

CHARACTERISATION OF PRIMITIVE IDEALS OF TOEPLITZ ALGEBRAS OF QUOTIENTS

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Abstract. Let Γ be a totally ordered abelian group, the topology on primitive ideal space of Toeplitz algebras $\text{Prim } \mathcal{T}(\Gamma)$ can be identified through the upwards-looking topology if and only if the chain of order ideals is well-ordered. Let I be an order ideal of such that the chain of order ideals of Γ/I is not well-ordered, we show that for any order ideal $J \not\supseteq I$, the topology on primitive ideal space can be identified through the upwards-looking topology. Also we discuss the closed sets in $\text{Prim } \mathcal{T}(\Gamma)$ with the upwards-looking topology and characterize maximal primitive ideals.

Key words: Toeplitz algebra, totally ordered group, primitive ideal, quotient, characterisation.

Abstrak. Misalkan Γ adalah grup abel terurut total, topologi pada ruang ideal primitif dari aljabar Toeplitz $\text{Prim } \mathcal{T}(\Gamma)$ dapat diidentifikasi melalui topologi *upwards-looking* jika dan hanya jika rantai dari ideal urutan adalah terurut dengan rapi (*well-ordered*). Misalkan I adalah sebuah ideal urutan sedemikian sehingga rantai dari ideal urutan dari Γ/I tidak terurut dengan rapi, diperlihatkan bahwa untuk sembarang ideal $J \not\supseteq I$, topologi pada ruang ideal primitif dapat diidentifikasi melalui topologi *upwards-looking*. Pada paper ini juga dibahas himpunan-himpunan tutup di $\text{Prim } \mathcal{T}(\Gamma)$ di bawah topologi *upwards-looking*, dan karakterisasi dari ideal primitif maksimal.

Kata kunci: Aljabar Toeplitz, grup terurut total, ideal primitif, kuosien, karakterisasi

1. Introduction

Suppose Γ is a totally ordered abelian group. Let $\Sigma(\Gamma)$ be the chain of order ideals of Γ , and $X(\Gamma)$ denotes the disjoint union

$$\bigsqcup \{\hat{I} : I \in \Sigma(\Gamma)\} = \{(I, \gamma) : I \in \Sigma(\Gamma), \gamma \in \hat{I}\}.$$

Adji and Raeburn shows that every primitive ideal of Toeplitz algebra $\mathcal{T}(\Gamma)$ of Γ is of the form

$$\ker Q_I \circ \alpha_\nu^{\Gamma^{-1}}$$

where I is an order ideal of Γ and $\nu \in \hat{\Gamma}$. They also showed [?, Theorem 3.1] that there is a bijection L of $X(\Gamma)$ onto the primitive ideal space $\text{Prim } \mathcal{T}(\Gamma)$ of Toeplitz algebra $\mathcal{T}(\Gamma)$ given by

$$L(I, \gamma) := \ker Q_I \circ \alpha_\nu^{\Gamma^{-1}} \text{ where } \nu \in \hat{\Gamma} \text{ satisfies } \nu|_I = \gamma.$$

Adji and Raeburn [?] introduced a topology in $X(\Gamma)$ which is called the upwards-looking topology. When $\Sigma(\Gamma)$ is isomorphic with a subset of $\mathbb{N} \cup \{\infty\}$, the bijection L is a homeomorphism [?, Proposition 4.7], so the usual hull-kernel topology of $\text{Prim } \mathcal{T}(\Gamma)$ can be identified through the upwards-looking topology in $X(\Gamma)$. Later, Raeburn and his collaborators [?] showed that L is a homeomorphism if and only if $\Sigma(\Gamma)$ is well-ordered, in the sense that every nonempty subset has a least element.

More recently, Rosjanuardi and Itoh [?] characterised maximal primitive ideals of $\mathcal{T}(\Gamma)$. A series of analysis on subsets of $\Sigma(\Gamma)$ implies that any singleton set $\{\gamma\}$ which consists of a character in $\hat{\Gamma}$ is closed. This implies that every maximal primitive ideal of $\mathcal{T}(\Gamma)$ is of the following form

$$L(\Gamma, \gamma) = \ker Q_\Gamma \circ \alpha_\gamma^{\Gamma^{-1}}.$$

Given a totally ordered abelian group Γ and an order ideal I . In this paper, we apply the method in [?] and [?] to characterise maximal primitive ideal of Toeplitz algebra $\mathcal{T}(\Gamma/I)$ of quotient Γ/I when the chain of order ideal $\Sigma(\Gamma/I)$ is not well-ordered.

2. UPWARDS-LOOKING TOPOLOGY

Let Γ be a totally ordered abelian group. The Toeplitz algebra $\mathcal{T}(\Gamma)$ of Γ is the C^* -subalgebra of $B(\ell^2(\Gamma^+))$ generated by the isometries $\{T_x = T_x^\Gamma : x \in \Gamma^+\}$ which are defined in terms of the usual basis by $T_x(e_y) = e_{y+x}$. This algebra is universal for isometric representation of Γ^+ [?, Theorem 2.9].

Let I be an order ideal of Γ . Then the map $x \mapsto T_{x+I}^{\Gamma/I}$ is an isometric representation of Γ^+ in $\mathbb{T}(\Gamma/I)$. Therefore by the universality of $\mathcal{T}(\Gamma)$, there is a homomorphism $Q_I : \mathcal{T}(\Gamma) \rightarrow \mathcal{T}(\Gamma/I)$ such that $Q_I(T_x) = T_{x+I}^{\Gamma/I}$, and that Q_I is surjective. Suppose $\mathcal{C}(\Gamma, I)$ denotes the ideal in $\mathcal{T}(\Gamma)$ generated by $\{T_u T_u^* - T_v T_v^* :$

$v - u \in I^\perp\}$ and $\text{Ind}_{I^\perp}^{\hat{\Gamma}}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I})$ is the closed subalgebra of $C(\hat{\Gamma}, \mathcal{T}(\Gamma/I))$ satisfying $f(xh) = \alpha_h^{\Gamma/I^{-1}}(f(x))$ for $x \in \hat{\Gamma}, h \in I^\perp$. It was proved in [?, Theorem 3.1] that there is a short exact sequence of C^* -algebras:

$$0 \rightarrow \mathcal{C}(\Gamma, I) \rightarrow \mathcal{T}(\Gamma) \xrightarrow{\phi_I} \text{Ind}_{I^\perp}^{\hat{\Gamma}}(\mathcal{T}(\Gamma/I), \alpha^{\Gamma/I}) \rightarrow 0. \quad (1)$$

in which $\phi_I(a)(\gamma) = Q_I \circ (\alpha_\gamma^\Gamma)^{-1}(a)$ for $a \in \mathcal{T}(\Gamma)$, $\gamma \in \hat{\Gamma}$, and α_γ^Γ is dual action of $\hat{\Gamma}$ on $\mathcal{T}(\Gamma)$ characterized by $\alpha_\gamma^\Gamma(T_x) = \gamma(x)T_x$. The identity representation $T^{\Gamma/I}$ of $\mathcal{T}(\Gamma/I)$ is irreducible [?], it follows from [?, Proposition 6.16] that $\ker Q_I \circ (\alpha_\gamma^\Gamma)^{-1}$ is a primitive ideal of $\mathcal{T}(\Gamma)$.

If $X(\Gamma)$ denotes the disjoint union

$$\bigsqcup \{\hat{I} : I \in \Sigma(\Gamma)\} = \{(I, \gamma) : I \in \Sigma(\Gamma), \gamma \in \hat{I}\},$$

it was showed in [?, Theorem 3.1] that

$$L(I, \gamma) := \ker Q_I \circ \alpha_\nu^{\Gamma^{-1}} \text{ where } \nu \in \hat{\Gamma} \text{ satisfies } \nu|_I = \gamma, \quad (2)$$

is a bijection of $X(\Gamma)$ onto $\text{Prim } \mathcal{T}(\Gamma)$.

Using the bijection L , Adji and Raeburn describe a new topology on X which corresponds to the hull-kernel topology on $\text{Prim } \mathcal{T}(\Gamma)$. This new topology, is later called the *upwards-looking* topology. They topologise X by specifying the closure operation as stated in the following definition.

Definition 2.1. [?] *The closure \overline{F} of a subset F of X is the set consisting of all pairs (J, γ) where J is an order ideal and $\gamma \in \hat{J}$ such that for every open neighbourhood N of γ in \hat{J} , there exists $I \in \Sigma(\Gamma)$ and $\chi \in N$ for which $I \subset J$ and $(I, \chi|_I) \in F$.*

Example 2.2. [?, Example 3] *We are going to discuss some description of sets in $X(\Gamma)$ by considering specific cases of Γ . An observation on $\Gamma := \mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$ gives interesting results. Let I be the ideal $\{(0, n) : n \in \mathbb{Z}\}$, since I is the only ideal, we have $X(\Gamma) = \hat{0} \sqcup \hat{I} \sqcup \hat{\Gamma}$. Suppose λ_0 is a character in \hat{I} defined by $(0, n) \mapsto e^{2\pi i n}$, and let $F = \{\lambda_0\}$. Next we consider a character γ in $\hat{\Gamma}$ defined by $(m, n) \mapsto e^{2\pi i(m+n)}$. It is clear that $\gamma|_I = \lambda_0$. Then $\gamma \in \overline{F}$, because every open neighbourhood N of γ in $\hat{\Gamma}$ contains an element λ (which is nothing but γ it self) such that its restriction on I gives a character in F . It is clear that $\gamma \notin F$, hence F is not closed in the upwards-looking topology for $X(\Gamma)$.*

Adji and Raeburn [?] proved that this is the correct topology to identify the hull-kernel topology of $\text{Prim } \mathcal{T}(\Gamma)$ when Γ is a group such that the set $\Sigma(\Gamma)$ of order ideal is order isomorphic to a subset of $\mathbb{N} \cup \{\infty\}$. In [?], Raeburn and his collaborators extended the results in [?]. Their main theorem, says that $\text{Prim } \mathcal{T}(\Gamma)$ is homeomorphic to $X(\Gamma)$ with the upwards-looking topology if and only if the totally ordered set $\Sigma(\Gamma)$ is well-ordered in the sense that every non-empty subset has a least element. Their technique uses classical Toeplitz operators as well as the

universal property of $\mathcal{T}(\Gamma)$ which was the main tool in [?]. Then they described $\text{Prim } \mathcal{T}(\Gamma)$ when parts of $\Sigma(\Gamma)$ are well-ordered.

Rosjanuardi in [?] improved the results in [?] to the case when $\Sigma(\Gamma)$ is not well ordered. In [?, Proposition 6] it is stated that when $\Sigma(\Gamma)$ is isomorphic to a subset of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, then we can use the upwards-looking topology on $X(\Gamma/I)$ to identify the topology on $\text{Prim } \mathcal{T}(\Gamma/I)$. For general totally abelian group Γ , as long as there is an order ideal I such that every order ideal $J \supseteq I$ has a successor, the upwards-looking topology is the correct topology for $\text{Prim } \mathcal{T}(\Gamma/I)$ [?, Proposition 8]. In [?, Theorem 9] it was proved that for any quotient Γ/I such that the chain $\Sigma(\Gamma/I)$ is isomorphic to a subset $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, for any order ideal $J \not\supseteq I$, the upwards-looking topology on $\Sigma(\Gamma/J)$ is the correct topology for $\text{Prim } \mathcal{T}(\Gamma/J)$.

3. CHARACTERISATION OF PRIMITIVE IDEALS

Example ?? implies that any closed set in the point wise topology is not necessarily closed in the upwards-looking topology. When it is applied to any complement F^C of a set F , it arrives to a conclusion that any open set in the point wise topology, is not necessarily open in the upwards-looking topology. This example motivated Rosjanuardi and Itoh [?] to prove more general cases.

Combining results in [?] with ones in [?] give characterisation results for more general cases than in [?].

Proposition 3.1. *Suppose that Γ is a totally ordered abelian group such that the chain $\Sigma(\Gamma)$ of order ideals in Γ is isomorphic to a subset of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. For any $I \in \Sigma(\Gamma)$, the maximal primitive ideals of $\mathcal{T}(\Gamma/I)$ are of the form*

$$\ker Q_{\Gamma/I} \circ (\alpha_{\gamma}^{\Gamma/I})^{-1}.$$

PROOF. Let $I \in \Sigma(\Gamma)$. The chain of order ideals in Γ/I is

$$I \subset J_1/I \subset J_2/I \subset \dots,$$

where $J_i \in \Sigma(\Gamma)$ and $I \subset J_i \subset J_{i+1}$ for all i . Hence $\Sigma(\Gamma/I)$ is well ordered. Give the set $X(\Gamma/I) := \bigsqcup \{\widehat{J/I} : J \in \Sigma(\Gamma), I \subset J\}$ the upwards-looking topology, hence $L^{\Gamma/I}$ is a homeomorphism of $X(\Gamma/I)$ onto $\text{Prim } \mathcal{T}(\Gamma/I)$ by Theorem 3.1 of [?]. Proposition 6 of [?], then implies that $X(\Gamma/I)$ is homeomorphic with $\text{Prim}(\mathcal{T}(\Gamma))$. Theorem 11 of [?] then gives the result.

Proposition 3.2. *Suppose that Γ is a totally ordered abelian group, and let I be an order ideal in Γ such that every order ideal $J \supseteq I$ has a successor. Then the maximal primitive ideals of $\mathcal{T}(\Gamma/I)$ are of the form*

$$\ker Q_{\Gamma/I} \circ (\alpha_{\gamma}^{\Gamma/I})^{-1}.$$

PROOF. Let $I \in \Sigma(\Gamma)$ such that every order ideal $J \supseteq I$ has a successor. Since each nontrivial element of $\Sigma(\Gamma/I)$ is of the form J/I for $J \in \Sigma(\Gamma)$ and $J \not\supseteq I$, every element of $\Sigma(\Gamma/I)$ has a successor. This implies that $\Sigma(\Gamma/I)$ is well ordered. Give the set $X(\Gamma/I) := \bigsqcup \{\widehat{J/I} : J \in \Sigma(\Gamma), I \subset J\}$ the upwards-looking topology, hence

$L^{\Gamma/I}$ is a homeomorphism by Theorem 3.1 of [?]. The result then follows from Theorem 9 of [?].

Theorem 3.3. *Suppose that Γ is a totally ordered abelian group, and $I \in \Sigma(\Gamma)$ such that $\Sigma(\Gamma/I) \cong \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. Let $J \in \Sigma(\Gamma)$ such that $J \supsetneq I$. Then the maximal primitive ideals of $\mathcal{T}(\Gamma/J)$ are of the form*

$$\ker Q_{\Gamma/J} \circ (\alpha_{\gamma}^{\Gamma/J})^{-1}.$$

PROOF. Since every nontrivial ideal of Γ/I is of the form J/I where $J \in \Sigma(\Gamma)$ and $J \supsetneq I$ and for ideals J_1, J_2 such that $J_1/I \subseteq J_2/I$ implies $J_1 \subseteq J_2$, then may write

$$\Sigma(\Gamma/I) := \{I = J_{-\infty} \subseteq \dots \subseteq J_k/I \subseteq J_{k+1}/I \subseteq \dots \subseteq \Gamma = J_{\infty}\}.$$

Now consider the subset

$$\mathcal{I} := \{I = J_{-\infty} \subseteq \dots \subseteq J_k \subseteq J_{k+1} \subseteq \dots \subseteq J_{\infty} = \Gamma\}$$

of $\Sigma(\Gamma)$. If $J \neq I$ is an element of \mathcal{I} , i.e $J \in \Sigma(\Gamma)$ such that $J \supsetneq I$, the set

$$\Sigma(\Gamma/J) = \{J \subseteq K_1/J \subseteq K_2/J \dots\}$$

is well ordered. Hence $L^{\Gamma/J}$ is a homeomorphism of $X(\Gamma/J)$ onto $\text{Prim } \mathcal{T}(\Gamma/J)$ by Theorem 3.1 of [?]. The result is then follow from Theorem 9 of [?].

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