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NUMERICAL ENTROPY PRODUCTION OF THE ONE-AND-A-HALF-DIMENSIONAL SHALLOW WATER EQUATIONS WITH TOPOGRAPHY

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Abstract. Numerical entropy production can be used as a smoothness indicator of solutions to conservation laws. By definition the entropy production is non-positive. However some authors, using a finite volume method framework, showed that positive overshoots of the numerical entropy production were possible for conservation laws (no source terms involved). Note that the one-and-a-half-dimensional shallow water equations without source terms are conservation laws. A report has been published regarding the behaviour of the numerical entropy production of the one-and-a-half-dimensional shallow water equations without source terms. The main result of that report was that positive overshoots of the numerical entropy production were avoided by use of a modified entropy flux which satisfies a discrete numerical entropy inequality. In the present article we consider an extension problem of the previous report. We take the one-and-a-half-dimensional shallow water equations involving topography. The topography is a source term in the considered system of equations. Our results confirm that a modified entropy flux which satisfies a discrete numerical entropy inequality is indeed required to have no positive overshoots of the entropy production.

Key words and Phrases: Numerical entropy production, shallow water equations, smoothness indicator, finite volume method.

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Abstrak. Produksi entropi numeris dapat digunakan sebagai indikator kehalusan fungsi penyelesaian hukum-hukum kekekalan. Berdasarkan definisi, produksi entropi adalah tak-positif. Namun demikian, menggunakan suatu kerangka metode volume hingga beberapa peneliti menunjukkan bahwa produksi entropi numeris bisa saja bernilai positif pada titik-titik tertentu untuk hukum-hukum kekekalan (tidak ada suku sumber dalam persamaan yang diselesaikan). Perlu dicatat bahwa persamaan air dangkal satu setengah dimensi yang tidak melibatkan suku sumber merupakan hukum kekekalan. Sebuah laporan tentang perilaku produksi entropi numeris untuk persamaan air dangkal satu setengah dimensi tanpa suku sumber telah diterbitkan. Hasil utama laporan tersebut adalah nilai positif produksi entropi numeris dapat dihindari menggunakan flux entropi termodifikasi yang memenuhi suatu pertidaksamaan entropi numeris diskret. Dalam artikel saat ini, dipandang suatu masalah yang lebih umum, yaitu persamaan air dangkal satu setengah dimensi yang melibatkan topografi. Topografi ini adalah suatu suku sumber dalam sistem persamaan yang dipandang. Hasil dalam artikel ini menegaskan bahwa suatu flux entropi termodifikasi yang memenuhi suatu pertidaksamaan entropi numeris diskret memang sungguh-sungguh diperlukan untuk menghindari timbulnya produksi entropi yang bernilai positif.

Kata kunci: Produksi entropi numeris, persamaan air dangkal, indikator kehalusan, metode volume hingga.

1. Introduction

Water flows are governed by a mathematical model, the system of shallow water equations (SWE). These equations admit discontinuous solutions, even when a smooth initial condition is given. Therefore, solutions to the SWE can be smooth and/or rough over a given spatial domain. This behaviour mimics physical phenomena, that is, water waves can be smooth and/or rough.

An indicator for the smoothness of solutions to the SWE is useful. For example, a smoothness indicator is always needed for an adaptive numerical method used to solve the SWE. The smoothness indicator detects which regions of the domain are smooth and which ones are rough. With this detection the adaptive technique takes some actions such that accurate solutions are obtained.

The numerical entropy production (NEP) is a well-known smoothness indicator for solutions to the SWE [6]. It is the local truncation error of the entropy. Its absolute values at rough regions is larger than its absolute values at smooth regions [10, 11]. On smooth regions, its values are zero analytically. This property makes the NEP able to detect the smoothness of solutions to the SWE. NEP has been successfully implemented in gas dynamics and other conservation laws by a number of authors, such as Ersoy et al. [2], Golay [3] and Puppo [12, 13]. NEP can also be used to investigate the accuracy of numerical methods [14]. Alternative types of smoothness indicators were discussed in [8, 9].

This article studies the numerical entropy production as a smoothness indicator of solutions to the one-and-a-half-dimensional shallow water equations (1.5D

SWE) involving topography. The 1.5D SWE is the one-dimensional SWE with an additional equation regarding a passive tracer or transverse velocity. The study follows from our previous results [7] on the homogeneuous 1.5D SWE. This article extend the previous results, since a source term (topography) is added in the system of equations.

The rest of this article is organised as follows. We recall the 1.5D SWE and the numerical method used to solve the 1.5D SWE in Section 2. Numerical results are presented in Section 3. Conclusions are given in Section 4.

2. Governing Equations and Numerical Methods

In this section we recall the 1.5D SWE and the numerical method that we use to solve the 1.5D SWE.

The 1.5D SWE involving a topography source term are

$$h_t + (hu)_x = 0\,, \tag{1}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghz_x\,, \tag{2}$$

$$(hv)_t + (huv)_x = 0\,. \tag{3}$$

Here, x is the one-dimensional space variable, t is the time variable, g represents the acceleration due to gravity, $h = h(x,t)$ denotes the water height, $u = u(x,t)$ denotes the water velocity in the x-direction, and $v(x,t)$ is the transverse velocity or the concentration of the passive tracer. The mass or water height h , momentum hu , and tracer-mass or transverse momentum hv are conserved.

According to Bouchut [1] the entropy inequality for (1)–(3) is

$$\eta_t + \psi_x \le 0\,, \tag{4}$$

where

$$\eta\left(\mathbf{q}(x,t)\right) = \frac{1}{2}h(u^2+v^2) + \frac{1}{2}gh^2 + ghz\,, \tag{5}$$

$$\psi\left(\mathbf{q}(x,t)\right) = \left(\frac{1}{2}h(u^2+v^2) + gh^2\right)u + ghzu\,, \tag{6}$$

are the entropy and the entropy flux respectively. The variable $\mathbf{q}(x,t) = [h\ hu\ hv]^T$ is the vector of conserved quantities of the 1.5D SWE. Note that the entropy inequality (4) is understood in the weak sense [4, 5].

To solve the 1.5D SWE we use the same method described in our previous work [7]. In this article we focus on the numerical method with a double-sided (local Lax-Friedrichs) flux for the mass and momentum evolutions, a single-sided (upwind) flux for the tracer-mass evolution, and the modified entropy flux for the entropy evolution. This was called Combination C in our previous work [7]. From our previous work [7] we have obtained that the use of a double-sided (local Lax-Friedrichs) stencil flux for the mass and momentum together with a single-sided stencil (upwind) flux for the tracer-mass results in a more accurate solution than the use of double-sided stencil fluxes for all conserved quantity. In addition, an entropy flux satisfying a discrete numerical entropy inequality leads to no positive overshoots of the NEP.

The 1.5D SWE form balance laws

$$q_t + f(q)_x = s\,, \tag{7}$$

where q is the analytical quantity and f is the analytical flux function. We solve the considered equations using a first order finite volume method

$$Q_j^{n+1} = Q_j^n - \lambda\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right) + S_j^n\,. \tag{8}$$

Here numerical fluxes $F_{j+\frac{1}{2}}$ and $F_{j-\frac{1}{2}}$ of the conserved quantities are computed in such a way that the method is stable with $\lambda = \Delta t/\Delta x$ is the mesh ratio. Variable Δt is the time step and Δx is the uniform cell-width. Variable Q_j^n approximates the average of the exact analytical quantity $q_j\,(x, t^n)$ in the jth cell (position $x = x_j$) at the nth time step (time $t = t^n$). In addition, variable S_j^n approximates s at position $x = x_j$ at time $t = t^n$. Note that $t^n = n\Delta t$ and $x_i = i\Delta x$, where n and i are finite nonnegative integers. More detailed explanation about finite volume methods can be found in a textbook of LeVeque [5].

The modified entropy flux, which is the focus of this article, is [1]

$$\Psi_{j+\frac{1}{2}}^n = \Psi_{j+\frac{1}{2}}^{n,1D} + \Psi_{j+\frac{1}{2}}^{n,hv}\,. \tag{9}$$

Variable $\Psi_{j+\frac{1}{2}}^{n,1D}$ is a numerical entropy flux Ψ of the 1D SWE part at position $x = x_{j+\frac{1}{2}}$ and time $t = t^n$. We compute $\Psi_{j+\frac{1}{2}}^{n,1D}$ using the Lax-Friedrichs entropy flux. Variable $\Psi_{j+\frac{1}{2}}^{n,hv}$ is a numerical entropy flux Ψ of the tracer (hv) at position $x = x_{j+\frac{1}{2}}$ and time $t = t^n$. We compute $\Psi_{j+\frac{1}{2}}^{n,hv}$ using the upwind entropy flux

$$\Psi_{j+\frac{1}{2}}^{n,hv} = \begin{cases} \frac{1}{2}F_{j+\frac{1}{2}}^{n,h}\,v_j^2 & \text{if} \quad F_{j+\frac{1}{2}}^{n,h} \geq 0\,, \\ \\ \frac{1}{2}F_{j+\frac{1}{2}}^{n,h}\,v_{j+1}^2 & \text{otherwise}\,. \end{cases} \tag{10}$$

Variable $F_{j+\frac{1}{2}}^{n,h}$ is the numerical flux F for the h quantity at position $x = x_{j+\frac{1}{2}}$ and time $t = t^n$. Note that if we simply used $\Psi_{j+\frac{1}{2}}^n = \Psi_{j+\frac{1}{2}}^{n,1.5D}$ instead of the modified entropy flux (9), then we could have positive overshoots of the NEP, as demonstrated in our previous work [7].

3. NUMERICAL RESULTS

In this section we present our numerical results using a test case to demonstrate that the modified entropy flux (9) indeed leads to no positive overshoots of the NEP. In the simulation we use SI units for all quantities. Therefore we omit the units as they are already clear.

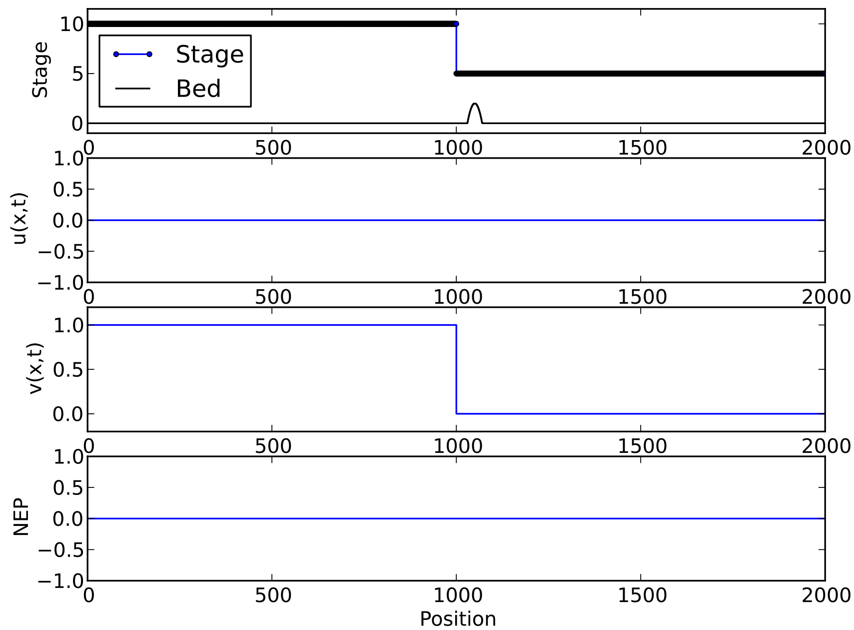

FIGURE 1. Initial condition of the dam-break problem.

As the test case we set up a dam break problem on a non-flat topography involving a passive tracer, as follows. The non-flat topography with $x \in [0, 2000]$ is

$$z(x) = \begin{cases} 2 - 0.005\,(x-1050)^2 & \text{if} \quad 1030 \le x \le 1070, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The initial condition is

$$u(x,0) = 0, \tag{12}$$

$$v(x,0) = \begin{cases} 1 & \text{if} \quad 0 < x < 1000, \\ 0 & \text{if} \quad 1000 < x < 2000, \end{cases} \tag{13}$$

$$w(x,0) = \begin{cases} 10 & \text{if} \quad 0 < x < 1000, \\ 5 & \text{if} \quad 1000 < x < 2000. \end{cases} \tag{14}$$

Here $w(x,t) := h(x,t) + z(x)$ is the absolute water level and is called the stage. The initial condition of this dam-break problem is shown in Figure 1.

Figures 2 and 3 show the simulation results at time $t = 30$ and $t = 90$, respectively, using the first order finite volume method described in Section 2. The implemented CFL number is 1.0. The acceleration due to gravity is $g = 9.81$. Here the spatial domain is descretised into 1600 cells uniformly. In each figure, the first subfigure contains the free water surface (stage) and bed topography. The second

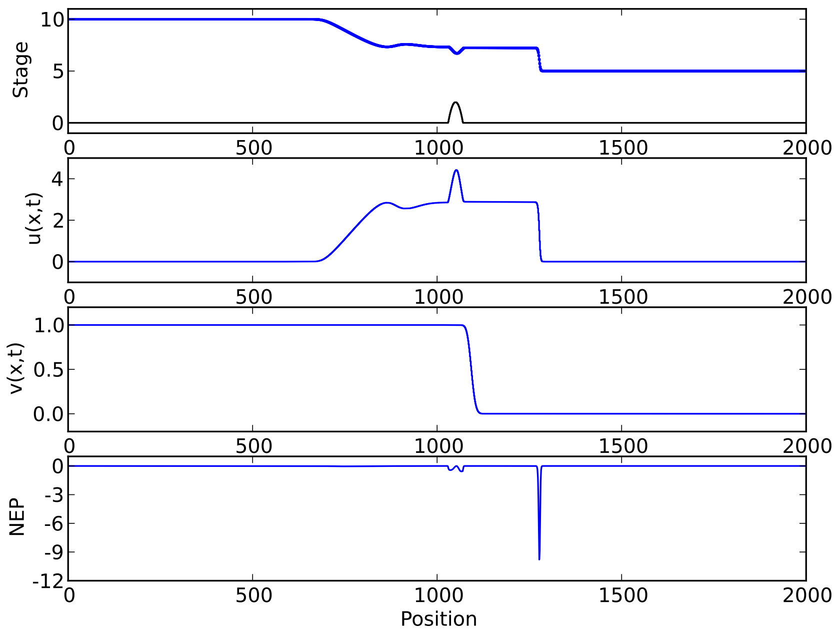

FIGURE 2. Dam-break results on a non-flat topography at $t = 30$ using a first order method and 1600 cells.

and third subfigures are the results for velocity $u(x, t)$ and tracer concentration $v(x, t)$ respectively. The fourth subfigure shows the NEP. We see that the NEP clearly detects the position of the shock discontinuity moving to the right. From these rusults, no positive overshoots of the NEP occur. This confirms that the modified entropy flux satisfying a discrete numerical entropy inequality is able to overcome positive overshoots of the entropy production.

Now we present our results on the behaviour of the NEP in relation to the mesh ratio $\lambda = \Delta t/\Delta x$ of the time step Δt and the cell width Δx. The mesh ratio is fixed and taken as $\lambda = 0.08$. The time step Δt and the cell width Δx are varied. The maximum values of $|\text{NEP}|$ at time $t = 30$ are recorded in Table 1.

We observe that for a fixed mesh ratio the values of $\Delta t \max |\text{NEP}|$ and $\Delta x \max |\text{NEP}|$ are about constant. This behaviour suggests that the NEP can be used as an effective refinement or coarsening indicator in adaptive mesh numerical methods. Here the fixed mesh ratio is chosen in such away that the numerical method is consistent and stable.

The NEP actually measures the error introduced at every time step of the finite volume method [10]. The positions of large values of NEP are the positions

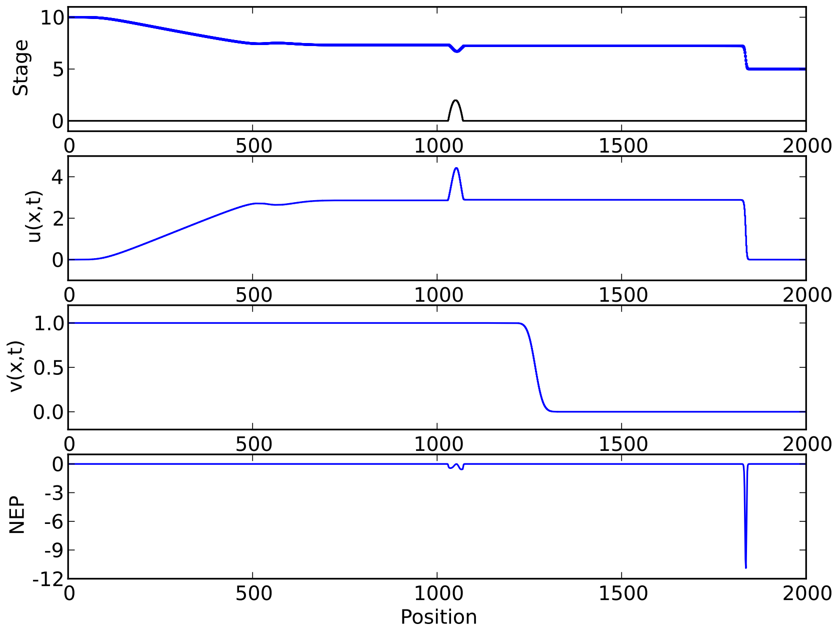

FIGURE 3. Dam-break results on a non-flat topography at $t = 90$ using a first order method and 1600 cells.

TABLE 1. The NEP in relation to a fixed value of the mesh ratio $\lambda = 0.08$, but varying Δt and varying Δx.

Δt	Δx	$\max\lvert\text{NEP}\rvert$	$\Delta t \max\lvert\text{NEP}\rvert$	$\Delta x \max\lvert\text{NEP}\rvert$
0.8	10	1.502	1.201	15.018
0.4	5	3.027	1.211	15.135
0.2	2.5	5.645	1.129	14.113
0.1	1.25	12.410	1.241	15.513
0.05	0.625	24.605	1.230	15.378

of the most dissipative regions in the space domain. The entropy is the energy dispersal of the flow. Therefore, the positions or regions with large values of NEP are the positions where some physical energy has lost due to flow mechanism. If an adaptive method were used, the method should have taken action at these dissipative regions, so that better accuracy of the solution to the SWE could be obtained. However, the topic of adaptive method is beyond the scope of this article.

4. Concluding Remarks

We have presented a study of the numerical entropy production as a smoothness indicator for the one-and-a-half-dimensional shallow water equations involving topography. In our study, no positive overshoots of the NEP occur and this is correct by definition of the numerical entropy production. To solve the one-and-a-half-dimensional shallow water equations, we recommend the use of the following flux combinations: (1) a double-sided (such as, local Lax-Friedrichs) stencil flux for the mass and momentum, (2) a single-sided stencil (such as, upwind) flux for the tracer-mass, and (3) an entropy flux defined in such a way that a discrete numerical entropy inequality is satisfied. In we only need to find the conserved quantities, then only fluxes (1) and (2) should be used. These results are promising to be extended to higher dimensional problems.

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