

COMPLEMENTARY DISTANCE SPECTRA AND COMPLEMENTARY DISTANCE ENERGY OF LINE GRAPHS OF REGULAR GRAPHS

H. S. RAMANE¹ AND K. C. NANDEESH²

¹Department of Mathematics, Karnatak University,
Dharwad - 580003, India
e-mail: hsrामane@yahoo.com

²Department of Mathematics, Karnatak University,
Dharwad - 580003, India
e-mail: nandeeshkc@yahoo.com

Abstract. The complementary distance (CD) matrix of a graph G is defined as $CD(G) = [c_{ij}]$, where $c_{ij} = 1 + D - d_{ij}$ if $i \neq j$ and $c_{ij} = 0$, otherwise, where D is the diameter of G and d_{ij} is the distance between the vertices v_i and v_j in G . The CD -energy of G is defined as the sum of the absolute values of the eigenvalues of CD -matrix. Two graphs are said to be CD -equienergetic if they have same CD -energy. In this paper we show that the complement of the line graph of certain regular graphs has exactly one positive CD -eigenvalue. Further we obtain the CD -energy of line graphs of certain regular graphs and thus constructs pairs of CD -equienergetic graphs of same order and having different CD -eigenvalues.

Key words and Phrases: Complementary distance eigenvalues, adjacency eigenvalues, line graphs, complementary distance energy.

Abstrak. Matriks complementary distance (CD) dari sebuah graph G didefinisikan sebagai $CD(G) = [c_{ij}]$, dimana $c_{ij} = 1 + D - d_{ij}$ jika $i \neq j$ dan $c_{ij} = 0$, atau yang lain, dimana D adalah diameter G dan d_{ij} adalah jarak antara titik-titik v_i dan v_j di G . Energi- CD dari G didefinisikan sebagai jumlahan dari nilai mutlak nilai-nilai eigen matriks- CD . Dua graf disebut ekuienergetik- CD jika mereka mempunyai energi- CD yang sama. Dalam paper ini kami menunjukkan komplemen graf garis dari graf-graf regular tertentu mempunyai tepat satu nilai eigen- CD positif. Lebih jauh, kami mendapatkan energi- CD graf garis dari graf-graf regular tertentu dan selanjutnya mengkonstruksi pasangan graf-graf ekuienergetik- CD -equienergetic berorde sama dan mempunyai nilai-nilai eigen- CD berbeda.

2000 Mathematics Subject Classification: 05C50, 05C12.

Received: 20-05-2015, revised: 17-02-2016, accepted: 19-02-2016.

Kata kunci: Nilai-nilai eigen complementary distance, Nilai-nilai eigen ketetanggaan, graf-graf garis, energi complementary distance.

1. INTRODUCTION

Let G be a simple, undirected, connected graph with n vertices and m edges. Let the vertex set of G be $V(G) = \{v_1, v_2, \dots, v_n\}$. The *adjacency matrix* of a graph G is the square matrix $A = A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the *adjacency eigenvalues* of G , and they are labeled as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. These form the *adjacency spectrum* of G [4].

The *distance* between the vertices v_i and v_j , denoted by d_{ij} , is the length of the shortest path joining v_i and v_j . The *diameter* of a graph G , denoted by $diam(G)$, is the maximum distance between any pair of vertices of G [3]. A graph G is said to be *r-regular graph* if all of its vertices have same degree equal to r .

The *complementary distance* between the vertices v_i and v_j , denoted by c_{ij} is defined as $c_{ij} = 1 + D - d_{ij}$, where D is the diameter of G and d_{ij} is the distance between v_i and v_j in G .

The *complementary distance matrix* or *CD-matrix* [7] of a graph G is an $n \times n$ matrix $CD(G) = [c_{ij}]$, where

$$c_{ij} = \begin{cases} 1 + D - d_{ij}, & \text{if } i \neq j \\ 0, & \text{if } i = j. \end{cases}$$

The complementary distance matrix is an important source of structural descriptors in the quantitative structure property relationship (QSPR) model in chemistry [7, 8].

The eigenvalues of $CD(G)$ labeled as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ are said to be the *complementary distance eigenvalues* or *CD-eigenvalues* of G and their collection is called *CD-spectra* of G . Two non-isomorphic graphs are said to be *CD-cospectral* if they have same *CD-spectra*.

The *complementary distance energy* or *CD-energy* of a graph G denoted by $CDE(G)$ is defined as

$$CDE(G) = \sum_{i=1}^n |\mu_i|. \quad (1)$$

The Eq. (1) is defined in full analogy with the *ordinary graph energy* $E(G)$, defined as [5]

$$E(G) = \sum_{i=1}^n |\lambda_i| . \tag{2}$$

Two graphs G_1 and G_2 are said to be *equienergetic* if $E(G_1) = E(G_2)$. Results on non cospectral equienergetic graphs can be found in [1, 2, 12, 13, 17]. For more details about ordinary graph energy one can refer [9].

Two connected graphs G_1 and G_2 are said to be *complementary distance equienergetic* or *CD-equienergetic* if $CDE(G_1) = CDE(G_2)$. Trivially, the CD-cospectral graphs are CD-equienergetic. In this paper we obtain the CD-energy of line graphs of certain regular graphs and thus construct CD-equienergetic graphs having different CD-spectra.

We need following results.

Theorem 1.1. [4] *If G is an r -regular graph, then its maximum adjacency eigenvalue is equal to r .*

The *line graph* of G , denoted by $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [6]. If G is a regular graph of order n and of degree r then the line graph $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$.

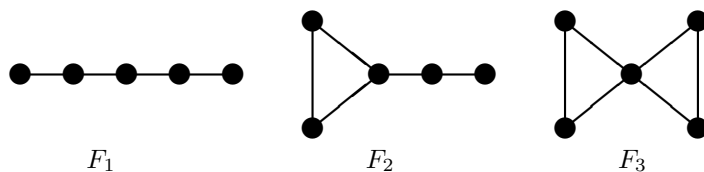


Figure 1: The forbidden induced subgraphs

Theorem 1.2. [10, 11] *For a connected graph G , $\text{diam}(L(G)) \leq 2$ if and only if none of the three graphs F_1 , F_2 and F_3 of Fig. 1 is an induced subgraph of G .*

Theorem 1.3. [15] *If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r , then the adjacency eigenvalues of $L(G)$ are*

$$\lambda_i + r - 2, \quad i = 1, 2, \dots, n, \quad \text{and}$$

$$-2, \quad n(r - 2)/2 \text{ times} .$$

Theorem 1.4. [14] *Let G be an r -regular graph of order n . If $r, \lambda_2, \dots, \lambda_n$ are the adjacency eigenvalues of G , then the adjacency eigenvalues of \overline{G} , the complement of G , are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \dots, n$.*

Lemma 1.5. [16] *If for any two adjacent vertices u and v of a graph G , there exists a third vertex w which is not adjacent to either u or v , then*

(i) \overline{G} is connected and

(ii) $\text{diam}(\overline{G}) = 2$.

2. CD-EIGENVALUES

Theorem 2.1. *Let G be an r -regular graph on n vertices and $\text{diam}(G) = 2$. If $r, \lambda_2, \dots, \lambda_n$ are the adjacency eigenvalues of G , then CD-eigenvalues of G are $n + r - 1$ and $\lambda_i - 1$, $i = 2, 3, \dots, n$.*

Proof. Since G is an r -regular graph, $\mathbf{1} = [1, 1, \dots, 1]'$ is an eigenvector of $A = A(G)$ corresponding to the eigenvalue r . Set $\mathbf{z} = \frac{1}{\sqrt{n}}\mathbf{1}$ and let P be an orthogonal matrix with its first column equal to \mathbf{z} such that $P'AP = \text{diag}(r, \lambda_2, \dots, \lambda_n)$. Since $\text{diam}(G) = 2$, the CD-matrix $CD(G)$ can be written as $CD(G) = J + A - I$, where J is the matrix whose all entries are equal to 1 and I is an identity matrix. Therefore

$$\begin{aligned} P'(CD)P &= P'(J + A - I)P \\ &= P'JP + P'AP - I \\ &= \text{diag}(n + r - 1, \lambda_2 - 1, \dots, \lambda_n - 1), \end{aligned}$$

where we have used the fact that any column of P other than the first column is orthogonal to the first column. Hence the eigenvalues of $CD(G)$ are $n + r - 1$ and $\lambda_i - 1$, $i = 2, 3, \dots, n$. \square

Theorem 2.2. *Let G be an r -regular graph of order n . Let $L(G)$ be the line graph of G such that for any two adjacent vertices u and v of $L(G)$, there exists a third vertex w in $L(G)$ which is not adjacent to either u or v . Then $\overline{L(G)}$, the complement of $L(G)$, has exactly one positive CD-eigenvalue, equal to $r(n - 2)$.*

Proof. Let the adjacency eigenvalues of G be $r, \lambda_2, \dots, \lambda_n$. From Theorem 1.3, the adjacency eigenvalues of $L(G)$ are

$$\left. \begin{array}{ll} 2r - 2, & \text{and} \\ \lambda_i + r - 2, & i = 2, 3, \dots, n, \quad \text{and} \\ -2, & n(r - 2)/2 \text{ times.} \end{array} \right\} \quad (3)$$

From Theorem 1.4 and the Eq. (3), the adjacency eigenvalues of $\overline{L(G)}$ are

$$\left. \begin{array}{ll} (nr/2) - 2r + 1, & \text{and} \\ -\lambda_i - r + 1, & i = 2, 3, \dots, n, \quad \text{and} \\ 1, & n(r - 2)/2 \text{ times.} \end{array} \right\} \quad (4)$$

The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2) - 2r + 1$. Since for any two adjacent vertices u and v of $L(G)$ there exists a third vertex w which is not adjacent to either u or v in $L(G)$, by Lemma 1.5, $\text{diam}(\overline{L(G)}) = 2$. Therefore by Theorem 2.1 and Eq. (4), the CD-eigenvalues of $\overline{L(G)}$ are

$$\left. \begin{array}{ll} nr - 2r, & \text{and} \\ -\lambda_i - r, & i = 2, 3, \dots, n, \quad \text{and} \\ 0, & n(r - 2)/2 \text{ times.} \end{array} \right\} \quad (5)$$

All adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$ [4]. Therefore $\lambda_i + r \geq 0$, $i = 1, 2, \dots, n$. The theorem follows from Eq. (5). \square

3. CD-ENERGY

Theorem 3.1. *Let G be an r -regular graph of order n . Let $L(G)$ be the line graph of G such that for any two adjacent vertices u and v of $L(G)$, there exists a third vertex w in $L(G)$ which is not adjacent to either u or v . Then $CDE(\overline{L(G)}) = 2r(n - 2)$.*

Proof. Bearing in mind Theorem 2.2 and Eq. (5), the CD-energy of $\overline{L(G)}$ is computed as:

$$\begin{aligned} CDE(\overline{L(G)}) &= nr - 2r + \sum_{i=2}^n (\lambda_i + r) + |0| \times \frac{n(r - 2)}{2} \\ &= 2r(n - 2) \quad \text{since} \quad \sum_{i=2}^n \lambda_i = -r. \end{aligned}$$

\square

Theorem 3.2. *Let G be a connected, r -regular graph with $n > 3$ vertices and let none of the three graphs F_1 , F_2 and F_3 of Fig. 1 is an induced subgraph of G .*

(i) *If the smallest adjacency eigenvalue of G is greater than or equal to $3 - r$, then $CDE(L(G)) = 3n(r - 2)$.*

(ii) *If the second largest adjacency eigenvalue of G is smaller than $3 - r$, then $CDE(L(G)) = nr + 4r - 6$.*

Proof. Let $r, \lambda_2, \lambda_3, \dots, \lambda_n$ be the adjacency eigenvalues of a regular graph G . Then from Theorem 1.3, the adjacency eigenvalues of $L(G)$ are

$$\left. \begin{array}{ll} 2r - 2 & \text{and} \\ \lambda_i + r - 2, & i = 1, 2, \dots, n, \quad \text{and} \\ -2, & n(r - 2)/2 \text{ times.} \end{array} \right\} \quad (6)$$

The graph G is regular of degree r and has order n . Therefore $L(G)$ is a regular graph on $nr/2$ vertices and of degree $2r - 2$. As none of the three graphs F_1 , F_2 and F_3 of Fig. 1 is an induced subgraph of G , from Theorem 1.2, $\text{diam}(L(G)) = 2$. Therefore from Theorem 2.1 and Eq. (6), the CD -eigenvalues of $L(G)$ are

$$\left. \begin{array}{l} (nr + 4r - 6)/2, \quad \text{and} \\ \lambda_i + r - 3, \quad i = 2, 3, \dots, n \quad \text{and} \\ -3, \quad n(r - 2)/2 \text{ times.} \end{array} \right\} \quad (7)$$

Therefore

$$CDE(L(G)) = \left| \frac{nr + 4r - 6}{2} \right| + \sum_{i=2}^n |\lambda_i + r - 3| + |-3| \frac{n(r-2)}{2}. \quad (8)$$

(i) By assumption, $\lambda_i + r - 3 \geq 0$, $i = 2, 3, \dots, n$, then from Eq. (8)

$$\begin{aligned} CDE(L(G)) &= \frac{nr + 4r - 6}{2} + \sum_{i=2}^n (\lambda_i + r - 3) + \frac{3n(r-2)}{2} \\ &= \frac{nr + 4r - 6}{2} + \sum_{i=2}^n \lambda_i + (n-1)(r-3) + \frac{3n(r-2)}{2} \\ &= 3n(r-2) \quad \text{since} \quad \sum_{i=2}^n \lambda_i = -r. \end{aligned}$$

(ii) By assumption, $\lambda_i + r - 3 < 0$, $i = 2, 3, \dots, n$, then from Eq. (8)

$$\begin{aligned} CDE(L(G)) &= \frac{nr + 4r - 6}{2} - \sum_{i=2}^n (\lambda_i + r - 3) + \frac{3n(r-2)}{2} \\ &= \frac{nr + 4r - 6}{2} - \sum_{i=2}^n \lambda_i - (n-1)(r-3) + \frac{3n(r-2)}{2} \\ &= nr + 4r - 6 \quad \text{since} \quad \sum_{i=2}^n \lambda_i = -r. \end{aligned}$$

□

Corollary 3.3. *Let G be a connected, cubic graph with n vertices and let none of the three graphs F_1 , F_2 and F_3 of Fig. 1 is an induced subgraph of G . Then $CDE(L(G)) = 3n + E(G)$.*

Proof. Substituting $r = 3$ in Eq. (8) we get

$$\begin{aligned}
 CD(L(G)) &= \left| \frac{3n+6}{2} \right| + \sum_{i=2}^n |\lambda_i| + |-3| \frac{n}{2} \\
 &= \frac{3n+6}{2} + (E(G) - 3) + \frac{3n}{2} \\
 &= 3n + E(G).
 \end{aligned}$$

□

4. CD-EQUIENERGETIC GRAPHS

Lemma 4.1. *Let G_1 and G_2 be regular graphs of the same order and of the same degree. Then following holds:*

(i) $L(G_1)$ and $L(G_2)$ are of the same order, same degree and have the same number of edges.

(ii) $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are of the same order, same degree and have the same number of edges.

Proof. Statement (i) follows from the fact that the line graph of a regular graph is a regular and that the number of edges of G is equal to the number of vertices of $L(G)$. Statement (ii) follows from the fact that the complement of a regular graph is a regular and that the number of vertices of a graph and its complement is equal. □

Lemma 4.2. *Let G_1 and G_2 be regular graphs of the same order and of the same degree. Let for $i = 1, 2$, $L(G_i)$ be the line graph of G_i such that for any two adjacent vertices u_i and v_i of $L(G_i)$, there exists a third vertex w_i in $L(G_i)$ which is not adjacent to either u_i or v_i . Then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are CD-cospectral if and only if G_1 and G_2 are cospectral.*

Proof. Follows from Eqs. (3), (4) and (5). □

Lemma 4.3. *Let G_1 and G_2 be connected, regular graphs of the same order $n > 3$ and of the same degree. Let none of the three graphs F_1 , F_2 and F_3 of Fig. 1 be an induced subgraph of G_i , $i = 1, 2$. Then $L(G_1)$ and $L(G_2)$ are CD-cospectral if and only if G_1 and G_2 are cospectral.*

Proof. Follows from Eqs. (6) and (7). □

Theorem 4.4. *Let G_1 and G_2 be regular, non CD-cospectral graphs of the same order and of the same degree. Let for $i = 1, 2$, $L(G_i)$ be the line graph of G_i such that for any two adjacent vertices u_i and v_i of $L(G_i)$, there exists a third vertex w_i in $L(G_i)$ which is not adjacent to either u_i or v_i . Then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ form a pair of non CD-cospectral, CD-equienenergetic graphs of equal order and of equal number of edges.*

Proof. Follows from Lemma 4.1, Lemma 4.2 and Theorem 3.1. □

Theorem 4.5. *Let G_1 and G_2 be connected, regular, non CD -cospectral graphs of the same order $n > 3$ and of the same degree r . Let none of the three graphs F_1 , F_2 and F_3 of Fig. 1 be an induced subgraph of G_i , $i = 1, 2$.*

(i) If the smallest adjacency eigenvalue of G_i , $i = 1, 2$ is greater than or equal to $3 - r$, then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non CD -cospectral, CD -equienergetic graphs of equal order and of equal number of edges.

(ii) If the second largest adjacency eigenvalue of G_i , $i = 1, 2$ is smaller than $3 - r$, then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non CD -cospectral, CD -equienergetic graphs of equal order and of equal number of edges.

Proof. Follows from Lemma 4.1, Lemma 4.3 and Theorem 3.2. □

Theorem 4.6. *Let G_1 and G_2 be connected, non CD -cospectral, cubic, equienergetic graphs of the same order. Let none of the three graphs F_1 , F_2 and F_3 of Fig. 1 be an induced subgraph of G_i , $i = 1, 2$. Then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non CD -cospectral, CD -equienergetic graphs of equal order and of equal number of edges.*

Proof. Follows from Lemma 4.1, Lemma 4.3 and Corollary 3.3. □

Acknowledgement. The authors H. S. Ramane and K. C. Nandeesh are thankful to the University Grants Commission (UGC), Govt. of India for support through research grant under UPE FAR-II grant No. F 14-3/2012 (NS/PE).

REFERENCES

- [1] Balakrishnan, R., "The energy of a graph", *Linear Algebra Appl.*, **387** (2004), 287–295.
- [2] Brankov, V., Stevanović, D. and Gutman, I., "Equienergetic chemical trees", *J. Serb. Chem. Soc.*, **69** (2004), 549–553.
- [3] Buckley, F. and Harary, F., *Distance in Graphs*, Addison–Wesley, Redwood, 1990.
- [4] Cvetković, D., Rowlinson, P. and Simić, S., *An Introduction to the Theory of Graph Spectra*, Cambridge Univ. Press, Cambridge, 2010.
- [5] Gutman, I., "The energy of a graph", *Ber. Math. Stat. Sect. Forschungsz. Graz*, **103** (1978), 1–22.
- [6] Harary, F., *Graph Theory*, Addison–Wesley, Reading, 1969.
- [7] Ivanciuc, O., Ivanciuc, T. and Balaban, A.T., "The complementary distance matrix, a new molecular graph metric", *ACH-Models Chem.* **137(1)** (2000), 57–82.
- [8] Jenežić, D., Miličević, A., Nikolić, S. and Trinajstić, N., *Graph Theoretical Matrices in Chemistry*, Uni. Kragujevac, Kragujevac, 2007.
- [9] Li, X., Shi, Y. and Gutman, I., *Graph Energy*, Springer, New York, 2012.
- [10] Ramane, H.S., Gutman, I. and Ganagi, A.B., "On diameter of line graphs", *Iranian J. Math. Sci. Inf.*, **8(1)** (2013), 105–109.
- [11] Ramane, H.S., Revankar, D.S., Gutman, I. and Walikar, H.B., "Distance spectra and distance energies of iterated line graphs of regular graphs", *Publ. Inst. Math. (Beograd)*, **85** (2009), 39–46.
- [12] Ramane, H.S. and Walikar, H.B., "Construction of equienergetic graphs", *MATCH Commun. Math. Comput. Chem.*, **57** (2007), 203–210.

- [13] Ramane, H.S., Walikar, H.B., Rao, S.B., Acharya, B.D., Hampiholi, P.R., Jog, S.R. and Gutman, I., "Equienergetic graphs", *Kragujevac J. Math.*, **26** (2004), 5–13.
- [14] Sachs, H., "Über selbstkomplementare Graphen", *Publ. Math. Debrecen*, **9** (1962), 270–288.
- [15] Sachs, H., "Über Teiler, Faktoren und charakteristische Polynome von Graphen, Teil II", *Wiss. Z. TH Ilmenau*, **13** (1967), 405–412.
- [16] Senbagamalar, J., Baskar Babujee, J. and Gutman, I., "On Wiener index of graph complements", *Trans. Comb.*, **3(2)** (2014), 11–15.
- [17] Xu, L. and Hou, Y., "Equienergetic bipartite graphs", *MATCH Commun. Math. Comput. Chem.*, **57** (2007), 363–370.

