

THE METRIC DIMENSION OF A GRAPH COMPOSITION PRODUCTS WITH STAR

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Abstract. A set of vertices W *resolves* a graph G if every vertex is uniquely determined by its coordinate of distances to the vertices in W . The minimum cardinality of a resolving set of G is called the *metric dimension* of G . In this paper, we consider a graph which is obtained by the composition product between two graphs. The *composition product* of graphs G and H , denoted by $G[H]$, is the graph with vertex set $V(G) \times V(H) = \{(a, v) | a \in V(G); v \in V(H)\}$, where (a, v) adjacent with (b, w) whenever $ab \in E(G)$, or $a = b$ and $vw \in E(H)$. We give a general bound of the metric dimension of a composition product of any connected graph G and a star. We also show that the bound is sharp.

Key words: Basis, composition product, metric dimension, resolving set, star.

Abstrak. Himpunan titik W membedakan graf G jika setiap titik ditentukan secara tunggal oleh koordinat jaraknya ke titik-titik di W . Kardinalitas minimum dari sebuah himpunan pembeda dari G disebut *dimensi metrik* of G . Pada paper ini, kami meneliti graf yang diperoleh dari hasil kali komposisi antara dua graf. The Hasil kali komposisi dari graf G dan H , dinyatakan dengan $G[H]$, adalah graf dengan himpunan titik $V(G) \times V(H) = \{(a, v) | a \in V(G); v \in V(H)\}$, dengan (a, v) bertetangga dengan (b, w) bilamana $ab \in E(G)$, atau $a = b$ dan $vw \in E(H)$. Kami menunjukkan sebuah batas umum dari dimensi metrik hasil kali komposisi dari sebarang graf terhubung G dengan graf bintang. Kami juga menunjukkan bahwa batas tersebut tajam.

Kata kunci: Basis, hasil kali komposisi, dimensi metrik, himpunan pembeda, graf bintang.

1. Introduction

Throughout this paper, all graphs are finite, simple, and connected. Let G be a graph. We denote by V_G or V the vertex set of G , and E_G or E the edge set of G .

The metric dimension were first studied by Harary and Melter [8], and independently by Slater [16, 17]. The basic parameter for this topic is the distance of two vertices in a graph. For any two distinct vertices $u, v \in V(G)$, the *distance* $d_G(u, v)$ between u and v is the length of a shortest (u, v) -path in G . In particular, if $d_G(u, w) \neq d_G(v, w)$, then we say that u and v are *resolved* by w . For an ordered k -tuple $W = \{w_1, w_2, \dots, w_k\}$ of $V(G)$ and a vertex $v \in V(G)$, the *representation* of v with respect to W is the k -tuple $r(v|W) = (d_G(v, w_1), \dots, d_G(v, w_k))$. We call W as a *resolving set* if $r(u|W) \neq r(v|W)$ for every two distinct vertices $u, v \in V(G)$. A resolving set of G with minimum cardinality is called *basis*, and its cardinality is called *metric dimension* of G , and denoted by $\beta(G)$.

Trivially, if G is a graph of order n , then $\beta(G) \leq n$, by taking all vertices of G to be a resolving set. However, we may obtain a resolving set whose cardinality is less than n . Chartrand *et al* [5] showed that $\beta(G) = 1$ if and only if $G \cong P_n$. Furthermore, they proved that K_n is the only graph G with $\beta(G) = n - 1$. They also proved that $\beta(G) = n - 2$ if and only if G is $K_{r,s}$ for $r, s \geq 1$, $K_r + \overline{K_s}$ for $r \geq 1, s \geq 2$, or $K_r + (K_1 \cup K_s)$ for $r, s \geq 1$.

Since there is no polynomial time algorithm which can be used to determine the metric dimension of any graph, many researchers consider this problem for some particular classes of graphs. Some classes of graph whose metric dimension have been known are cycles [6], trees [5, 8, 10], stars [5, 8, 10], wheels [2, 3, 15], complete multipartite [5, 14], unicyclic graphs [12], Cayley graphs [7], and regular graphs [1].

Determining a relation, in terms of metric dimension, between the origin graph and the resulting graph under a graph operation is also interesting to be considered. Some results on joint product graph have been proved in [2, 4, 3, 15]. Caceres *et. al.* [3], Khuler *et. al.* [10], Melter [11], and Saputro *et. al.* [14]

showed the metric dimension of some graphs which obtained from the cartesian product between two or more graphs. Some graphs which constructed from the corona product between two graphs, have been investigated in [9, 18]. The metric dimension of the composition product graphs have been studied in [13].

In the previous result [13], we proved the general bounds of the metric dimension of a *composition product* $G[H]$ of a connected graphs G and H . We also show that the bounds are tight. In this paper, we determine a general bound of the metric dimension of a composition product of any connected graph G and a star. We can show that the upper bound of a composition product of any graph G and a star is less than the upper bound of $G[H]$. We also show that the bound is sharp.

2. Main Results

The composition product of graph G and H , denoted by $G[H]$, is the graph with vertex set $V(G) \times V(H) = \{(a, v) \mid a \in V(G), v \in V(H)\}$, where (a, v) adjacent with (b, w) whenever $ab \in E(G)$, or $a = b$ and $vw \in E(H)$. For any vertex $a \in V(G)$, we define $H(a) = \{(a, v) \mid v \in V(H)\}$, and for any vertex $b \in V(H)$, we define $G(b) = \{(v, b) \mid v \in V(G)\}$. For example, let $G \cong P_3$ with $V(G) = \{a, b, c\}$, and $H \cong S_3$ with $V(H) = \{w, x, y, z\}$. The illustration of $G[H]$ can be seen in Figure 1.

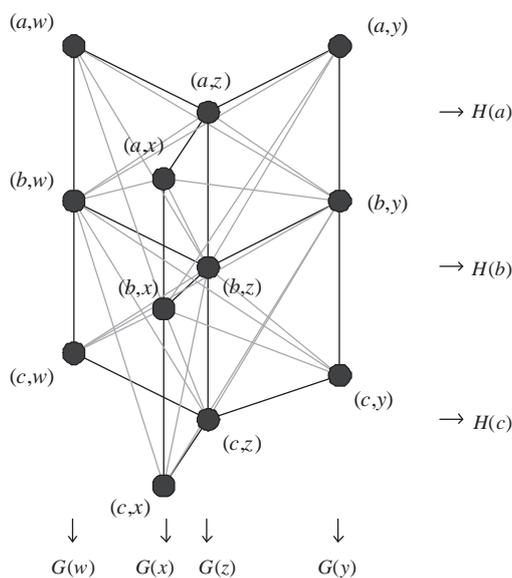


FIGURE 1. The composition product of P_3 and S_3

Saputro *et al.* [13] have proved the general bounds of $\beta(G[H])$ for any connected graphs G and H . They also show that the bounds are tight.

Theorem 2.1. [13] *Let G and H be connected graphs with $|V(G)| \geq 1$ and $|V(H)| \geq 2$. If $|V(G)| = n$, then $n \cdot \beta(H) \leq \beta(G[H]) \leq n \cdot \beta(H + K_1) + (n - 2)$.*

Now, for $m \geq 3$, we consider $H \cong S_m$, a star with $m + 1$ vertices. We show that we can reduce the general bounds in Theorem 2.1 above, for H is a star.

By definition of $G[H]$, for $a, b \in V(G)$ and $a \neq b$, every two different vertices $x, y \in H(a)$ satisfy $d_{G[H]}(x, z) = d_{G[H]}(y, z)$ whenever $z \in H(b)$. Therefore, there exist $S_a \subseteq H(a)$ such that every two distinct vertices $x, y \in H(a)$ satisfies $r(x|S_a) \neq r(y|S_a)$.

Lemma 2.2. [13] *Let G and H be connected graphs with $|V(G)| \geq 1$ and $|V(H)| \geq 2$. For any vertex $a \in V(G)$, let $H(a) = \{(a, v) | v \in V(H)\}$ be the subset of $V(G[H])$. Let W be a basis of $G[H]$. If $S_a = W \cap H(a)$, then $S_a \neq \emptyset$. Moreover, if B is a basis of H , then $|S_a| \geq |B|$.*

Considering Lemma 2.2 and $H \cong S_m$, then $\beta(G[H])$ is equal to the lower bound of Theorem 2.1.

Next, we consider a joint graph $H + K_1$ where $H \cong S_m$. A joint graph of G and H , denoted by $G + H$, is a graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. The following lemma show the metric dimension of $H + K_1$.

Lemma 2.3. *For $m \geq 3$, let H be a star S_m with $m+1$ vertices. Then $\beta(H + K_1) = m$.*

Proof. Let $V(H + K_1) = \{u, v\} \cup \{s_i | 1 \leq i \leq m\}$ and $E(G) = \{uv\} \cup \{us_i, vs_i | 1 \leq i \leq m\}$.

Let $W = \{u\} \cup \{s_i | 1 \leq i \leq m - 1\}$. Since $vs_1 \in E(G)$ but $s_m s_1 \notin E(G)$, we obtain $r(v|W) \neq r(s_m|W)$. Therefore, W is a resolving set of $H + K_1$.

Now, suppose that $\beta(H + K_1) \leq m - 1$. Let S be a basis of $H + K_1$. Then we have two cases of S as follows.

(1) $u, v \notin S$

Since $uz, vz \in E(G)$ for $z \in V(G \setminus \{u, v\})$, we obtain $r(u|S) = r(v|S)$, a contradiction.

(2) $s_p, s_q \notin S$ for $p, q \in \{1, 2, \dots, m\}$ and $p \neq q$

Since $s_p a, s_q a \in E(G)$ for $a \in \{u, v\}$ and $s_p b, s_q b \notin E(G)$ for $b \in V(G \setminus \{u, v, s_p, s_q\})$, we obtain $r(s_p|S) = r(s_q|S)$, a contradiction.

□

In the theorem below, we show the general upper bounds for $\beta(G[H])$ where H is a star. We can see that the upper bound is less than the general upper bound in Theorem 2.1.

Theorem 2.4. For $m \geq 3$, let G be a connected graph, and $H \cong S_m$ of $m + 1$ vertices. If $|V(G)| = n$, then $\beta(G[H]) \leq n \cdot \beta(H + K_1) - 1$.

Proof. By Lemma 2.3, $n \cdot \beta(H + K_1) - 1 = nm - 1$. Now, we will show that there exists a resolving set of $nm - 1$ vertices.

Let $V(H) = \{u\} \cup \{s_i | 1 \leq i \leq m\}$ and $E(H) = \{us_i | 1 \leq i \leq m\}$. Let P be a longest shortest path between two vertices in G . Let b be an end point of P . We define $W = V(G[H]) \setminus (G(s_m) \cup \{(b, u)\})$. We will show that W is a resolving set.

Let $x, y \in V(G)$ and $x \neq y$. We consider two conditions.

(1) (x, s_m) and (y, s_m)

If $xy \in E(G)$, then $d_{G[H]}((y, s_m), (y, s_1)) = d_{G[H]}((x, s_m), (y, s_1)) + 1$. Otherwise, $d_{G[H]}((x, s_m), (y, u)) = d_{G[H]}((y, s_m), (y, u)) + d - 1$ where $d = d_G(x, y) \geq 2$. Therefore $r((x, s_m)|W) \neq r((y, s_m)|W)$.

(2) (x, s_m) and (b, u)

Let $c \in V(P)$ and $bc \in E(G)$. Since $(b, u)(b, s_i), (b, u)(c, s_i) \in E(G[H])$ for $1 \leq i \leq m - 1$, then $r((b, u)|W) \neq r((x, s_m)|W)$ where $x \in V(G)$. Therefore, W is a resolving set of $G[H]$.

□

By applying Theorems 2.1 and 2.4 above, we obtain the general bounds of $\beta(G[H])$ with $H \cong S_m$ as stated below.

Theorem 2.5. For $n \geq 1$, let G be a connected graph with n vertices. If $H \cong S_m$ of $m + 1$ vertices where $m \geq 3$, then $n \cdot \beta(H) \leq \beta(G[H]) \leq n \cdot \beta(H + K_1) - 1$. □

In the next two theorems, we prove that the upper bound and the lower bound in Theorem 2.5 above are the best possible.

Theorem 2.6. For $m \geq 3$, let G be a connected graph, and $H \cong S_m$ of $m + 1$ vertices. If $|V(G)| = n$, then there exist G such that $\beta(G[H]) = n \cdot \beta(H)$.

Proof. For $n \geq 4$, let $G \cong P_n$ with $V(G) = \{p_1, p_2, \dots, p_n\}$. By Theorem 2.5, we only need to show that $\beta(G[H]) \leq n \cdot \beta(H)$. In [5, 8, 10], the metric dimension of star of $m + 1$ vertices is $m - 1$.

Let $V(H) = \{u\} \cup \{s_i | 1 \leq i \leq m\}$ and $E(H) = \{us_i | 1 \leq i \leq m\}$. Let $W = V(G[H]) \setminus (G(u) \cup G(s_m))$. We will show that W is a resolving set. For $a_1, a_2 \in V(G)$ and $b_1, b_2 \in V(H)$, we consider two possibilities of (a_1, b_1) and (a_2, b_2) for $b_1, b_2 \in \{s_m, u\}$.

(1) $a_1 = a_2$

Since $(a_1, u)(a_1, s_1) \in E(G[H])$ but $(a_1, s_m)(a_1, s_1) \notin E(G[H])$, then $r((a_1, u)|W) \neq r((a_1, s_m)|W)$.

(2) $a_1 \neq a_2$

For $b_1, b_2 \in \{s_m, u\}$, we consider $p = d_G(a_1, a_2)$ in G . Since $n \geq 4$ and $\text{diam}(G) = n - 1$, we consider two cases.

(a) $1 \leq p \leq n - 2$

Then there exists $a_3 \in V(G)$ such that $a_2 a_3 \in E(G[H])$ and $d_G(a_1, a_3) = p + 1$. Therefore, $d_{G[H]}((a_1, b_1), (a_3, b_1)) = d_{G[H]}((a_1, b_1), (a_2, b_1)) + 1$, which implies $r((a_1, b_1)|W) \neq r((a_2, b_1)|W)$.

(b) $p = n - 1$

Then there exists $a_3 \in V(G)$ such that $a_2 a_3 \in E(G[H])$ and $d_G(a_1, a_3) = p - 1$. Therefore, $d_{G[H]}((a_1, b_1), (a_3, b_1)) \neq d_{G[H]}((a_1, b_1), (a_2, b_1))$, which implies $r((a_1, b_1)|W) \neq r((a_2, b_1)|W)$.

□

Theorem 2.7. *For $m \geq 3$, let G be a connected graph, and $H \cong S_m$ of $m + 1$ vertices. If $|V(G)| = n$, then there exist G such that $\beta(G[H]) = n \cdot \beta(H + K_1) - 1$.*

Proof. For $n \geq 3$, let $G \cong K_n$. By Theorem 2.5, we only need to show that $\beta(G[H]) \geq n \cdot \beta(H + K_1) - 1$.

Suppose that $\beta(G[H]) \leq n \cdot \beta(H + K_1) - 2$ and W is a basis of $G[H]$. By Lemma 2.3, $\beta(G[H]) \leq n \cdot \beta(H + K_1) - 2 = nm - 2$. Note that, $\beta(H) = m - 1$ (see [5, 8, 10]). By Lemma 2.2, there exists two distinct vertices $a, b \in V(G)$ such that $H(a)$ contributes $m - 1$ vertices to W and so does $H(b)$.

Let $V(H) = \{u\} \cup \{s_i | 1 \leq i \leq m\}$ and $E(H) = \{us_i | 1 \leq i \leq m\}$. If there exists two distinct vertices $(a, s_j), (a, s_k) \notin W$, then by definition of $G[H]$, we have $r((a, s_j)|W) = r((a, s_k)|W)$, a contradiction. Otherwise, since (a, u) and (b, u) are adjacent to every vertex in $V(G[H]) \setminus \{(a, u), (b, u)\}$, then $r((a, u)|W) = r((b, u)|W)$, a contradiction. □

We also show that there exist graphs G and H such that the metric dimension of $G[H]$ is not equal to both the lower and the upper bounds in Theorem 2.5.

Theorem 2.8. *For $m \geq 3$, there exist connected graphs G of order $n \geq 1$ and $H \cong S_m$ of order $m + 1$ such that $\beta(G[H]) = c$ where $n \cdot \beta(H) < c < n \cdot \beta(H + K_1) - 1$.*

Proof. Let $G \cong P_3$ where $V(G) = \{p_1, p_2, p_3\}$ and $E(G) = \{p_1 p_2, p_2 p_3\}$, and $H \cong S_m$ where $V(H) = \{u\} \cup \{s_i | 1 \leq i \leq m\}$ and $E(H) = \{us_i | 1 \leq i \leq m\}$. We will show that $\beta(G[H]) = n \cdot \beta(H) + 1$. By Lemma 2.3 and the metric dimension of star S_m [5, 10], we have

$$\begin{aligned} n \cdot \beta(H) &< n \cdot \beta(H) + 1 \\ &= n(m - 1) + 1 \\ &= nm - (n - 1) \\ &< nm - 1 \\ &= n \cdot \beta(H + K_1) - 1. \end{aligned}$$

Suppose that $\beta(G[H]) \leq n \cdot \beta(H)$. By Theorem 2.5, we have $\beta(G[H]) = n \cdot \beta(H)$. Let S be a basis of $G[H]$. By Lemma 2.2, for every $a \in V(G)$, $H(a)$ contributes $\beta(H) = m - 1$ vertices to W . If there exists two distinct vertices $(a, s_j), (a, s_k) \notin S$, then by definition of $G[H]$, we have $r((a, s_j)|S) = r((a, s_k)|S)$, a contradiction. Otherwise, we consider $(p_1, s_j), (p_3, s_k) \notin S$ where $j, k \in \{1, 2, \dots, m\}$. Since $d_{G[H]}((p_1, s_j), v) = 2 = d_{G[H]}((p_3, s_k), v)$ for $v \in S \cap (H(p_1) \cup H(p_3))$, and $d_{G[H]}((p_1, s_j), y) = 1 = d_{G[H]}((p_3, s_k), y)$ for $y \in S \cap H(p_2)$, we obtain $r((p_1, s_j)|S) = r((p_3, s_k)|S)$, a contradiction.

Now, we define $W = G(s_1) \cup G(s_2) \cup \dots \cup G(s_{m-1}) \cup \{(p_1, s_m)\}$. We will show that W is a resolving set.

For $a_1, a_2 \in V(G)$ and $b_1, b_2 \in V(H)$, we consider three possibilities of (a_1, b_1) and (a_2, b_2) for $b_1, b_2 \in \{s_m, u\}$.

- (1) $a_1 = a_2$ where $a_1 \in \{p_2, p_3\}$
 Since $(a_1, u)(a_1, s_1) \in E(G[H])$ but $(a_1, s_m)(a_1, s_1) \notin E(G[H])$, then $r((a_1, u)|W) \neq r((a_1, s_m)|W)$.
- (2) $a_1 \neq a_2$
 For $b_1, b_2 \in \{s_m, u\}$, we consider $p = d(a_1, a_2)$ in G . Since $n \geq 4$ and $\text{diam}(G) = n - 1$, we consider two cases.
 - (a) $1 \leq p \leq n - 2$
 Then there exists $a_3 \in V(G)$ such that $a_2 a_3 \in E(G[H])$ and $d_G(a_1, a_3) = p + 1$. Therefore, $d_{G[H]}((a_1, b_1), (a_3, b_1)) = d_{G[H]}((a_1, b_1), (a_2, b_1)) + 1$, which implies $r((a_1, b_1)|W) \neq r((a_2, b_1)|W)$.
 - (b) $p = n - 1$
 Then there exists $a_3 \in V(G)$ such that $a_2 a_3 \in E(G[H])$ and $d_G(a_1, a_3) = p - 1$. Therefore, $d_{G[H]}((a_1, b_1), (a_3, b_1)) \neq d_{G[H]}((a_1, b_1), (a_2, b_1))$, which implies $r((a_1, b_1)|W) \neq r((a_2, b_1)|W)$.

□

An interesting question is whether all the values between the lower and the upper bounds are achievable, as stated in the following problem.

Problem 2.9. *Let H be a star S_m of order $m + 1$. For every integer c with $n \cdot \beta(H) < c < n \cdot \beta(H + K_1) - 1$, does there exist a graph G of order n such that $\beta(G[H]) = c$?*

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