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THE GOODNESS OF LONG PATH WITH RESPECT TO MULTIPLE COPIES OF COMPLETE GRAPHS

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Abstract. Let H be a graph with the chromatic number $\chi(H)$ and the chromatic surplus $s(H)$. A connected graph G of order n is called *good* with respect to H, H-*good*, if $R(G,H) = (n-1)(\chi(H)-1) + s(H)$. The notation tK_m represents a graph with t identical copies of complete graphs on m vertices, K_m. In this note, we discuss the goodness of path P_n with respect to tK_m. It is obtained that the path P_n is tK_m-good for $m, t \geq 2$ and sufficiently large n. Furthermore, it is also obtained the Ramsey number $R(G, tK_m)$, where G is a disjoint union of paths.

Key words and Phrases: (G,H)-free, H-good, complete graph, path, Ramsey number.

Abstrak. Notasi H menyatakan graf dengan bilangan kromatik $\chi(H)$ dan surplus kromatik $s(H)$. Graf G yang memiliki n titik disebut *elok* terhadap H, H-*elok*, jika $R(G,H) = (n-1)(\chi(H)-1)+s(H)$. Notasi tK_m merepresentasikan t rangkap graf lengkap identik dengan m titik, K_m. Dalam makalah ini dapat ditunjukkan bahwa graf lintasan P_n adalah tK_m-elok untuk semua $m, t \geq 2$ dan n cukup besar. Menggunakan sifat elok tersebut hasil lebih jauh juga diperoleh, yaitu bilangan Ramsey $R(G, tK_m)$ dapat ditentukan jika G adalah gabungan graf lintasan sebarang.

Kata kunci: (G,H)-kritis, H-elok, graf lengkap, lintasan, bilangan Ramsey.

1. Introduction

All graphs in this paper are finite, undirected and simple. Let G and H be two graphs, where H is a subgraph of G, we define $G - H$ as a graph obtained from G by deleting the vertices of H and all edges incident to them. Let t be a

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natural number and G_i be a connected graph with the vertex set V_i and the edge set E_i for every $i = 1, 2, ..., t$. *The disjoint union of graphs*, $\bigcup_{i=1}^{t} G_i$, has the vertex set $\bigcup_{i=1}^{t} V_i$ and the edge set $\bigcup_{i=1}^{t} E_i$. Furthermore, if each G_i is isomorphic to a connected graph G then we denote by tG the disjoint union of t copies of G.

For graphs G and H, the *Ramsey number* $R(G, H)$ is the minimum n such that in every coloring of the edges of the complete graph K_n with two colors, say red and blue, there is a red copy of G or a blue copy of H. A graph F is called (G, H)-*free* if F contains no subgraph isomorphic to G and its complement $\overline{F}$ contains no subgraph isomorphic to H. The Ramsey number $R(G, H)$ can be equivalently defined as the smallest natural number n such that no (G, H)-free graph on n vertices exists.

Determining $R(G, H)$ is a notoriously hard problem. Burr [4] showed that the problem of determining whether $R(G, H) \leq n$ for a given n is NP-hard. Furthermore in Shaeffer [8] one can find a rare natural example of a problem higher than NP-hard in the polynomial hierarchy of computational complexity theory, that is, Ramsey arrowing is $\prod_2^p$-complete. The few known values of $R(G, H)$ are collected in the dynamic survey of Radziszowski [7].

Burr [3] proved the general lower bound

$$R(G, H) \geq (n-1)(\chi(H) - 1) + s(H), \qquad (1)$$

where G is a connected graph of order n, $\chi(H)$ denotes the chromatic number of H and $s(H)$ is its *chromatic surplus*, namely, the minimum cardinality of a color class taken over all proper colorings of H with $\chi(H)$ colors. Motivated by this inequality, the graph G is said to be H-*good* if equality holds in (1). Chvátal [5] proved that trees are K_m-good graphs. Sudarsana et al. [10] showed that path is a good graph with respect to $2K_m$, and P_n is also tW_4-good in [12]. Other result concerning the goodness of graphs with the chromatic surplus one can be found in Lin et al. [6]. However, the goodness of path P_n with respect to tK_m for $t \geq 2$ is still open. In this paper, we establish that P_n is tK_m-good for $t \geq 2$ and sufficiently large n.

2. Known Results

For the proof of our new result, Theorem 3.1, we use the following results.

Theorem 2.1 (Chvátal [5])**.** *Let $n, m \geq 2$ be integers and T_n is a tree of order n. Then, $R(T_n, K_m) = (n-1)(m-1) + 1$.*

Note that the chromatic surplus of K_m, $s(K_m)$, is equal to one and path P_n is a tree of order n. Therefore, $R(P_n, K_m) = (n-1)(m-1) + 1$.

Theorem 2.2 (Sudarsana et al. [10])**.** *Let $m \geq 2$ and $n \geq 3$ be integers. Then, $R(P_n, 2K_m) = (n-1)(m-1) + 2$.*

Lemma 2.3 (Sudarsana et al. [10])**.** *Let n and t be positive integers. Then,*

$$R(P_n, tK_2) = \begin{cases} n + t - 1, & t \leq \lfloor \frac{n}{2} \rfloor; \\ 2t + \lceil \frac{n}{2} \rceil - 1, & t > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

3. The Main Result

The following theorem deals with the goodness of path P_n with respect to t identical copies of complete graphs, tK_m.

Theorem 3.1. *Let $m, t \geq 2$ be integers and $g(t,m) = (t-2)((tm-2)(m-1)+1)+3$. If $n \geq g(t,m)$ then $R(P_n, tK_m) = (n-1)(m-1)+t$.*

Proof of Theorem 3.1: The lower bound $R(P_n, tK_m) \geq (n-1)(m-1)+t$ follows from the fact that $(m-1)K_{n-1} \cup K_{t-1}$ is a (P_n, tK_m)–free graph of order $(n-1)(m-1)+t-1$.

To prove the upper bound $R(C_n, tK_m) \leq (n-1)(m-1)+t$ we use inductions on t and m. For $t = 2$, we have $g(2,m) = 3$ and therefore Theorem 2.2 implies that $R(P_n, 2K_m) = (n-1)(m-1)+2$ for $n \geq g(2,m) = 3$. Hence, the assertion holds for $n \geq g(2,m) = 3$. Assume that the theorem is true for $n \geq g(t-1,m)$, that is $R(P_n, (t-1)K_m) \leq (n-1)(m-1)+t-1$.

From Lemma 2.3, we have $R(P_n, tK_2) = n+t-1$ for $n \geq 2t$. Note that if $t \geq 2$ then $n \geq g(t,2) > 2t$. Therefore, the theorem holds for $m = 2$. Assume that $m \geq 3$ and the theorem is true for $n \geq g(t,m-1)$, that is $R(P_n, tK_{m-1}) \leq (n-1)(m-2)+t$.

Now we will show that the theorem is also valid for $n \geq g(t,m)$. Let F be an arbitrary graph on $(n-1)(m-1)+t$ vertices. We shall show that F contains P_n or $\overline{F}$ contains tK_m. Note that Theorem 2.1 guarantees that F contains P_n or $\overline{F}$ contains K_m. If F contains P_n then we are done. Thus we may assume that $\overline{F}$ contains K_m. Since the subgraph $F - \overline{K}_m$ of F has $(n-2)(m-1)+t-1$ vertices and $n-1 \geq g(t,m)-1 > g(t-1,m)$, by the induction hypothesis on t we know that $F - \overline{K}_m$ contains P_{n-1} or the complement of $F - \overline{K}_m$ contains $(t-1)K_m$. If the complement of $F - \overline{K}_m$ contains $(t-1)K_m$ then by companying with the first ones we have a tK_m in $\overline{F}$ and hence the proof is done. Thus, F has a path P_{n-1}. Therefore, the subgraph $F - P_{n-1}$ of F has $(n-1)(m-2)+t$ vertices. Note that $n \geq g(t,m) > g(t,m-1)$. By the induction hypothesis on m, we know that $F - P_{n-1}$ contains P_n or the complement of $F - P_{n-1}$ contains tK_{m-1}. If $F - P_{n-1}$ contains P_n then we are done. Hence we may assume that F contains a path P_{n-1} with vertex set, say $p_1, p_2, \ldots, p_{n-1}$ and edges $p_i p_{i+1}$ (subscripts modulo $(n-1)$), and that $\overline{F}$ contains t disjoint copies $K^1_{m-1}, K^2_{m-1}, \ldots, K^t_{m-1}$ of the complete graph with $m-1$ vertices. It is clear that the subgraphs P_{n-1} and tK_{m-1} have no vertices in common.

Assume that F contains no P_n. We will show that $\overline{F}$ contains tK_m. Thus, the end vertices p_1 and p_{n-1} of path P_{n-1} must not be adjacent to any vertices in $K^1_{m-1}, K^2_{m-1}, \ldots, K^t_{m-1}$. Therefore, the set $D = \{\{p_1\} \cup V(K^1_{m-1})\} \cup \{\{p_{n-1}\} \cup V(K^2_{m-1})\}$ forms a $2K_m$ in $\overline{F}$. Let us now consider the relation between the vertices in $A' = \{p_2, p_3, \ldots, p_{n-2}\}$ and in $B' = V(K^3_{m-1}) \cup V(K^4_{m-1}) \cup \ldots \cup V(K^t_{m-1})$.

Since there is no P_n in F, it follows that every two consecutive vertices p_i, p_{i+1} in A' can not be adjacent to any vertices in B' for every $i \in \{2, 3, \ldots, n-2\}$. Suppose that the neighborhood $N_{A'}(u)$ in A' of a vertex $u \in B'$ satisfies $|N_{A'}(u) \cap V(P_{n-1})| \geq tm-1$. Let $p_i, p_j \in N_{A'}(u) \cap V(P_{n-1})$ with $i < j$. Note that $j-i > 1$ since otherwise we can extend P_{n-1} to a path of order n containing u. If $p_{i+1}p_{j+1}$ is an edge in F then we also have a new path $\{p_1p_2....p_iup_jp_{j-1}p_{j-2}...p_{i+1}p_{j+1}p_{j+2}....p_{n-1}\}$ of length $n-1$ in F. If $p_{i+1}p_{j+1}$ is not an edge for every pair $p_i, p_j \in N_{A'}(u) \cap V(P_{n-1})$ then $\{p_{i+1} : p_i \in N_{A'}(u) \cap V(P_{n-1})\} \cup \{u\}$ is a set of tm independent vertices in F and we obtain that $\overline{F}$ contains tK_m. Hence, for each $u \in B'$ we have $|N_{A'}(u) \cap V(P_{n-1})| \leq tm-2$. Therefore,

$$\left| A \setminus \bigcup_{u \in B'} N_{A'}(u) \right| \geq n-3-(t-2)(tm-2)(m-1). \tag{2}$$

Since $n \geq g(t,m)$, it follows that there are at least $t-2$ vertices in A' which are adjacent to no vertex in B' and hence together with D we have that $\overline{F}$ contains tK_m. This concludes the proof of Theorem 3.1. □

By extending previous results of Baskoro et al. [1] and Stahl [9], Bielak [2] and Sudarsana et al. [11] independently proved a formula for $R(G,H)$ when every connected component of G is an H-good graph. This result motivates the study of general families of H-good graphs. In particular, Theorem 3.1 provides the following computation of $R(G,tK_m)$, if G is a set of disjoint paths (linear forest).

Corollary 3.2. *Let $m, t \geq 2$ be integers and $g(t,m) = (t-2)((tm-2)(m-1)+1)+3$. Let $G \simeq \bigcup_{i=1}^{k} l_iP_{n_i}$, where $l_i \geq 1$ and each P_{n_i} is a path of order n_i.*

If $n_1 \geq n_2 \geq ... \geq n_k \geq g(t,m)$ then

$$R(G,tK_m) = \max_{1 \leq i \leq k} \left\{ (n_i-1)(m-2) + \sum_{j=1}^{i} l_jn_j \right\} + t - 1. \tag{3}$$

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