha_{j+1}$ for every $1 \leq j \leq n-2$ then $End^{op}(T) = KQ/\mathcal{I}$ where $\mathcal{I} = \langle \rho_1, \rho_2, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_{n-2} \rangle$.

Proof. Suppose that $T' = \{T_1, T_2, \ldots, T_{n-1}\}$ then it is clear that T' is the set of m-diagonals that are not crossing each other in $\mathcal{P}_{m(n+1)+2}$. We have that $X_{1,1} = (1, 2m+2)$ and $X_{1,i} = (m(n-(i-2))-(i-4), 2m-(i-3))$ for $1 \leq i \leq n-2 = m$. Hence,

$$m+2 < 2m-(i-3) < 2m+3$$

It follows that the set $T' \cup \{X_{1,i}\}$ of m-diagonals in $\mathcal{P}_{m(n+1)+2}$ is as in Figure 15. We conclude that T is an m-cluster tilting object of $C^m_{A_n}$. From Figure 15 we

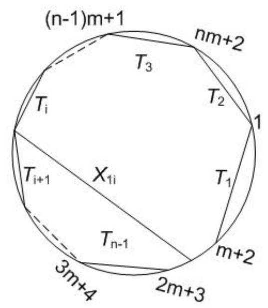


FIGURE 15. m-diagonal $T' \cup X_{1,i}$

obtain easily that quiver of $End^{op}(T)$ is Q. Note that m-diagonals $T_i, X_{1,i}, T_{i+1}$ satisfy case 2, hence the composition $\rho_i = \alpha_i\alpha_{i+1}$ is not zero. But all ρ_j with $j \neq i$ is zero since the corresponding m-diagonals with ρ_j satisfy case 1. We conclude $\mathrm{End}^{op}(T) \cong KQ/\mathcal{I}$, as required. □

Lemma above gives us how to construct other m-cluster tilting objects which have different relations. We know that the number of paths of length two in A_n is $(n-2)$, where the relations are $\rho_1, \rho_2, \ldots, \rho_{n-2}$. In Lemma 4.2 ideal $\mathcal{I}$ is generated by a combination of $(n-3)$ relations of paths of length two from $(n-2)$ relations. We can get the m-CTA $\mathrm{End}^{op}(T) \cong KQ/\mathcal{I}$ where $\mathcal{I}$ generated by $(n-4)$ relations of paths of length two from $(n-2)$ relations by the following lemma.

Lemma 4.3. *Suppose that $m \geq n-2$ and $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{n-2} \oplus X_{1,i} \oplus X_{2,j}$ where $1 \leq i \leq j \leq n-3$ then T is an m-cluster tilting object of $\mathcal{C}^m_{A_n}$. Furthermore, the algebra $End^{op}(T) \cong KQ/\mathcal{I}$ where $\mathcal{I}$ generated by $(n-4)$ relations of paths of length two. If $\mathfrak{T}$ be the collection of such T then $|\mathfrak{T}| = \binom{n-2}{n-4}$.*

Proof. It is clear that m-diagonal $T_1, T_2, \ldots, T_{n-2}$ are not crossing each other in $\mathcal{P}_{m(n+1)+2}$. Now we just need to consider m-diagonals $X_{1,i}$ and $X_{2,j}$ in $\mathcal{P}_{m(n+1)+2}$. We have that

$$\begin{aligned}
X_{1,1} &= (1, 2m+2),\\
X_{2,1} &= (1, 3m+2),\\
X_{1,i} &= (m(n-(i-2)) - (i-4), 2m-(i-3)) \text{ and}\\
X_{2,j} &= (m(n-(j-2)) - (j-4), 3m-(j-3))
\end{aligned}$$

where $i > 1$ and $j > 1$. It is easy to see that for $i = 1$ and $j = 1$, m-diagonals $T_1, T_2, \ldots, T_{n-2}, X_{1,1}, X_{2,1}$ are not crossing each other. Next, we consider endpoints of $X_{1,i}$ and $X_{2,j}$ for every $i \geq 1, j > 1$. If $i = j$ then $3m-(j-3)-(2m-(i-3)) = m = n-2$. Since $j \leq n-3$ then

$$m+2 < m+4 \leq 2m-(i-3) < 3m-(j-3) \leq 3m+2 < 3m+4.$$

It follows that one end point of $X_{1,i}$ and $X_{2,j}$ is in arc $D_{m+2,3m+4}$. While other point both of $X_{1,i}$ and $X_{2,j}$ coincides with one of endpoint of $T_1, T_2, \dots, T_{n-2}$. It turns out that $X_{1,i}$is not crossing with $T_1, T_2, \dots, T_{n-2}$ as well as also for $X_{2,j}$. It remains to prove that $X_{1,i}$ and $X_{2,k}$ are not crossing each other. If $i = 1$ and $j = 1$ then it is clear that $X_{1,1}$ and $X_{2,1}$ are not crossing each other. If $i = 1$ and $1 < j \leq n-3$ then $X_{1,1} = (1, 2m+2)$ and $X_{2,j} = (m(n-(j-2))-(j-4), 3m-(j-3))$ are not crossing each other. If $j \geq i > 1$, we have $X_{1,i} = (m(n-(i-2))-(i-4), 2m-(i-3))$ and $X_{2,j} = (m(n-(j-2))-(j-4), 3m-(j-3))$. Since

$$m(n-(j-2))-(j-4) \leq m(n-(i-2))-(i-4) \text{ and } 2m-(i-3) < 3m-(j-3)$$

then $X_{1,i}$ and $X_{2,j}$ are not crossing each other. We deduce that $T_1, T_2, T_{n-2}, X_{1,i}, X_{2,j}$ is the set of m-diagonals which are not crossing each other. Thus, $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{n-2} \oplus X_{1,i} \oplus X_{2,j}$ is an m-cluster tilting object. Observe that paths of length two $X' \to X_{1,i} \to X''$ and $Y' \to X_{2,j} \to Y''$ with X', Y', X'', Y'' are m-diagonals of T which satisfy case 2 in Lemma 2.1. Beside these two paths, all other path of length two in quiver $\mathrm{End}(T)$ satisfy case 1 in Lemma 2.1. Furthermore, for such T there are exactly two paths of length two in Q which composition in $\mathrm{End}^{op}(T)$ is not zero .

We can compute the number of such T by compute the number of all combinations (i, j) where $1 \leq i \leq n-3$ and $i \leq j \leq n-3$.

TABLE 3. Pair of (i, j)

i	1	2	3	$\dots$	$n-2$	$n-3$
j	1					
	2	2				
	3	3	3			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$n-2$	
	$n-3$	$n-3$	$n-3$	$n-3$	$n-3$	$n-3$

The number of such T is

$$1 + 2 + \cdots + (n-4) + (n-3) = \frac{1}{2}(n-3)(n-2) = \frac{(n-2)!}{(n-4)!2!}.$$

□

We combine two lemmas above into a more general result, that is m-CTA $\mathrm{End}^{op}(T) \cong KQ/\mathcal{I}$ where $\mathcal{I}$ is an ideal generated by $(n-2-t)$ relations of paths of length two from $(n-2)$ relations and $1 \leq t \leq n-2$.

Lemma 4.4. *Suppose that $m \geq n-2$ and $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{n-t} \oplus X_{1,j_1} \oplus X_{2,j_2} \oplus \cdots \oplus X_{t,j_t}$ with $1 \leq j_1 \leq j_2 \leq \cdots \leq j_t \leq n-t-1$ and $1 \leq t \leq n-2$, then T is an m-cluster tilting object of $\mathcal{C}^m_{A_n}$. The m-cluster tilted algebra $\mathrm{End}^{op}(T) \cong kQ/\mathcal{I}$ where $\mathcal{I}$ is generated by $(n-2-t)$ relations of paths of length two. If $\mathfrak{T}$ be the collection of such T then $|\mathfrak{T}| = \binom{n-2}{n-2-t}$.*

Proof. For $t = 1$ and $t = 2$, it has been proved in Lemma 4.2 and Lemma 4.3. In general, we have that m-diagonals $T_1, T_2, \ldots, T_{n-t}$ are not crossing each other in regular gon $\mathcal{P}_{m(n+1)+2}$. Now consider m-diagonals $X_{1,j_1}, X_{2,j_2}, \ldots, X_{t,j_t}$ in $\mathcal{P}_{m(n+1)+2}$. If $m = n-2$ then

$$T_{n-t} = ((t+2)m-n+t+4, (t+3)m-n+t+5) = ((t+1)m+t+2, (t+2)m+t+3).$$

We will see all cases of $j_1, j_2, \ldots, j_t$ in $\mathcal{P}_{m(n+1)+2}$. To show this we first consider the case $j_1 = j_2 = \cdots = j_t = 1$ with the picture of this case in $\mathcal{P}_{m(n+1)+2}$ is

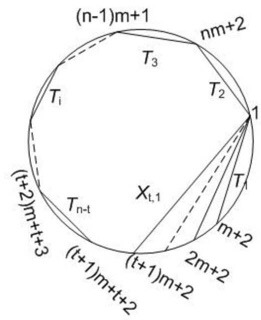


FIGURE 16. m-diagonals of T in Lemma 4.4

We get that

$$\begin{aligned} X_{1,j_1} &= (1, 2m+2) \\ X_{2,j_2} &= (1, 3m+2) \\ &\vdots \\ X_{t-1,j_{t-1}} &= (1, tm+2) \\ X_{t,j_t} &= (1, (t+1)m+2). \end{aligned}$$

The configuration of these m-diagonals in $\mathcal{P}_{m(n+1)+2}$ can be illustrated as in Figure 17. We will use that picture to see the other cases of $j_1, j_2, \ldots, j_t$. The upper line has $(n-t-1)$ black dots while the bottom line has t black dots. Let us observe the m-diagonal $X_{i,j_i} = (x_i, y_i)$ where x_i is one of the black dots on the upper line and y_i one of the points (not necessarily black dot) on the bottom line. We have that $X_{k,1} = (1, (k+1)m+2)$ with $1 \leq k \leq t$. We can conclude that $X_{i,j_i} = (x_i, y_i)$ where x_i is the j_i-th black dot on the upper line counted from the right-hand side, and $y_i = (i+1)m+2-(j_i-1) = (i+1)m+3-j_i$. Suppose that $1 \leq i \leq t-1$ and $X_{i,j_i} = (x_i, y_i), X_{i+1,j_{i+1}} = (x_{i+1}, y_{i+1})$ then

$$y_i = (i+1)m+3-j_i < y_{i+1} = (i+1)m+3+m-j_{i+1}.$$

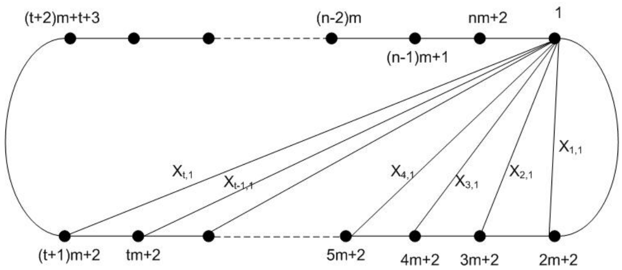


FIGURE 17. m-diagonals $X_{1,1}, X_{2,1}, \ldots, X_{t,1}$

Since $j_i \leq j_{i+1} \leq n-t-1 \leq m$ then either $x_i = x_{i+1}$ or x_{i+1}'s position is on the left of x_i. Moreover $im+2 < x_i \leq (i+1)m+2$. We describe this situation as in Figure 18.

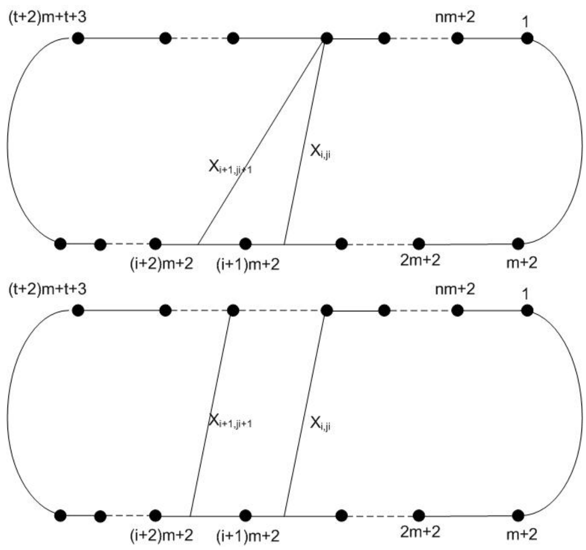


FIGURE 18. m-diagonals X_{i,j_i} and $X_{i+1,j_{i+1}}$

Since $X_{i,j_i} = (x_i, y_i), X_{i+1,j_{i+1}} = (x_{i+1}, y_{i+1})$ satisfy this condition(see Figure 18) for every i then $X_{1,j_1}, X_{2,j_2}, \ldots, X_{t,j_t}$ are not crossing each other in $\mathcal{P}_{m(n+1)+2}$. Finally we conclude that m-diagonals $T_1, T_2, \ldots, T_{n-t}, X_{1,j_1}, X_{2,j_2}, \ldots, X_{t,j_t}$ are not crossing each other in regular gon $\mathcal{P}_{m(n+1)+2}$, it proves that T is an m-cluster tilting object. Next we show the last statement. Every m-diagonal X_{i,j_i} represent one path of length two which is not zero in $\mathrm{End}^{op}(T)$. Hence, there exists $(n-2-t)$ relations of paths of length two in $\mathrm{End}^{op}(T)$. Now we compute the number of

T in this theorem. This number equal to the number of possibilities of t-tuple $(j_1, j_2, \ldots, j_t)$ where $1 \le j_1 \le j_2 \le \cdots \le j_t \le n-t-1$. This problem is equivalent to counting the number of distinct shortest routes from point A to point B in the the following diagram :

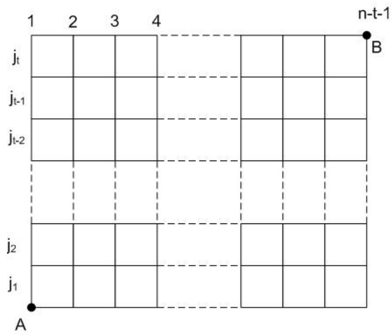


FIGURE 19. Map of routes from A to B

Here j_i interpreted as a step up to the i-th and for every j_i there is $(n-t-1)$ positions can be chosen. It is easy to see that the number of distinct shortest route is combination $(n-2-t)$ from $(n-2)$, that is

$$\binom{n-2}{n-2-t} = \frac{(n-2)!}{t!(n-2-t)!}.$$

$\square$

Proposition 4.5. *Let $m = n-2$ and $H = KQ/\mathcal{I}$ where Q is quiver*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to n-1 \xrightarrow{\alpha_{n-1}} n.$$

Let $W = \{\rho_j = \alpha_j\alpha_{j+1} | 1 \le j \le n-2\}$ and $B \subseteq W$, $|B| \neq n-2$. If $\mathcal{I} = \langle B \rangle$ then H is an m-CTA.

Proof. If $B = \emptyset$ then $I = 0$, we choose T in Lemma 4.4 with $t = n-2$ hence we get $\mathrm{End}^{op}(T) = KQ$. If $|B| = k > 1$, by Lemma 4.4 we can choose T with $t = n-2-k$ such that $\mathrm{End}^{op}(T) \cong H$. $\square$

So far we have obtain some m-CTAs in case $m = n-2$. By Theorem 3.1 it remains to find m-CTAs whose number of relations is $n-2$. But we will show that there is no such m-CTA.

Lemma 4.6. *If $m = n-2$ then there is no m-cluster tilting object T of $\mathcal{C}^m_{A_n}$ such that $End^{op}(T) \cong KQ/\mathcal{I}$ with $\mathcal{I} = \langle \rho_1, \rho_2, \ldots, \rho_{n-1}, \rho_{n-2} \rangle$.*

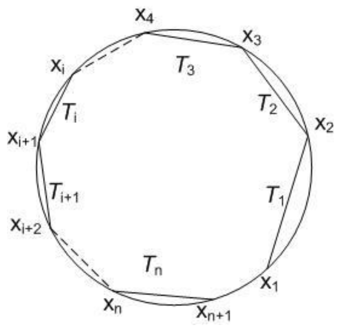


FIGURE 20. m-diagonals $T_1, T_2, \dots, T_n$

Proof. Let $T_1, T_2, \dots, T_n$ be m-diagonals corresponding to T, then by Lemma 2.5, these m- diagonal in $\mathcal{P}_{m(n+1)+2}$ should be as in Figure 20. It means that $x_{n+1} \neq x_1$ or equivalently arc $D_{x_1 x_{n+1}}$ has at least one side. Note that arc $D_{x_{i+1} x_i}$ has at least $m+1$ side. If all T_i are short then without loss of generality, suppose that $x_1 = m+2$ and $x_2 = 1$. Consequently, $T_1 = (1, m+2), T_2 = (1, nm+2), T_3 = ((n-1)m+1, nm+2), T_4 = ((n-1)m+1, (n-2)m)$ and for $5 \leq i \leq n$,

$$T_i = ((n-(i-2))m - (i-4), (n-(i-3))m - (i-5)).$$

The number of sides in arc $D_{x_{n+1} x_1}$ is $(m+1)n = mn+n$. Hence, the number of sides in arc $D_{x_1 x_{n+1}}$ is

$$m(n+1) + 2 - (mn+n) = m - (n-2).$$

However if $m = n-2$ then there is no side in arc $D_{x_1 x_{n+1}}$, a contradiction. Now suppose that there exists T_j which is not short. It follows that the number of sides in arc $D_{x_{n+1} x_1}$ is more than $(m+1)n$. If x is the number of sides in arc $D_{x_{n+1} x_1}$ then $x > mn+n$. We have that $(m(n+1)+2-x)$ is the number of side in arc $D_{x_1 x_{n+1}}$. Consequently

$$m(n+1) + 2 - x < m(n+1) + 2 - (mn+n) = m - (n-2) = 0$$

since $m = n-2$, a contradiction. We conclude that there is no such T. □

We end this section by giving all m-CTAs which are Nakayama algebras with acyclic quiver in the case $m \geq n-2$.

Proposition 4.7. *Let $m = n-2$ and $H \cong KQ/\mathcal{I}$ be an algebra with Q is*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to n-1 \xrightarrow{\alpha_{n-1}} n.$$

The algebra H is an m-CTA of $\mathcal{C}^m_{A_n}$ if only if $\mathcal{I}$ is generated by at most $(n-3)$ paths of length two.

Proof. Use Theorem 2.6, Corollary 4.5 and Lemma 4.6. □

Proposition 4.8. *Let $m > n-2$ and $H = KQ/\mathcal{I}$ with Q is the quiver*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to n-1 \xrightarrow{\alpha_{n-1}} n.$$

Suppose that $W = \{\rho_j = \alpha_j\alpha_{j+1} | 1 \le j \le n-2\}$ and $B \subseteq W$. If $\mathcal{I} = \langle B \rangle$ then H is an m-CTA.

Proof. If $B \neq W$, we choose m-cluster tilting object T in Lemma 4.4. If $B = W$ then we choose the m-cluster tilting object T in Lemma 4.1. □

Theorem 4.9. *Let $m > n-2$ and $H \cong KQ/\mathcal{I}$ be an algebra with Q is the quiver*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n.$$

The algebra H is an m-CTA of $\mathcal{C}^m_{A_n}$ if only if $\mathcal{I}$ is generated by any collection of paths of length two.

Proof. Apply Theorem 2.6 and Proposition 4.8. □

4.2. **Case** $m < n-2$.

Just like in the two previous cases to characterize Nakayama m-CTA, in this case it is sufficient to simply consider the relations of path of length two that appear on this algebra. If the number of relations is at most m, then there is m-cluster tilting object such that the corresponding m-CTA is Nakayama algebra. If the ideal generated by more than m relations of paths of length two we have not been able to guarantee which algebras are Nakayama m-CTA. This happens because we get different cases depending on the difference between m and $n-2$ (we denote by a). In the first part we put forward some Nakayama algebra which are not m-CTA in the case $m < n-2$. This class of algebra are given in Lemma 4.10, Lemma 4.11, Lemma 4.12 and Lemma 4.13. Next, we provide all the Nakayama m-CTA algebras which have at most m relation of path of length two in Lemma 4.14 parts (ii), (iii) and Lemma 4.16 parts (ii). In Theorem 4.18 we give a characterization of Nakayama m-CTA which have at most m relations. In the last part we try to find the possibility of more than m relations of path of length two. In Proposition 4.19 there are Nakayama algebras with more than m relation which are not m-CTA for some certain condition of a. We also give Nakayama algebras with more than m relation which are m-CTA for some certain condition in Proposition 4.20.

We begin by giving Nakayama algebras acyclic type which are not m-CTAs.

Lemma 4.10. *If $m < n-2$ then there is no m-cluster tilting object T in $\mathcal{C}^m_{A_n}$ such that $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n$$

and $\mathcal{I} = \langle \rho_1, \rho_2, \ldots, \rho_{n-3}, \rho_{n-2} \rangle$, where $\rho_i = \alpha_i\alpha_{i+1}$ for every i.

Proof. We utilize the same methods as in the proof of Lemma 4.6. If $T_1, T_2, \ldots, T_n$ are m-diagonals correspond to T then by Lemma 2.5, these n m-diagonals in $\mathcal{P}_{m(n+1)+2}$ should be as Figure 21, and it turns out that arc $D_{x_1x_{n+1}}$ at least

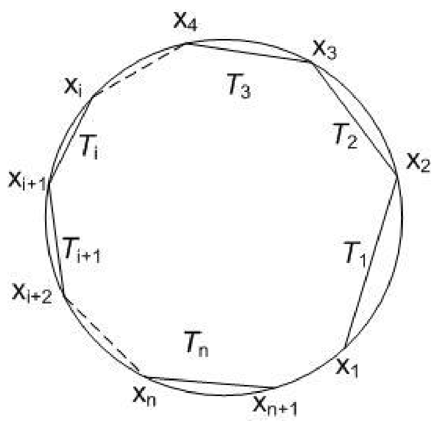


FIGURE 21. m-diagonals $T_1, T_2, \dots, T_n$

has one side. Note that the number of sides in arc $D_{x_{i+1}x_i}$ is at least $m+1$. Therefore, arc $D_{x_{n+1}x_1}$ has at least $(mn+n)$ sides. Let x be the number of sides in arc $D_{x_{n+1}x_1}$, then $x \geq mn+n$. We also have that $(m(n+1)+2-x)$ is the number of sides in arc $D_{x_1x_{n+1}}$. Therefore

$$m(n+1)+2-x \leq m(n+1)+2-mn-n = m-(n-2) < 0,$$

because $m < n-2$, a contradiction. The proof is complete. $\square$

Next lemma shows that the Nakayama algebra whose relations are $m+1$ consecutive relation paths of length two starting from ρ_{a+1} is not m-CTA.

Lemma 4.11. *Suppose that $m < n-2$ and $a = n-2-m$ then there is no m-cluster tilting object T of $\mathcal{C}^m_{A_n}$ such that $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n$$

and $\mathcal{I} = \langle \rho_{a+1}, \rho_{a+2}, \dots, \rho_{n-3}, \rho_{n-2} \rangle$, where $\rho_i = \alpha_i\alpha_{i+1}$ for every i.

Proof. Suppose that there exists such T. By Lemma 2.1, the configuration of m-diagonals correspond to T in $\mathcal{P}_{m(n+1)+2}$ is as in Figure 22. Hence we may write $T = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_{m+2} \oplus X_1 \oplus X_2 \oplus \cdots \oplus X_a$. It follows that arc $D_{x_{a+1}y_{m+2}}$ has at least one side. By the definition of m-diagonal, arc $D_{y_{i+1}y_i}$ and arc $D_{x_1y_1}$ have at least $m+1$ sides, while arc $D_{x_jx_{j+1}}$ has at least m side. Hence, arc $D_{y_{m+2}x_{a+1}}$ has at least

$$(m+2)(m+1)+am = (m+2)(m+1)+(n-2-m)m = m(n+1)+2$$

sides. A contradiction since $\mathcal{P}_{m(n+1)+2}$ has $m(n+1)+2$ sides and arc $D_{x_{a+1}y_{m+2}}$ has at least one side. $\square$

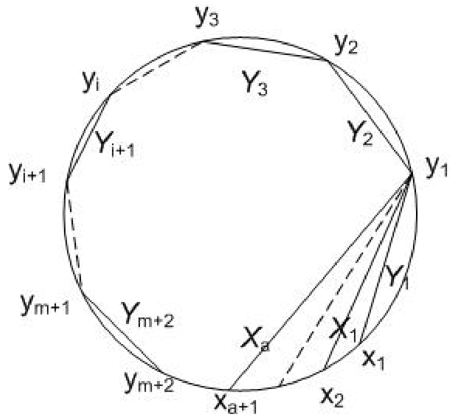


FIGURE 22. m-diagonals $Y_1, Y_2, \ldots, Y_{m+2}, X_1, X_2, \ldots, X_a$

We have that Nakayama algebra with m consecutive relations of path of length two is not m-CTA of type A_n.

Lemma 4.12. *Suppose that $m < n-2$ and $a = n-2-m$ then there is no m-cluster tilting object T of $\mathcal{C}^m_{A_n}$ such that End$^{op}(T) \cong KQ/\mathcal{I}$ with Q is*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n$$

and $\mathcal{I} = \langle \rho_t, \rho_{t+1}, \ldots, \rho_{t+m-1} \rangle$ where $1 \le t \le a$, where $\rho_i = \alpha_i \alpha_{i+1}$ for every i.

Proof. Assume that there exists such T, then we have m paths of length two in End$^{op}(T)$ whose composition is zero. Therefore we need exactly m triplets of m-diagonals satisfy case 1 in Lema 2.1. Since the quiver of End$^{op}(T)$ is a path then there exist $(m+2)$ m-diagonals in $\mathcal{P}_{m(n+1)+2}$, where the configuration is as in Figure 23. Thus it remains a m-diagonals. Because $\mathcal{I} = \langle \rho_t, \rho_{t+1}, \ldots, \rho_{t+m-1} \rangle$ then we should have $(t-1)$ m-diagonals whose endpoint is y_1 and the other endpoint in arc $D_{x_1 x_{m+2}}$ while the remaining $(a-(t-1))$ m-diagonals have one endpoint at y_{m+1} and the other point in arc $D_{x_1 y_{m+2}}$. More precisely, the picture of all m-diagonals should be like Figure 24. From Figure 24, m-diagonals which correspond to T are $T_1, T_2, \ldots, T_{m+2}, X_1, X_2, \ldots, X_{t-1}$,
$Y_1, Y_2, \ldots, Y_{a-t+1}$ with $X_i = (y_1, x_{i+1})$ and $Y_j = (y_{m+1}, z_j)$. Note that for every $1 \le i \le t-1$, arc $D_{x_i x_{i+1}}$ has at least m sides. We also have that either arc $D_{x_j x_{j-1}}$ or arc $D_{z_1 y_{m+1}}$ has at least m sides. Hence, the number of sides in arc $D_{z_{a-t+1} x_t}$ is at least

$$(m+1)(m+2) + (t-1)m + (a-t+1)m = (m+1)(m+2) + am = m(n+1) + 2,$$

this contradicts the fact that arc $D_{x_t z_{a-t+1}}$ has at least one side. $\square$

The following lemma states that Nakayama algebra with consecutive relations of path of length two ending in ρ_{n-2} is not m-CTA of type A_n.

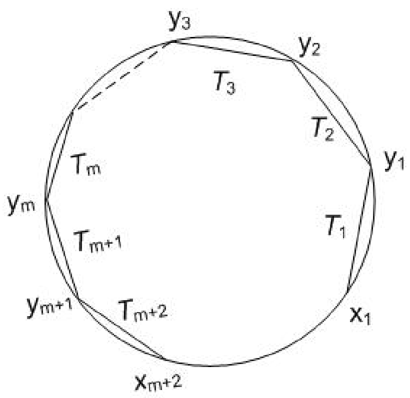


FIGURE 23. m-diagonals $T_1, T_2, \dots, T_{m+2}$

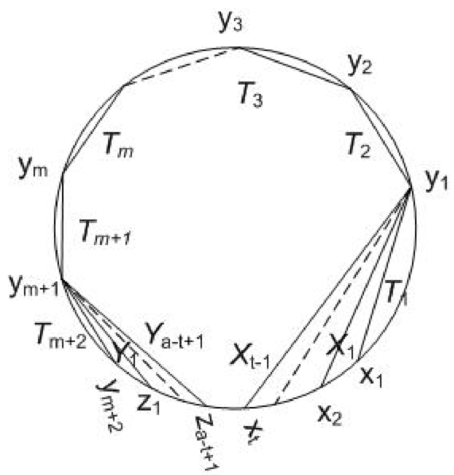


FIGURE 24. m-diagonals $T_1, \dots, T_{m+2}, X_1, X_2, \dots, X_{t-1}, Y_1, \dots, Y_{a-t+1}$

Lemma 4.13. *Suppose that $m < n-2$ and $a = n-2-m$ then there is no m-cluster tilting object T of $\mathcal{C}^m_{A_n}$ such that $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n$$

and $\mathcal{I} = \langle \rho_{j+1}, \rho_{j+2}, \dots, \rho_{n-3}, \rho_{n-2} \rangle$ for every $0 \le j \le a$, where $\rho_i = \alpha_i \alpha_{i+1}$ for every i.

Proof. The cases $j = 0$ and $j = a$ have been proved in Lemma 4.10 and Lemma 4.11. Now assume that $1 < j < a$, then the picture of m-diagonals which corresponds to

T in $\mathcal{P}_{m(n+1)+2}$ is Observe that arc $D_{y_{n-j}x_1}$ has at least $(m+1)(n-j)$ sides, while

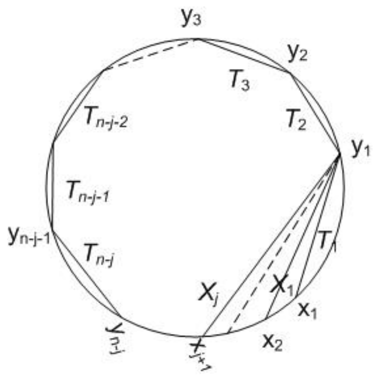


FIGURE 25. m-diagonals $T_1, T_2, \ldots, T_{n-j}, X_1, \ldots, X_j$

arc $D_{x_1x_{j+1}}$ has at least jm sides. Thus, the number of sides in arc $D_{y_{n-j}x_{j+1}}$ is at least

$$(m+1)(n-j) + jm = mn - jm + n - j + jm = n(m+1) - j.$$

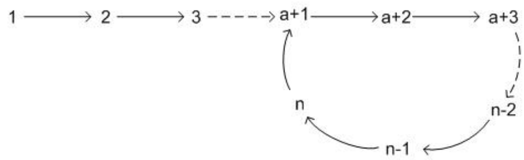
Since $j < a$ we have

$$m(n+1) - j > m(n+1) - a = n(m+1) - (n-2-m) = m(n+1) + 2.$$

This contradicts the fact that $\mathcal{P}_{m(n+1)+2}$ has $(m(n+1)+2)$ sides. $\square$

Lemma 4.14. *Suppose that $m < n-2$, $a = n-2-m$ and $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{n-t} \oplus X_{1,j_1} \oplus X_{2,j_2} \oplus \cdots \oplus X_{t,j_t}$ with $1 \leq j_1 \leq j_2 \leq \cdots \leq j_t \leq min\{m, n-t-1\}$ and $a \leq t \leq n-2$ then T is an m-cluster tilting object of $\mathcal{C}^m_{A_n}$.*

(i) *If $t = a$ and $j_t = 1$ then the algebra $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is*



and $\mathcal{I}$ generated by all paths of length two in the cycle.

(ii) *If $t > a$ then $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n$$

and $\mathcal{I}$ generated by $(n-2-t)$ relations of paths of length two from $(n-2)$ relations of paths of length two.

(iii) *If $t = a$ and $j_t \neq 1$ then $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is the quiver in part (ii) and $\mathcal{I}$ generated by m relations of paths of length two where $\rho_{n-2} \in \mathcal{I}$.*

Proof. First, consider case $t = a$ and $j_t = 1$, we get $j_1 = j_2 = \cdots = j_t = 1$. Consequently $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{m+2} \oplus X_{1,1} \oplus X_{2,1} \oplus \cdots \oplus X_{a,1}$. We have that $T_{m+2} = (m(a+1)+2, m(a+2)+3)$ and $X_{a,j_a} = X_{a,1} = (1, m(a+1)+2)$, it follows that T_{m+2} and $X_{a,1}$ have a common endpoint $m(a+1)+2$. Hence, the picture of m-diagonals that corresponds to T is as in Figure 26. It is clear

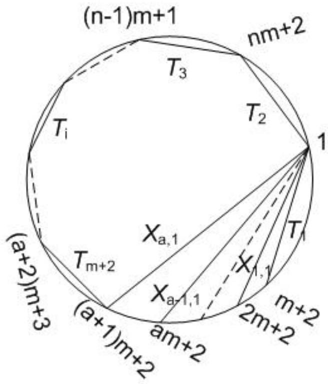


FIGURE 26. m-diagonals $T_1, T_2, \ldots, T_{m+1}, X_{1,1}, \ldots, X_{a,1}$

that the algebra $\mathrm{End}^{op}(T)$ satisfies the first part of the lemma. Furthermore, if $t > a$ then $(t+1)m + 2 + t + m - n - 2 > (t+1)m + 2$. Therefore $T_{n-t} = ((t+1)m+2+t+m-n-2, (t+2)m+3+t+m-n-2)$ and $X_{t,1} = (1, (t+1)m+2)$ either are not crossing each other or have a common endpoint in $\mathcal{P}_{m(n+1)+2}$. Since $t \geq a+1$ then $\min\{m, n-t-1\} = m+1$ or $\min\{m, n-t-1\} = n-t-1$. It follows that

$$1 \leq j_1 \leq j_2 \leq \cdots \leq j_t \leq \min\{m, n-t-1\} \leq m$$

and then we may use the same way as in the proof of Lemma 4.4. If $t = a$ then $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{m+2} \oplus X_{1,j_1} \oplus X_{2,j_2} \oplus \cdots \oplus X_{a,j_a}$. The fact that m-diagonals which correspond to T are not crossing each other can be obtained by the same argument as in the proof of Lemma 4.4. Because $2 \leq j_a \leq m$ we have that X_{a,j_a} does not have a common endpoint neither with T_{m+2} nor at the point $am+2$. Thus, we have the quiver of $\mathrm{End}^{op}(T)$ is A_n. Next, we will prove that $\rho_{n-2} \in \mathcal{I}$. Consider m-diagonals T_m, T_{m+1} and T_{m+2} in $\mathcal{P}_{m(n+1)+2}$ in Figure 27.

Since $j_i \leq m$ then there is no m-diagonal X_{i,j_i} that have a common endpoint at y_{m+1}. So there exists an irreducible map $T_m \to T_{m+1} \to T_{m+2}$. Because at the point x_{m+2} there is only one m-diagonal T_{m+2} then this irreducible map corresponds to the path $\alpha_{n-2}\alpha_{n-1}$ in Q. But this path satisfies case 1 in Lemma 2.1, hence by Lemma 2.5, $\rho_{n-2} = 0$ in $\mathrm{End}^{op}(T)$. $\square$

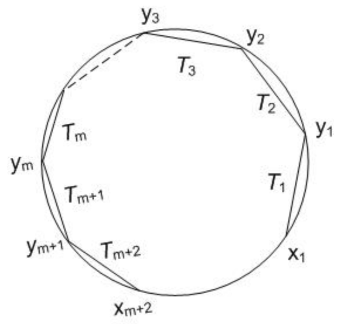


FIGURE 27. m-diagonals $T_1, T_2, \ldots, T_{m+2}$

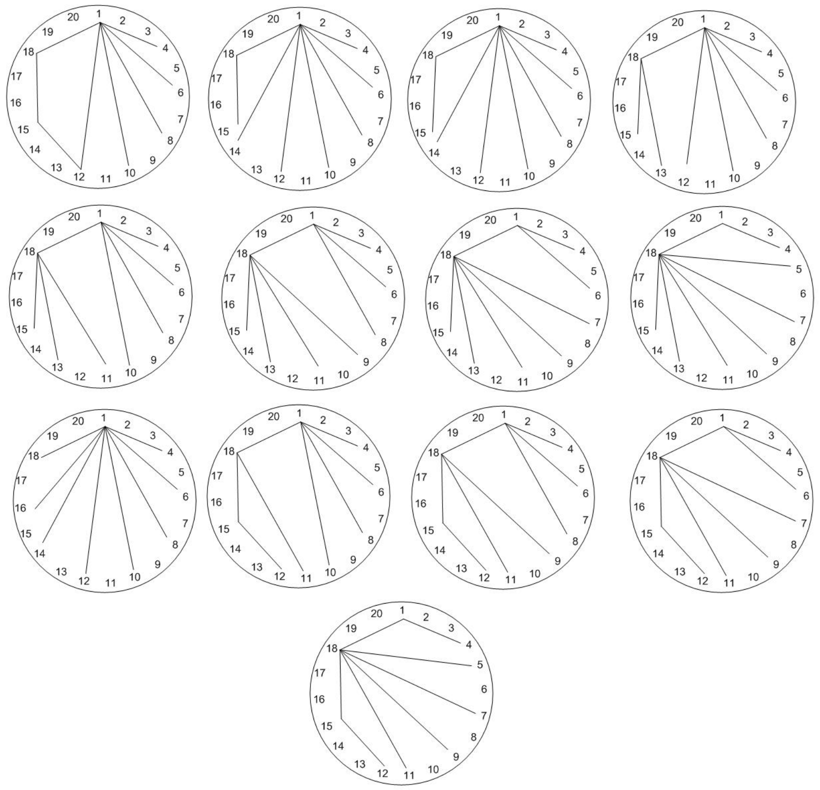


FIGURE 28. m-diagonals of T for $m = 2$ and $n = 8$

Example 4.15. *Let $m = 2$ and $n = 8$ then $a = 8-2-2 = 4$ and $m(n+1)+2 = 20$. All m-diagonals which correspond to T in Lemma 4.14 are as in Figure 28*

Lemma 4.16. *Suppose that $m < n-2$, $a = n-2-m$ and*
$T = T_1 \oplus \cdots \oplus T_{m+2} \oplus X_{1,j_1} \oplus X_{2,j_2} \oplus \cdots \oplus X_{k,j_k} \oplus X_{k+1,m+1} \oplus \cdots \oplus X_{a-1,m+1} \oplus X_{a,m+1}$
with $1 \leq j_1 \leq j_2 \cdots \leq j_k \leq m$ and $1 \leq k < a$ then T is an m-cluster tilting object of $\mathcal{C}^m_{A_n}$.

(i) If $j_k = 1$ then the algebra $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is and $\mathcal{I}$ generated by

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r, dashed] & k+1 \arrow[r] & k+2 \arrow[r, dashed] & k+m+1 \arrow[r] & k+m+2 \arrow[r, dashed] \arrow[lll, bend left] & n-1 \arrow[r] & n \end{tikzcd}$$

all paths of length two in the cycle.

(ii) If $j_k \neq 1$ and $j_k \leq m$ then the algebra $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \rightarrow \cdots \rightarrow (n-1) \xrightarrow{\alpha_{n-1}} n$$

and $\mathcal{I}$ generated by m relations of paths of length two with $\rho_{k+m+1}, \rho_{k+m+2}, \ldots, \rho_{n-3}, \rho_{n-2} \notin \mathcal{I}$ and $\rho_{k+m} \in \mathcal{I}$.

Proof. Note that for $k = a$, T is the m-cluster tilting object in Lemma 4.14 part 1. Assume that $k < a$, if $j_k = 1$ then $T = T_1 \oplus \cdots \oplus T_{m+2} \oplus X_{1,1} \oplus X_{2,1} \oplus \cdots \oplus X_{k,1} \oplus X_{k+1,m+1} \oplus \cdots \oplus X_{a-1,m+1} \oplus X_{a,m+1}$. We have that m-diagonal $X_{k,1} = (1, (k+1)m+2)$ and $X_{k+1,m+1} = X_{k+1,2}$ if $m = 1$ or

$$\begin{aligned} X_{k+1,m+1} &= ((n-(m+1-2))m-(m+1-4), (k+1+1)m-(m+1-3)) \\ &= ((n-m)m+3, (k+1)m+2) \end{aligned}$$

if $m \neq 1$. If $m \neq 1$ then $X_{k,1}$ and $X_{k+1,m+1}$ have a common endpoint at $(k+1)m+2$. If $m = 1$ then $k = 1$ and hence $X_{k+1,m+1} = X_{2,2} = (n+2,4)$, $X_{k,1} = X_{1,1} = (1,4)$. It turns out that $X_{k+1,m+1}$ and $X_{k,1}$ have a common endpoint if $m = 1$. So the picture of m-diagonals which correspond to $T = T_1 \oplus \cdots \oplus T_{m+2} \oplus X_{1,1} \oplus X_{2,1} \oplus \cdots \oplus X_{k,1} \oplus X_{k+1,m+1} \oplus \cdots \oplus X_{a-1,m+1} \oplus X_{a,m+1}$ in $\mathcal{P}_{m(n+1)+2}$ is as in Figure 29. For $j_k \neq 1$ the configuration of m-diagonals $T_1, T_2, \ldots, T_{m+2}$ and $X_{k+1,m+1}, \ldots, X_{a-1,m+1}, X_{a-1,m+1}$ in $\mathcal{P}_{m(n+1)+2}$ is the same as in the Figure 29. It remains to consider the position of $X_{1,j_1}, X_{2,j_2}, \ldots, X_{k,j_k}$ in $\mathcal{P}_{m(n+1)+2}$ if $j_k \neq 1$ and $j_k \leq m$. By the same arguments as in the proof of Lemma 4.4 then for X_{i,j_i} and $X_{i+1,j_{i+1}}$ in $\mathcal{P}_{m(n+1)+2}$ will be one of the following pictures in Figure 30. If $j_k \leq m$ then the number of black dots on the top line that can be the end point of X_{i,j_i} except point 1 is m (see Figure 30). Consequently the leftmost black dot on the top line is $(a+3)m+4$. We claim that the ideal $\mathcal{I}$ generated by m relations of paths of length two. From Figure 29 we have that $T_2, T_3, \ldots, T_{m+1}$ are m-diagonals that correspond to a midpoint of a path of length two that satisfies case 1 in Lemma 2.1 while others m-diagonal satisfy case 2 in Lemma 2.1. So the number of relations that generate $\mathcal{I}$ is only m.

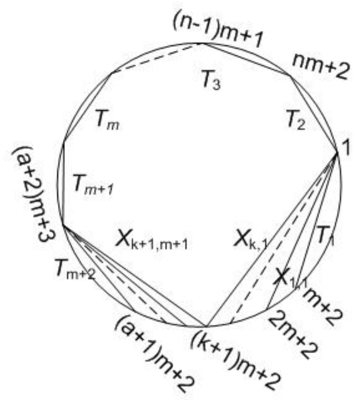


FIGURE 29. m-diagonals $T_1, \ldots, T_{m+2}, X_{1,1}, X_{2,1}, \ldots, X_{k,1}, X_{k+1,m+1}, \ldots, X_{a-1,m+1}, X_{a,m+1}$

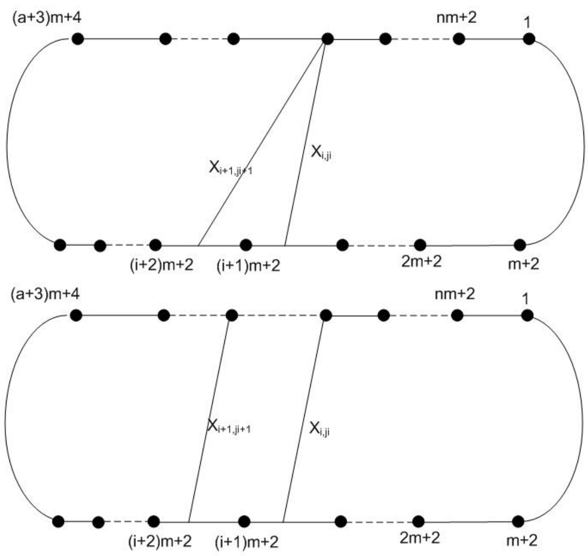


FIGURE 30. m-diagonals $X_{i,j_i}, X_{i+1,j_{i+1}}$

Note that m-diagonals $T_{m+1}, X_{k+1,m+1}, X_{k+2,m+1}, \ldots, X_{a,m+1}, T_{m+2}$ have a common endpoint at $(a+2)m+3$. Therefore there exists a composition of irreducible maps

$$T_{m+1} \to X_{k+1,m+1} \to X_{k+2,m+1} \to \cdots \to X_{a,m+1} \to T_{m+2}.$$

Since there is no other m-diagonal whose one endpoint is $(a+2)m+3$ and in the arc $D_{(a+1)m+2,(a+2)m+3}$ then this composition of irreducible maps correspond to

$$(k+m+1) \xrightarrow{\alpha_{k+m+1}} (k+m+2) \xrightarrow{\alpha_{k+m+2}} \cdots \to (n-2) \xrightarrow{\alpha_{n-2}} (n-1) \xrightarrow{\alpha_{n-1}} n.$$

We conclude that $\rho_{k+m+1}, \rho_{k+m+2}, \ldots, \rho_{n-3}, \rho_{n-2} \notin \mathcal{I}$. The path

$$(k+m) \xrightarrow{\alpha_{k+m}} (k+m+1) \xrightarrow{\alpha_{k+m+1}} (k+m+2)$$

in Q correspond to the composition of irreducible maps $X \to T_{m+1} \to X_{k+1,m+1}$ where $X = T_m$ or $X = X_{k,m}$. Because either m-diagonals $T_m, T_{m+1}, X_{k+1,m+1}$ or $X_{k,m}, T_{m+1}, X_{k+1,m+1}$ always satisfy case 1 in Lemma 2.1, then $\rho_{k+m} \in \mathcal{I}$. □

Example 4.17. *Let $m = 3$ and $n = 7$ then $a = n-m-2 = 2$ and $m(n+1)+2 = 26$. The figure of m-diagonals that correspond to T in Lemma 4.16 for this case is*

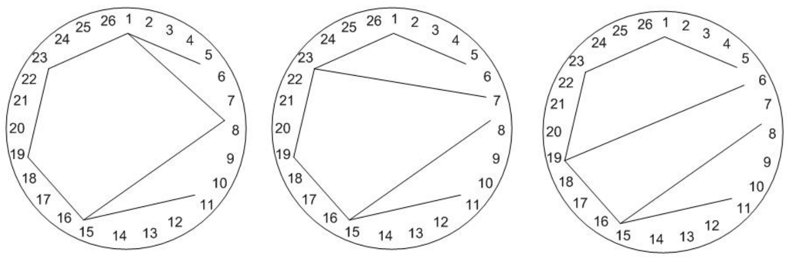


FIGURE 31. m-diagonals of T for $m = 3$ and $n = 7$

Lemma 4.16 gives us the information of m-CTA from type A_n which is a Nakayama algebra of acyclic type and have m relations. Therefore we can compute the number of m-CTA from type A_n which has less than or equal to m relations. By the second part of Lemma 4.14, the number of m-CTA which have less than m relations of paths of length two is

$$\binom{n-2}{0} + \binom{n-2}{1} + \cdots + \binom{n-2}{m-2} + \binom{n-2}{m-1}.$$

Next, the possibility of the number of m-CTAs that have exactly m relations of path of length two is $\binom{n-2}{m}$. But, by Lemma 4.12 there are a m-CTAs who have m relations which are not Nakayama algebras of acyclic type and from Lemma 4.13 we get one more this kind. So the number of m-CTAs which have m relations and whose quiver is A_n for this case is at most $\binom{n-2}{m} - (a+1)$. We compute the number of m-cluster tilting objects in Lemma 4.14 part (iii) together with Lemma 4.16 part (ii). Since $1 \leq j_1 \leq j_2 \cdots \leq j_k < m$ and $j_k \neq 1$ then for every k the number of m-cluster tilting objects is $\binom{m+k}{k} - 1$. Because $1 \leq k \leq a$ then the total number of m-cluster tilting objects in Lemma 4.14 part (iii) and Lemma 4.16 part (ii) is

$$\sum_{k=1}^{a} \binom{m+k}{k} - a.$$

Using Pascal's identity it can be proved that

$$\sum_{k=0}^{a} \binom{m+k}{k} = \binom{n+a+1}{a}.$$

We know that $a = n-2-m$, hence

$$\begin{aligned}
\sum_{k=1}^{a} \binom{m+k}{k} - a &= \sum_{k=1}^{a} \binom{m+k}{k} + 1 - (a+1) \\
&= \sum_{k=1}^{a} \binom{m+k}{k} + \binom{m+0}{0} - (a+1) \\
&= \sum_{k=0}^{a} \binom{m+k}{k} - (a+1) \\
&= \binom{m+a+1}{a} - (a+1) \\
&= \binom{n-2}{n-2-m} - (a+1) \\
&= \binom{n-2}{m} - (a+1).
\end{aligned}$$

We conclude that all m-CTAs which are Nakayama algebras of acyclic type and have m relations of paths of length two are the algebras in Lemma 4.14 part (iii) and Lemma 4.16 part (ii). We write the results so far for the case $m < n-2$ in the following theorem.

Theorem 4.18. *Let $H \cong KQ/\mathcal{I}$ be an m-CTA of $\mathcal{C}^m_{A_n}$ with $m < n-2$, and let $\mathcal{I}$ be an ideal generated by less than or equal to m relations of paths of length two and Q is*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n.$$

Suppose that $W = \{\rho_j = \alpha_j\alpha_{j+1} | 1 \leq j \leq n-2\}$ then the generator of $\mathcal{I}$ is one of the following

(i) $B \subseteq W$ for any B with $0 \leq |B| < m$.
(ii) $B \subseteq W$ for any B with $|B| = m$ and $B \neq \{\rho_t, \rho_{t+1}, \ldots, \rho_{t+m-1}\}$ for every $1 \leq t \leq a+1$.

Proof. Apply Lemma 4.11, 4.12, 4.13, 4.14, 4.16. □

Until here we have known all m-CTAs $H = KQ/\mathcal{I}$ with $Q = A_n$ and I generated by at most m relations of path of length two for the case $m < n-2$.

Next we will give some m-CTAs whose ideal is generated by more than m relations of paths of length two.

Proposition 4.19. *Suppose that $m < n-2$ and $km \leq a$ with $1 \leq k \leq (a-1)$ then there is no m-cluster tilting object T of $\mathcal{C}^m_{A_n}$ such that $End^{op}(T) \cong KQ/\mathcal{I}$ with Q is*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n$$

and ideal $\mathcal{I}$ generated by at least $(n-2-k)$ paths of length two.

Proof. Assume that such m-cluster tilting object T exists. First assume that $a \geq k(m+1)$. Since $1 \leq k \leq (a-1)$ and $\mathcal{I}$ generated by at least $(n-2-k)$ relations of paths of length two then there exist $(m+2)$ m-diagonals which configuration is as in Figure 32. Observe that $D_{y_{m+2}x_1}$ has at least $(m+2)(m+1)$ sides. Hence

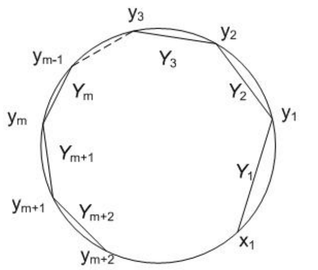


FIGURE 32. m-diagonals $Y_1, Y_2, \ldots, Y_{m+2}$

arc $D_{x_1y_{m+2}}$ has at least

$$m(n+1)+2-(m+2)(m+1) = am$$

sides. Now there are a m-diagonals which are not shown in the Figure 32. Since $\mathcal{I}$ is generated by at least $(n-2-k)$ paths of length two, there exist m-diagonals $X_1, X_2, \ldots X_{a-k}$ together with $(m+2)$ m-diagonal in the Figure 32 such that the configuration as in the Figure 33. Note that arc $D_{xy_{m+2}}$ at least has $(a-k)(m+1)$ sides. Since $a \geq k(m+1)$ we get

$$(a-k)(m+1) = am + (a-k(m+1)) \geq am,$$

a contradiction. Now assume that $km \leq a < k(m+1)$. Consider Figure 32, we obtain that arc D_{x_1x} has at least $(k(m+1)-a)$ sides. Hence

$$k(m+1) - a \leq k(m+1) - km \leq k.$$

But there exist k m-diagonals of T besides $Y_1, Y_2, \ldots, Y_{m+1}, Y_{m+2}, X_1, X_2, \ldots, X_{a-k}$. Each of them has one endpoint outside arc D_{x_1x} and the other endpoint should be in arc D_{x_1x} and different from x_1, x. Since arc D_{x_1x} has at most k sides then there exist two m-diagonals from these k m-diagonals whose common endpoint is

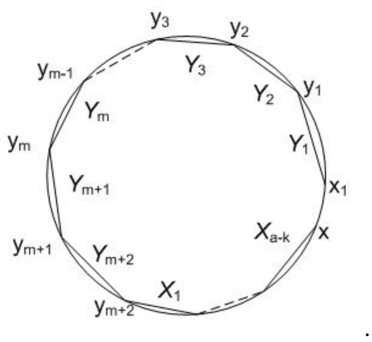


FIGURE 33. m-diagonals $Y_1, Y_2, \ldots, Y_{m+2}, X_1, X_2, \ldots X_{a-k}$

in arc Dx_1x. Consequently the quiver of $\text{End}^{op}(T)$ has a cycle, a contradiction. This completes the proof. $\square$

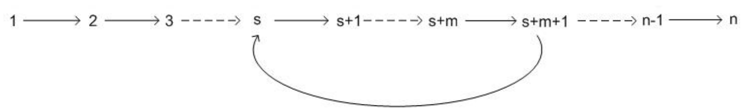
Consider Proposition 4.19 for the case $k = a - 1$. If $k = a - 1$ then

$$a \geq (a-1)(m+1) \Leftrightarrow a \leq 1 + \frac{1}{m}.$$

We get that a must be equal to 1. If $a = 1$ or equivalently $n - 2 = m + 1$ then by Lemma 4.10 the ideal I is generated by at most m relations of paths of length two.

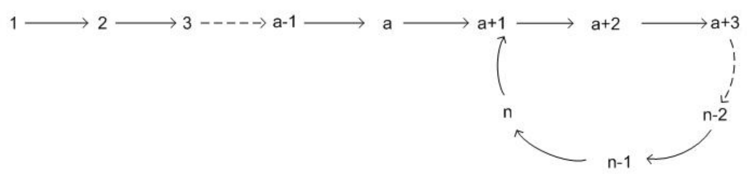
Proposition 4.20. *Suppose that $2 \leq m < n-2$, $1 < a = (n-2-m) < m$ and $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{m+2} \oplus T_{m+3} \oplus T_{m+4} \oplus \cdots \oplus T_{m+2+t} \oplus X_{1,j_1} \oplus X_{2,j_2} \oplus \cdots \oplus X_{a-t,j_{a-t}}$ with $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{a-t} \leq m+1$, $1 \leq t \leq a-1$ and $j_{a-t} > t$ then T is an m-cluster tilting object of $\mathcal{C}^m_{A_n}$.*

(i) *if $j_{s-1} = 1$ and $j_s = m+1$ for $1 \leq s \leq a-t$ then the algebra $End^{op}(T) \cong KQ/\mathcal{I}$ where Q is*



and $\mathcal{I}$ generated by all paths of length two in the cycle and t paths of length two from the right.

(ii) *If $j_{a-t} = t+1$ then $End^{op}(T) \cong KQ/\mathcal{I}$ where Q is*



and $\mathcal{I}$ generated by all paths of length two in the cycle and t paths of length two in the path $1 \to 2 \to 3 \to \cdots \to a \to a+1$.

(iii) Otherwise $End^{op}(T) \cong KQ/\mathcal{I}$ where Q is

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to (n-1) \xrightarrow{\alpha_{n-1}} n$$

and $\mathcal{I}$ generated by $(m+t)$ relations of paths of length two with $\rho_{n-t-1}, \ldots, \rho_{n-3}, \rho_{n-2} \in \mathcal{I}$.

Proof. It is clear that m-diagonals which correspond to $T_1, T_2, \ldots, T_{m+2+t}$ are not crossing each other in $\mathcal{P}_{m(n+1)+2}$. Now consider case (1) that is $j_{s-1} = 1$ and $j_s = m+1$ for $1 \leq s \leq a-t$. We get that $T = T_1 \oplus T_2 \oplus \cdots \oplus T_{m+2+t} \oplus X_{1,1} \oplus X_{2,1} \oplus \cdots \oplus X_{s-1,1} \oplus X_{s,m+1} \oplus X_{s+1,m+1} \oplus \cdots \oplus X_{a-t,m+1}$. We have that

$$\begin{aligned}
X_{1,1} &= (1, 2m+2) \\
X_{2,1} &= (1, 3m+2) \\
&\vdots \\
X_{s-1,1} &= (1, sm+2) \\
X_{s,m+1} &= (am+3, sm+2) \\
X_{s+1,m+1} &= (am+3, (s+1)m+2) \\
&\vdots \\
X_{a-t,m+1} &= (am+3, (a-t)m+2) \\
T_{m+2+t} &= ((a-t)m+m+2-t, (a-t+1)m+m+3-t).
\end{aligned}$$

It follows that $X_{s-1,1}$ and $X_{s,m+1}$ have a common endpoint. Since $t \leq a-1 < m$ then m diagonals T_{m+2+t}, $X_{a-t,m+1}$ are not crossing each other and do not have a common endpoint. We get the figure of m-diagonals which correspond to T for this case is as in Figure 34. Now we come to the case 2, let $j_{a-t} = t+$

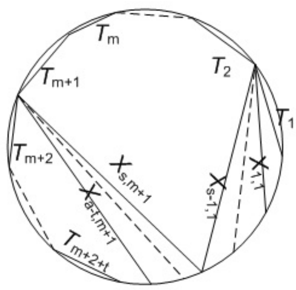


FIGURE 34. m-diagonals of T

1. Note that $X_{a-t,t+1} = ((n-t+1)m+3-t, (n-m-t-1)m+2-t)$ and

$T_{m+2+t} = ((n-m-t-1)m+2-t, (n-m-t)m+3-t)$. It turns out that $X_{a-t,t+1}$ and T_{m+2+t} have a common endpoint and $T_1, T_2, \ldots, T_{m+2+t}, X_{a-t,t+1}$ are not crossing each other in $\mathcal{P}_{m(n+1)+2}$. We obtain the figure of m-diagonals $T_1, T_2, \ldots, T_{m+2+t}, X_{a-t,t+1}$ in $\mathcal{P}_{m(n+1)+2}$ as in Figure 35. It is easy to check that

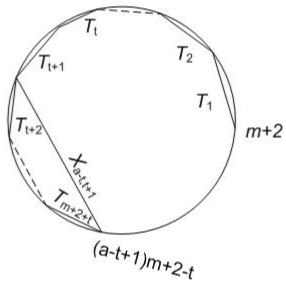


FIGURE 35. m-diagonals $T_1, T_2, \ldots, T_{m+2}, X_{a-t,t+1}$

$X_{1,j_t}, X_{2,j_2}, \ldots, X_{a-t-1,j_{a-t-1}}$ are not crossing each other since $t \leq a-1 < m$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{a-t} = t+1$. $\square$

We end the case $m < n-2$ by the above proposition. We have not been able to find all m-CTAs which is Nakayama algebra type A_n. This is because many cases on the value of a have to be considered and have different characteristics in some cases of the value of a. However, Proposition 4.19 gives some m-CTAs which are not Nakayama algebras in the case $km \leq a$ with $1 \leq k \leq a-1$. While Proposition 4.20 part (3) give some m-CTAs which are Nakayama algebras in the case $1 < a < m$ and have more than m relations. A way to find all m-CTAs which are Nakayama algebras in this case is by investigating all m-CTAs in each case $km \leq a$ where $1 \leq k \leq a-1$.

Example 4.21. *The following figure shows m-diagonals correspond to m-cluster tilting objects in Proposition 4.20 in the case $m = 4$ and $n = 9$.*

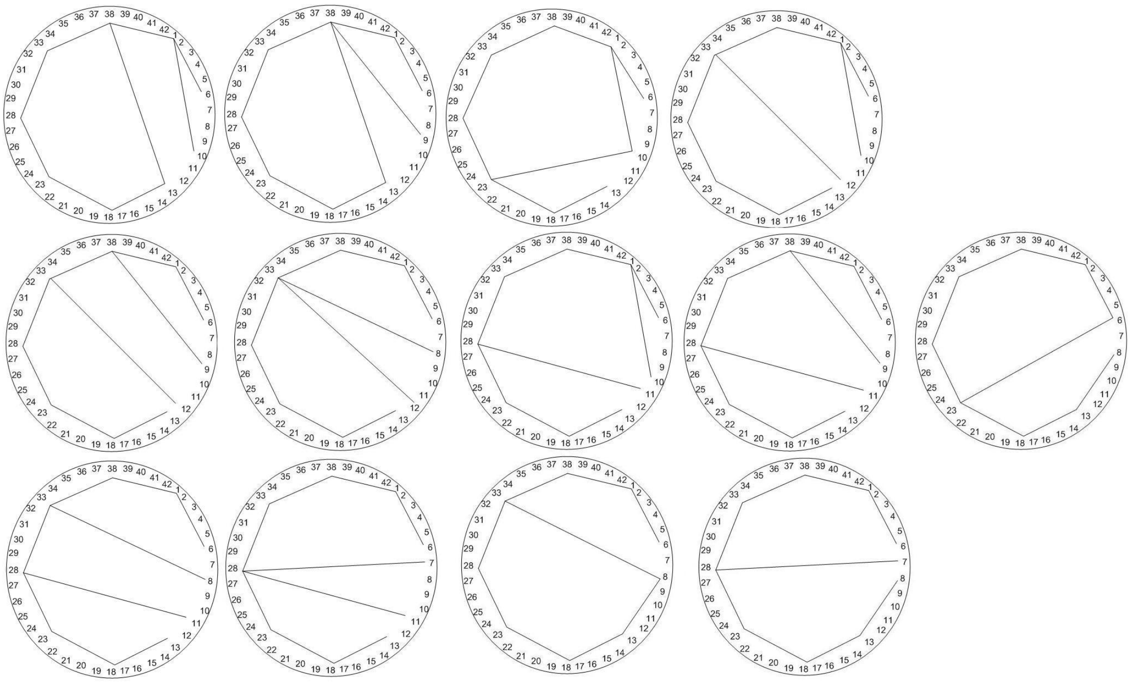


FIGURE 36. m-diagonals of T for $m = 4$ and $n = 9$

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REFERENCES

[1] Baur, K. and Marsh, R.J., "A Geometric description of m-Cluster categories", *Trans. Amer. Math. Soc.*, **360**(2008), 5789-5803.

[2] Buan, A., Marsh, R., Reineke, M., Reiten, I., and Todorov, G., "Tilting theory and cluster combinatorics", *Advances in mathematics*, **204** (2006), 572-68.

[3] Caldero, P., Chapoton, F., and Schiffler, R., "Quivers with relations arising from clusters (A_n case)", *Trans. Amer. Math. Soc.*, **358**(2006), no. 3, 1347-1364.

[4] Faisal and Muchtadi-Alamsyah, I., "On cyclic Nakayama m-cluster tilted algebra of type A_n", *Proc. International Conference on Mathematical Research, Education and Application*, (2013), 119-127.

[5] Keller, B., "On triangulated orbit categories", *Doc. Math.*, **10**(2005), 551-581.

[6] Murphy, G.J., "Derived equivalence classification of m-cluster tilted algebra of type A_n", *Journal of Algebra*, (2009).

[7] Zhu, B., "Generalized cluster complexes via quiver representations", *J. Algebraic Combin.*, **27**(2008), 25-54.