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A CERTAIN SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS FOR OPERATOR ON HILBERT SPACE

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Abstract. By making use of the operators on Hilbert space, we introduce and study a subclass $\mathcal{A}k_p(\alpha, \beta, \delta, T)$ of multivalent analytic functions with negative coefficients. Also we obtain some geometric properties.

Key words: Hilbert space, analytic function, convex set, extreme points.

Abstrak. Dengan menggunakan operator-operator di ruang Hilbert, kami memperkenalkan dan mempelajari suatu subclass dari fungsi analitik multivalen $\mathcal{A}k_p(\alpha, \beta, \delta, T)$ dengan koefisien negarif. Kami juga mendapatkan beberapa sifat geometri mereka.

Kata kunci: Ruang Hilbert, fungsi analitik, himpunan konveks, titik-titik ekstrim.

1. Introduction

Let $\mathcal{A}_p$ be the class of functions f of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \ (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let k_p denote the subclass of $\mathcal{A}_p$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \ (a_{n+p} \geq 0, p \in \mathbb{N} = \{1, 2, ...\}), \tag{2}$$

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Definition 1.1. *A function $f \in k_p$ is said to be in the class $\mathcal{A}k_p(\alpha, \beta, \delta)$ if it satisfies*

$$\left| \frac{f'(z) - pz^{p-1}}{\alpha(f'(z) - \beta) + p - \beta} \right| < 0,$$

where $0 \leq \alpha < 1$, $0 \leq \beta < p$, $0 < \delta \leq 1$ dan $z \in U$.

Let H be a Hilbert space on the complex field. Let T be a linear operator on H. For a complex analytic function f on the unit disk U, we denoted $f(T)$, the operator on H defined by the usual Riesz-Dunford integral [2]

$$f(T) = \frac{1}{2\pi i} \int_c f(z)(zI - T)^{-1} dz,$$

where I is the identity operator on H, c is a positively oriented simple closed rectifiable contour lying in U and containing the spectrum $\sigma(T)$ of T in its interior domain [3]. Also $f(T)$ can be defined by the series

$$f(T) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} T^n,$$

which converges in the norm topology [4].

Definition 1.2. *Let H be a Hilbert space and T be an operator on H such that $T \neq \emptyset$ and $\|T\| < 1$. Let α, β be real numbers such that $0 \leq \alpha < 1$, $0 \leq \beta < p$, $0 < \delta \leq 1$. An analytic function f on the unit disk is said to belong to the class $\mathcal{A}k_p(\alpha, \beta, \delta, T)$ if it satisfy the inequality*

$$\|f'(T) - pT^{p-1}\| < \delta \|\alpha(f'(T) - \beta) + p - \beta\|,$$

where $\emptyset$ denote the zero operator on H.

The operator on Hilbert space were consider recently be Xiaopei [8], Joshi [6], Chrakim et al. [1], Ghanim and Darus [5], and Selvaraj et al. [7].

2. Main Results

Theorem 2.1. *Let $f \in k_p$ be defined by (2). Then $f \in \mathcal{A}k_p(\alpha, \beta, \delta, T)$ for all $T \neq \emptyset$ if and only if*

$$\sum_{n=1}^{\infty} (n+p)(1+\delta\alpha) a_{n+p} \leq \delta(p-\beta)(1+\alpha). \tag{3}$$

where $0 \leq \alpha < 1$, $0 \leq \beta < p$, $0 < \delta \leq 1$.
The result is sharp for the function f given by

$$f(z) = z^p - \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)} z^{n+p}, \ n \geq 1. \tag{4}$$

Proof. Suppose that the inequality (3) holds. Then, we have

$$\begin{aligned}
&\|f'(T) - pT^{p-1}\| - \delta\|\alpha(f'(T) - \beta) + p - \beta\| \\
&\qquad = \left\| -\sum_{n=1}^{\infty}(n+p)a_{n+p}T^{n+p-1} \right\| \\
&\qquad - \delta \left\| \alpha pT^{p-1} - \sum_{n=1}^{\infty}\alpha(n+p)a_{n+p}T^{n+p-1} + p - \beta(1+\alpha) \right\| \\
&\qquad \le \sum_{n=1}^{\infty}(n+p)(1+\delta\alpha)a_{n+p} - \delta(p-\beta)(a+\alpha) \le 0.
\end{aligned}$$

Hence, $f \in \mathcal{A}k_p(\alpha, \beta, \delta, T)$.
To show the converse, let $f \in \mathcal{A}k_p(\alpha, \beta, \delta, T)$. Then

$$\|f'(T) - pT^{p-1}\| < \delta\|\alpha(f'(T) - \beta) + p - \beta\|,$$

gives

$$\begin{aligned}
&\left\| -\sum_{n=1}^{\infty}(n+p)a_{n+p}T^{n+p-1} \right\| \\
&\qquad < \delta \left\| \alpha pT^{p-1} - \sum_{n=1}^{\infty}\alpha(n+p)a_{n+p}T^{n+p-1} + p - \beta(1+\alpha) \right\|
\end{aligned}$$

Setting $T = rI(0 < r < 1)$ in the above inequality, we get

$$\frac{\sum_{n=1}^{\infty}(n+p)a_{n+p}r^{n+p-1}}{\alpha pr^{p-1} - \sum_{n=1}^{\infty}\alpha(n+p)a_{n+p}r^{n+p-1} + p - \beta(1+\alpha)} < \delta \tag{5}$$

Upon clearing denominator in (5) and letting $r \to 1$, we obtain

$$\sum_{n=1}^{\infty}(n+p)a_{n+p} < \delta(p-\beta)(1+\alpha) - \sum_{n=1}^{\infty}\delta\alpha(n+p)a_{n+p}.$$

Thus

$$\sum_{n=1}^{\infty}(n+p)(1+\delta\alpha)a_{n+p} \le \delta(p-\beta)(1+\alpha),$$

which completes the proof. □

Corollary 2.2. *If $f \in \mathcal{A}k_p(\alpha, \beta, \delta, T)$, then*

$$a_{n+p} \le \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}, n \ge 1.$$

Theorem 2.3. *If $f \in \mathcal{A}k_p(\alpha, \beta, \delta, T)$ and $\|T\| < 1$, $T \neq \emptyset$, then*

$$\|T\|^p - \frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}\|T\|^{p+1} \le \|f(T)\| \le \|T\|^p + \frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}\|T\|^{p+1}$$

and

$$p\|T\|^{p-1}-\frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}\|T\|^{p}\leq\|f'(T)\|\leq p\|T\|^{p-1}+\frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}\|T\|^{p}$$

The result is sharp for the function f given by

$$f(z)=z^{p}-\frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}z^{p+1}.$$

Proof. According to the Theorem 2.1, we get

$$\sum_{n=1}^{\infty}a_{n+p}\leq\frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}.$$

Hence

$$\begin{aligned}\|f(T)\|&\geq\|T\|^{p}-\sum_{n=1}^{\infty}a_{n+p}\|T\|^{n+p}\\&\geq\|T\|^{p}-\|T\|^{p+1}\sum_{n=1}^{\infty}a_{n+p}\\&\geq\|T\|^{p}-\frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}\|T\|^{p+1}.\end{aligned}$$

Also,

$$\begin{aligned}\|f(T)\|&\leq\|T\|^{p}+\sum_{n=1}^{\infty}a_{n+p}\|T\|^{n+p}\\&\leq\|T\|^{p}+\frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}\|T\|^{p+1}\end{aligned}$$

In view of Theorem 2.1, we have

$$\sum_{n=1}^{\infty}(n+p)a_{n+p}\leq\frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}.$$

Thus

$$\begin{aligned}\|f'(T)\|&\geq p\|T\|^{p-1}-\sum_{n=1}^{\infty}(n+p)a_{n+p}\|T\|^{n+p-1}\\&\geq p\|T\|^{p-1}-\|T\|^{p}\sum_{n=1}^{\infty}(n+p)a_{n+p}\\&\geq p\|T\|^{p-1}-\frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}\|T\|^{p}\end{aligned}$$

and

$$\begin{aligned}\|f'(T)\| &\leq p\|T\|^{p-1} + \|T\|^p \sum_{n=1}^{\infty}(n+p)a_{n+p}\\ &\leq p\|T\|^{p-1} + \frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}\|T\|^p\end{aligned}$$

Therefore the proof is complete. □

Theorem 2.4. *Let* $f_0(z) = z^p$ *and*

$$f_n(z) = z^p - \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}z^{n+p}, n \geq 1.$$

Then $f \in \mathcal{A}k_p(\alpha, \beta, \delta, T)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z), \tag{6}$$

where $\lambda_n \geq 0$ *and* $\sum_{n=0}^{\infty} \lambda_n = 1$.

Proof. Assume that f can be expressed by (6). Then, we have

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = z^p - \sum_{n=0}^{\infty} \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}\lambda_n z^{n+p}. \tag{7}$$

Thus

$$\sum_{n=0}^{\infty} \frac{(n+p)(1+\delta\alpha)}{\delta(p-\beta)(1+\alpha)} \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}\lambda_n = \sum_{n=0}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1, \tag{8}$$

and so $f \in \mathcal{A}k_p(\alpha, \beta, \delta, T)$.

Conversely, suppose that f given by (2) in in the class $\mathcal{A}k_p(\alpha, \beta, \delta, T)$. Then by Corollary 2.2, we have

$$a_{n+p} \leq \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}.$$

Setting

$$\lambda_n = \frac{(n+p)(1+\delta\alpha)}{\delta(p-\beta)(1+\alpha)}a_n, n \geq 1,$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$. Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z),$$

This completes the proof of the theorem. □

Theorem 2.5. *The class* $\mathcal{A}k_p(\alpha, \beta, \delta, T)$ *is a convex set.*

Proof. Let f_1 and f_2 be the arbitrary elements of $\mathcal{A}k_p(\alpha,\beta,\delta,T)$. Then for every $t(0 \le t \le 1)$, we show that $(1-t)f_1 + tf_2 \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$. Thus, we have

$$(1-t)f_1 + tf_2 = z^p - \sum_{n=1}^{\infty} \left((1-t)a_{n+p} + tb_{n+p}\right) z^{n+p}.$$

Hence,

$$\sum_{n=1}^{\infty}(n+p)(1+\delta\alpha)\left((1-t)a_{n+p} + tb_{n+p}\right)$$
$$= (1-t)\sum_{n=1}^{\infty}(n+p)(1+\delta\alpha)a_{n+p} + t\sum_{n=1}^{\infty}(n+p)(1+\delta\alpha)b_{n+p}$$
$$\le (1-t)\delta(p-\beta)(1+\alpha) + t\delta(p-\beta)(1+\alpha).$$

This completes the proof. □

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