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SUPER EAT LABELINGS OF SUBDIVIDED STARS

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Abstract. Kotzig and Rosa (1970) conjectured that every tree admits edge-magic total labeling. Enomoto et al. (1998) proposed the conjecture that every tree is super edge-magic total. In this paper, we describe super (a, d)-edge-antimagic total labelings on a subclass of the subdivided stars denoted by $T(n, n, n, n, n_5, n_6..., n_r)$ for $d \in \{0, 1, 2\}$, where $n \geq 3$ odd, $r \geq 5$ and $n_m = 2^{m-4}(n-1)+1$ for $5 \leq m \leq r$.

Key words: Super (a, d)-EAT labelings, subdivision of stars.

Abstrak. Kotzig dan Rosa (1970) telah membuat konjektur bahwa setiap *tree* dapat menghasilkan *edge-magic total labeling*. Enomoto et al. (1998) telah membuat konjektur bahwa setiap *tree* adalah *super edge-magic total*. Di dalam makalah ini, kami menjelaskan *super* (a, d)*-edge-antimagic total labeling* pada sebuah sub-kelas dari *star* yang terbagi yang dinyatakan oleh $T(n, n, n, n, n_5, n_6..., n_r)$ untuk $d \in \{0, 1, 2\}$, dimana $n \geq 3$ ganjil, $r \geq 5$ dan $n_m = 2^{m-4}(n-1)+1$ untuk $5 \leq m \leq r$.

Kata kunci: *Super* (a, d)*-EAT labelings*, pembagian *stars*.

1. INTRODUCTION

All graphs in this paper are finite, undirected and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. A (v, e)-graph G is a graph such that $|V(G)| = v$ and $|E(G)| = e$. A general reference for graph-theoretic ideas can be seen in [29]. A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called *a total labeling*. Some labelings use the vertex-set only or the

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edge-set only and we shall call them *vertex-labelings* or *edge-labelings*, respectively. A number of classification studies on edge antimagic total graphs has been intensively investigated. For further studies on antimagic labelings, reader can see [13, 5].

Definition 1.1 A (s, d)-*edge-antimagic vertex* ((s, d)-EAV) labeling of a graph G is a bijective function $\lambda : V(G) \to \{1, 2, \dots, v\}$ such that the set of edge-sums of all edges in G, $\{w(xy) = \lambda(x) + \lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{s, s + d, s + 2d, \dots, s + (e - 1)d\}$, where $s > 0$ and $d \geq 0$ are two fixed integers.

Definition 1.2. An (a, d)-*edge-antimagic total* ((a, d)-EAT) labeling of a graph G is a bijective function $\lambda : V(G) \cup E(G) \to \{1, 2, \dots, v + e\}$ such that the set of edge-weights of all edges in G, $\{w(xy) = \lambda(x) + \lambda(xy) + \lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{a, a + d, a + 2d, \dots, a + (e - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a, d)-*EAT graph*. Additionally, if $\lambda(V(G)) = \{1, 2, ..., v\}$ then λ is called a super (a, d)-edge-antimagic total (super (a, d)-EAT) labeling and G becomes a super (a, d)-EAT graph.

In the above definition, if $d = 0$ then $(a, 0)$-EAT labeling is called edge-magic total (EMT) labeling and super $(a, 0)$-EAT labeling is called super edge-magic total (SEMT) labeling. The subject of edge-magic total (EMT) labeling of graphs has its origin in the works of Kotzig and Rosa [20, 21] on what they called magic valuations of graphs. The definition of (a, d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [27] as a natural extension of EMT labeling defined by Kotzig and Rosa. A super (a, d)-EAT labeling is a natural extension of the notion of SEMT labeling defined by Enomoto, Lladó, Nakamigawa and Ringel in [9]. Moreover, they proposed the following conjecture:

Conjecture 1.1 *Every tree admits SEMT labeling* [9].

In the favour of this conjecture, many authors have proved the existence of SEMT labelings for various particular classes of trees for examples [1-8, 10-12, 14-17, 24, 25, 28, 29]. Lee and Shah [22] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still open. Bača et al. investigated the following relationship between (s, d)-EAV labeling and (a, d)-EAT labeling [3]:

Proposition 1.1. If a (v, e)-graph G has a (s, d)-EAV labeling then G admits

(i) a super $(s + v + 1, d + 1)$-EAT labeling,
(ii) a super $(s + v + e, d - 1)$-EAT labeling. ■

The notion of dual labeling has been introduced by Wallis [30]. The next lemma follows from the principal of duality, which is first studied by Baskoro [8].

Lemma 1.1 If g is a super edge-magic total labeling of G with the magic constant c, then the function $g_1 : V(G) \cup E(G) \to \{1, 2, ..., v + e\}$ defined by

$$g_1(x) = \begin{cases} v + 1 - g(x), & \text{for} \quad x \in V(G), \\ \\ 2v + e + 1 - g(x), & \text{for} \quad x \in E(G), \end{cases}$$

is also a super-magic total labeling of G with the magic constant $c_1 = 4v+e+3-c$. ■

Definition 1.3 For $n_i \geq 1$ and $r \geq 2$, let $G \cong T(n_1, n_2, ..., n_r)$ be a graph obtained by inserting $n_i - 1$ vertices to each of the i^{th} edge of the star $K_{1,r}$, where $1 \leq i \leq r$. Thus, the graph $T\underbrace{(1, 1, ..., 1)}_{r-times}$ is a star $K_{1,r}$.

Subdivided stars form a particular class of trees and many authors have proved the antimagicness for various subclasses of subdivided stars as follows:

- Lu [23, 24] has called the subdivided star $T(m, n, k)$ as a three-path tree. Moreover, he has proved that it is a SEMT graph if n and m are odd with $k = n + 1$ or $k = n + 2$.
- Ngurah et al. [25] have proved that $T(m, n, k)$ is a SEMT graph if n and m are odd with $k = n + 3$ or $k = n + 4$.
- In [26], Salman et al. have found the results related to SEMT labelings on the subdivision of stars S_n^m for $m = 1, 2$, where $S_n^1 \cong T\underbrace{(2, 2, ..., 2)}_{n-times}$ and $S_n^2 \cong T\underbrace{(3, 3, ..., 3)}_{n-times}$.
- In [16], Javaid et al. have formulated SEMT labelings on the subdivision of star $K_{1,4}$ and w-trees.
- Javaid and Akhlaq [17] have proved that the subdivided stars $T(n, n, n + 2, n + 2, n_5, ..., n_p)$ admit super (a, d)-EAT labelings, where $n \geq 3$ is odd, $r \geq 5$ and $n_m = 1 + (n + 1)2^{m-4}$ for $5 \leq m \leq r$.

However, the problem to find super (a, d)-EAT labelings on $T(n_1, n_2, n_3, ..., n_r)$ for different $\{n_i : 1 \leq i \leq r\}$ is still open. In this paper, for $d \in \{0, 1, 2\}$, we find super (a, d)-EAT labelings on the subdivided stars $T(n, n, n, n, n_5, n_6, ..., n_r)$, where $n \geq 3$ is odd, $r \geq 5$ and $n_m = 2^{m-4}(n - 1) + 1$ for $5 \leq m \leq r$.

2. BOUNDS OF MAGIC CONSTANT

In this section, we present different lemmas related to lower and upper bounds of the magic constant a for various subclasses of subdivided stars.

Ngurah et al. [25] found the following lower and upper bounds of the magic constant a for a particular subclass of the subdivided stars denoted by $T(m,n,k)$, which is given below:

Lemma 2.1. If $T(m,n,k)$ is a super $(a,0)$-EAT graph, then $\frac{1}{2l}(5l^2+3l+6) \leq a \leq \frac{1}{2l}(5l^2+11l-6)$, where $l = m+n+k$.

The lower and upper bounds of the magic constant a for a particular subclass of the subdivided stats $T\underbrace{(n,n,...,n)}_{n-times}$ are established by Salman et al. [26] as follows:

Lemma 2.2. If $T\underbrace{(n,n,...,n)}_{n-times}$ is a super $(a,0)$-EAT graph, then $\frac{1}{2l}(5l^2+(9-2n)l+n^2-n) \leq a \leq \frac{1}{2l}(5l^2+(2n+5)l+n-n^2)$, where $l = n^2$. ■

Javaid [19] has proved lower and upper bounds of the magic constant a for the most extended subclasses of the subdivided stars denoted by $T(n_1,n_2,n_3,...,n_r)$ with any $n_i \geq 1$ for $1 \leq i \leq r$, which is presented in the following lemma:

Lemma 2.3. If $T(n_1,n_2,n_3,...,n_r)$ is a super $(a,0)$-EAT graph, then $\frac{1}{2l}(5l^2+r^2-2lr+9l-r) \leq a \leq \frac{1}{2l}(5l^2-r^2+2lr+5l+r)$, where $l = \sum\limits_{i=1}^{r} n_i$. ■

3. SUPER (a,d)-EAT LABELINGS OF SUBDIVIDED STARS

In this section, we prove the main results related to super (a,d)-EAT labelings on a particular subclass of the subdivided stars for different values of the parameter d.

Theorem 2.1. For any odd $n \geq 3$, $G \cong T(n,n,n,n,2n-1)$ admits a super $(a,0)$-EAT labeling with $a = 2v+s-1$ and a super $(\acute{a},2)$-EAT labeling with $\acute{a} = v+s+1$, where $v = |V(G)|$ and $s = 3n+4$.

Proof. Let us denote the vertices and edges of G, as follows:
$V(G) = \{c\} \cup \{x_i^{l_i} \mid 1 \leq i \leq 5\ ;\ 1 \leq l_i \leq n_i\}$, $E(G) = \{c\, x_i^1 \mid 1 \leq i \leq 5\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \leq i \leq 5\ ;\ 1 \leq l_i \leq n_i - 1\}$. If $v = |V(G)|$ and $e = |E(G)|$ then $v = 6n$, and $e = 6n-1$. Now, we define the labeling $\lambda : V(G) \to \{1,2,...,v\}$ as follows:

$$\lambda(c) = 4n+2.$$

For $\quad 1 \le l_i \le n_i$ odd;

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+2) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (n+1) + \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}, \\ (2n+3) - \frac{l_4+1}{2}, & \text{for } u = x_4^{l_4} \\ (3n+3) - \frac{l_5+1}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

For $\quad 2 \le l_i \le n_i - 1$ even;

$$\lambda(u) = \begin{cases} (3n+2) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (4n+2) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (4n+2) + \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (5n+2) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (6n+1) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}. \end{cases}$$

The set of all edge-sums generated by the above formulas forms a consecutive integer sequence $s = 3n+4, 3n+5, \cdots, 3n+3+e$. Therefore, by Proposition 1.1, λ can be extended to a super $(a,0)$-EAT labeling with magic constant $a = 2v+s-1 = 15n+3$ and to a super $(\acute{a},2)$-EAT labeling with minimum edge-weight $\acute{a} = v+1+s = 9n+5$. ■

Theorem 2.2. For any odd $n \ge 3$, $G \cong T(n,n,n,n,2n-1)$ admits a super $(a,1)$-EAT labeling with $a = s + \frac{3v}{2}$, where $v = |V(G)|$ and $s = 3n+4$.

Proof. Let us consider the vertex and edge set of G and the labeling $\lambda : V(G) \to \{1,2,...,v\}$ by the same manner as in Theorem 2.1. It follows that edge-sums of all the edges of G constitute an arithmetic sequence $3n+4, 3n+5, \cdots, 3n+3+e$, with common difference 1. We denote it by $A = \{a_i; 1 \le i \le e\}$. Now to show that λ is an $(a,1)$-EAT labeling of G, define the set of edge-labels as $B = \{b_j = v+j \ ; \ 1 \le j \le e\}$. The set of edge-weights can be obtained as $C = \{a_{2i-1} + b_{e-i+1} \ ; \ 1 \le i \le \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1} \ ; \ 1 \le j \le \frac{e+1}{2} - 1\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and $a = s + \frac{3v}{2} = 12n+4$. Since all vertices receive the smallest labels, λ is a super $(a,1)$-EAT labeling. ■

Theorem 2.3. For any odd $n \ge 3$, $G \cong T(n,n,n,n,2n-1,4n-3)$ admits a super $(a,0)$-EAT labeling with $a = 2v+s-1$ and a super $(\acute{a},2)$-EAT labeling with $\acute{a} = v+s+1$, where $v = |V(G)|$ and $s = 5n+3$.

Proof. Let us denote the vertices and edges of G, as follows:
$V(G) = \{c\} \cup \{x_i^{l_i} \mid 1 \le i \le 6\ ;\ 1 \le l_i \le n_i\}$, $E(G) = \{c\, x_i^1 \mid 1 \le i \le 6\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \le i \le 6\ ;\ 1 \le l_i \le n_i - 1\}$. If $v = |V(G)|$ and $e = |E(G)|$ then $v = 10n-3$, and $e = 10n-4$. Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c) = 6n + 1.$$

For $\quad 1 \le l_i \le n_i$ odd;

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+2) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (n+1) + \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}, \\ (2n+3) - \frac{l_4+1}{2}, & \text{for } u = x_4^{l_4} \\ (3n+3) - \frac{l_5+1}{2}, & \text{for } u = x_5^{l_5}, \\ (5n+2) - \frac{l_6+1}{2}, & \text{for } u = x_6^{l_6}. \end{cases}$$

For $\quad 2 \le l_i \le n_i - 1$ even;

$$\lambda(u) = \begin{cases} (5n+1) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (6n+1) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (6n+1) + \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (7n+1) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ 8n - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}, \\ (10n-2) - \frac{l_6}{2}, & \text{for } u = x_6^{l_6}. \end{cases}$$

The set of all edge-sums generated by the above formulas forms a consecutive integer sequence $s = 5n+3, 5n+4, \cdots, 5n+2+e$. Therefore, by Proposition 1.1, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = 2v + s - 1 = 25n - 4$ and to a super $(\acute{a}, 2)$-EAT labeling with minimum edge-weight $\acute{a} = v + 1 + s = 15n + 1$. ■

Theorem 2.4. For any odd $n \ge 3$, $G \cong T(n, n, n, n, 2n-1, 4n-3, 8n-7)$ admits a super $(a, 0)$-EAT labeling with $a = 2v + s - 1$ and a super $(\acute{a}, 2)$-EAT labeling with $\acute{a} = v + s + 1$, where $v = |V(G)|$ and $s = 9n$.

Proof. Let us denote the vertices and edges of G, as follows:
$V(G) = \{c\} \cup \{x_i^{l_i} \mid 1 \le i \le 7\ ;\ 1 \le l_i \le n_i\}$, $E(G) = \{c\, x_i^1 \mid 1 \le i \le 7\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \le i \le 7\ ;\ 1 \le l_i \le n_i - 1\}$. If $v = |V(G)|$ and $e = |E(G)|$ then $v =$

$18n-10$, and $e=18n-11$. Now, we define the labeling $\lambda : V(G) \to \{1,2,...,v\}$ as follows:

$$\lambda(c) = 10n-2.$$

For $\quad 1 \le l_i \le n_i$ odd;

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+2) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (n+1) + \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}, \\ (2n+3) - \frac{l_4+1}{2}, & \text{for } u = x_4^{l_4} \\ (3n+3) - \frac{l_5+1}{2}, & \text{for } u = x_5^{l_5}, \\ (5n+2) - \frac{l_6+1}{2}, & \text{for } u = x_6^{l_6} \\ (9n-1) - \frac{l_7+1}{2}, & \text{for } u = x_7^{l_7}. \end{cases}$$

For $\quad 2 \le l_i \le n_i - 1$ even;

$$\lambda(u) = \begin{cases} (9n-2) + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (10n-2) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ (10n-2) + \frac{l_3}{2}, & \text{for } u = x_3^{l_3}, \\ (11n-2) - \frac{l_4}{2}, & \text{for } u = x_4^{l_4}, \\ (12n-3) - \frac{l_5}{2}, & \text{for } u = x_5^{l_5}, \\ (14n-5) - \frac{l_6}{2}, & \text{for } u = x_6^{l_6} \\ (18n-9) - \frac{l_7}{2}, & \text{for } u = x_7^{l_7}. \end{cases}$$

The set of all edge-sums generated by the above formulas forms a consecutive integer sequence $s = 9n, 9n+1, \cdots, 9n-1+e$. Therefore, by Proposition 1.1, λ can be extended to a super $(a,0)$-EAT labeling with magic constant $a = 2v+s-1 = 45n-21$ and to a super $(\acute{a},2)$-EAT labeling with minimum edge-weight $\acute{a} = v+1+s = 27n-9$. ∎

Theorem 2.5. For any odd $n \ge 3$, $G \cong T(n,n,n,n,2n-1,4n-3,8n-7)$ admits a super $(a,1)$-EAT labeling with $a = s + \frac{3v}{2}$, where $v = |V(G)|$ and $s = 9n$.

Proof. Let us consider the vertex and edge set of G and the labeling $\lambda : V(G) \to \{1,2,...,v\}$ by the same manner as in Theorem 2.4. It follows that edge-sums of all the edges of G constitute an arithmetic sequence $9n, 9n+1, \cdots, 9n-1+e$, with common difference 1. We denote it by $A = \{a_i; 1 \leq i \leq e\}$. Now to show that λ is an $(a, 1)$-EAT labeling of G, define the set of edge-labels as $B = \{b_j = v + j\ ;\ 1 \leq j \leq e\}$. The set of edge-weights can be obtained as $C = \{a_{2i-1} + b_{e-i+1}\ ;\ 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1}\ ;\ 1 \leq j \leq \frac{e+1}{2} - 1\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and $a = s + \frac{3v}{2} = 36n - 15$. Since all vertices receive the smallest labels, λ is a super $(a, 1)$-EAT labeling. ■

Theorem 2.6. For any $n \geq 3$ odd, $G \cong T(n, n, n, n, n_5, ..., n_r)$ admits a super $(a, 0)$-EAT labeling with $a = 2v + s - 1$ and a super $(\acute{a}, 2)$-EAT labeling with $\acute{a} = v + s + 1$ where $v = |V(G)|$, $s = (2n + 4) + \sum\limits_{m=5}^{r} [2^{m-5}(n - 1) + 1]$, $r \geq 5$ and $n_m = 2^{m-4}(n - 1) + 1$ for $5 \leq m \leq r$.

PROOF. Let us denote the vertices and edges of G, as follows: $V(G) = \{c\} \cup \{x_i^{l_i} \mid 1 \leq i \leq r\ ;\ 1 \leq l_i \leq n_i\}$, $E(G) = \{c\ x_i^1 \mid 1 \leq i \leq r\} \cup \{x_i^{l_i} x_i^{l_i+1} \mid 1 \leq i \leq r\ ;\ 1 \leq l_i \leq n_i - 1\}$. If $v = |V(G)|$ and $e = |E(G)|$ then $v = (4n + 1) + \sum\limits_{m=5}^{r} [2^{m-4}(n - 1) + 1]$ and $e = v - 1$. Now, we define the labeling $\lambda : V(G) \to \{1, 2, ..., v\}$ as follows:

$$\lambda(c) = (3n + 2) + \sum_{m=5}^{r} [2^{m-5}(n - 1) + 1].$$

For $1 \leq l_i \leq n_i$ odd, where $i = 1, 2, 3, 4$ and $5 \leq i \leq r$, we define

$$\lambda(u) = \begin{cases} \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (n + 2) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (n + 1) + \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}, \\ (2n + 3) - \frac{l_4+1}{2}, & \text{for } u = x_4^{l_4}. \end{cases}$$

$$\lambda(x_i^{l_i}) = (2n + 3) + \sum_{m=5}^{i} [2^{m-5}(n - 1) + 1] - \frac{l_i + 1}{2} \text{ respectively.}$$

Let $\alpha = (2n + 2) + \sum\limits_{m=5}^{r} [2^{m-5}(n - 1) + 1]$. For $2 \leq l_i \leq n_i$ even, and $1 \leq i \leq r$, we define

$$\lambda(u) = \begin{cases} \alpha + \frac{l_1}{2}, & \text{for} \quad u = x_1^{l_1}, \\ (\alpha + n) - \frac{l_2}{2}, & \text{for} \quad u = x_2^{l_2}, \\ (\alpha + n) + \frac{l_3}{2}, & \text{for} \quad u = x_3^{l_3}, \\ (\alpha + 2n) - \frac{l_4}{2}, & \text{for} \quad u = x_4^{l_4}. \end{cases}$$

and

$$\lambda(x_i^{l_i}) = (\alpha + 2n) + \sum_{m=5}^{i} [2^{m-5}(n-1)] - \frac{l_i}{2}.$$

The set of all edge-sums generated by the above formulas forms a consecutive integer sequence $s = \alpha + 2, \alpha + 3, \cdots, \alpha + 1 + e$. Therefore, by Proposition 1.1, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = v + e + s = 2v + (2n + 3) + \sum_{m=5}^{r} [2^{m-5}(n-1) + 1]$ and to a super $(\acute{a}, 2)$-EAT labeling with minimum edge-weight $\acute{a} = v + 1 + s = v + (2n + 5) + \sum_{m=5}^{r} [2^{m-5}(n-1) + 1]$. ■

Theorem 2.7. For any $n \geq 3$ odd, $G \cong T(n, n, n, n, n_5, ..., n_r)$ admits super $(a, 1)$-EAT labeling with $a = s + \frac{3v}{2}$ if v is even, where $v = |V(G)|$, $s = (2n + 4) + \sum_{m=5}^{r} [2^{m-5}(n-1) + 1]$, $r \geq 5$, and $n_m = 2^{m-4}(n-1) + 1$ for $5 \leq m \leq r$.

Proof. Let us consider the vertex and edge set of G and the labeling $\lambda : V(G) \rightarrow \{1, 2, ..., v\}$ by the same manner as in Theorem 2.6. It follows that edge-sums of all the edges of G constitute an arithmetic sequence $s = \alpha + 2, \alpha + 3, \cdots, \alpha + 1 + e$ with common difference 1, where $\alpha = (2n + 2) + \sum_{m=5}^{r} [2^{m-5}(n-1) + 1]$. We denote it by $A = \{a_i; 1 \leq i \leq e\}$. Now to show that λ is an $(a, 1)$-EAT labeling of G, define the set of edge-labels as $B = \{b_j = v + j \ ; \ 1 \leq j \leq e\}$. The set of edge-weights can be obtained as $C = \{a_{2i-1} + b_{e-i+1} \ ; \ 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1} \ ; \ 1 \leq j \leq \frac{e+1}{2} - 1\}$. It is easy to see that C constitutes an arithmetic sequence with $d = 1$ and $a = s + \frac{3v}{2}$. Since, all vertices receive the smallest labels, λ is a super $(a, 1)$-EAT labeling. ■

From Theorems 2.1, 2.3, 2.4 and 2.6 by the principal of duality it follows that we can find the super $(a, 0)$-EAT labelings with different magic constant. Thus, we have the following corollaries:

Corollary 2.1. For any odd $n \geq 3$, $T(n, n, n, n, 2n - 1)$ admits a super $(a, 0)$-EAT total labeling with magic constant $a = 15n - 1$.

Corollary 2.2. For any odd $n \geq 3$, $T(n, n, n, n, 2n-1, 4n-3)$ admits a super $(a, 0)$-EAT labeling with magic constant $a = 25n - 9$.

Corollary 2.3. For any odd $n \geq 3$, $T(n, n, n, n, 2n-1, 4n-3, 8n-7)$ admits a super $(a, 0)$-EAT labeling with magic constant $a = 45n - 27$.

Corollary 2.4. For any $n \geq 3$ odd, and $r \geq 5$, $T(n, n, n, n, n_5, ..., n_r)$ admits a super $(a, 0)$-EAT total labeling with $a = 3v - (2n+1) - \sum\limits_{m=5}^{r}[2^{m-5}(n-1)+1]$, where $n_m = 2^{m-4}(n-1)+1$ for $5 \leq m \leq r$.

4. CONCLUSION

In this paper, we have proved that a subclass of subdivided stars denoted by $T(n, n, n, n, n_5, ..., n_r)$, admits super (a, d)-EAT labelings for $d = 0, 1, 2$, when $n \geq 3$ is odd, $r \geq 5$ and $n_m = 2^{m-4}(n-1)+1$ for $5 \leq m \leq r$.

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