COMPLETE CONVERGENCES AND STRONG LAWS OF LARGE NUMBERS FOR WEIGHTED SUMS PAIRWISE NQD RANDOM VARIABLES SEQUENCE

Guang-hui Cai

Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables. Some complete convergences and strong laws of large numbers for a weighted sums sequence of pairwise NQD random variables are obtained. The results obtainted generalize the results of Cabrera and Volodin (see [3]).

1. INTRODUCTION

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d) random variables. The Marcinkiewicz-Zygmund strong laws of large numbers (SLLN) provides that

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} (X_i - EX_i) \to 0 \, a.s. \, for \, 1 \leq \alpha < 2$$

and

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} X_i \to 0 \, a.s. \, for \, 0 < \alpha < 1$$

if and only if $E|X|^\alpha < \infty$. The case $\alpha = 1$ is due to Kolmogorov.

As for negatively associated (NA) random variables, Joag and Proschan [8] gave the following definition.

Definition 1.1. [8] *A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, ..., n\}$, we have*

$$Cov(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

Let $\{X, X_n, n \geq 1\}$ be an identically distributed NA sequence, Matula [10] proved the Kolmogorov strong laws of large numbers. Two random variables X and Y are negative quadrant dependent (NQD), if for all $x, y \in R$, we have

$$P(X > x, Y > y) \leq P(X > x)P(Y > y).$$

A random variables sequence $\{X_k, k \in N\}$ is said to be pairwise NQD, if for all $i \neq j$, X_i and X_j are NQD.

The concept of NQD was given by Lehmann [9]. We known that NQD is more general than NA. So NQD is very general. As for pairwise NQD random variables sequences, Wu [13] Proved Kolmogorov strong law of large numbers and complete convergence for pairwise NQD random sequences. Cabrera and Volodin [3] obtained mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability. The result of Cabrera and Volodin [3] sees the following Theorem.

Theorem A *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of rowwise pairwise NQD random variables and $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of constants with $\sum_{k=u_n}^{v_n} |a_{nk}| \leq C$ for all $n \in N$ and some constant $C > 0$. Let moreover $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h_n \uparrow \infty$ as $n \uparrow \infty$. Suppose that*
(a) $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is h- integrable concerning the array of constants $\{a_{nk}\}$,
(b) $h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0$ as $n \to \infty$.
Let $T_n = \sum_{k=u_n}^{v_n} a_{nk} X_{nk}$, for all $n \geq 1$. Then $\lim_{n \to \infty} |T_n| = 0$ in Probability.

The main purpose of this paper is to establish some complete convergences and strong laws of large numbers for a weighted sums sequence of pairwise NQD random variables. The results obtained generalize the results of Cabrera and Volodin [3].

2. MAIN RESULTS

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$. In order to prove our results, we need the concept of the Hsu-Robbins-Erdös law of large numbers (see [6], [7]) and the following lemma. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d) random variables and denote $S_n = \sum_{i=1}^{n} X_i$. The Hsu-Robbins-Erdös law of large numbers (see [6], [7]) states that

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty$$

is equivalent to $EX = 0$, $EX^2 < \infty$.

This is a foundamental theorm in probability theory and has been intensively investigated by many authors in the past decades. One of the most important results is Baum and Katz [1] law of large numbers, which states that, for $p < 2$ and $r \geq p$,

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P(|S_n| > \varepsilon n^{\frac{1}{p}}) < \infty$$

if and only if $E|X|^r < \infty, r \geq 1$, and $EX = 0$. There have been many extensions in various directions. Two of them are Chow and Lai ([4], [5]) proposed a two sided estimate and Petrov [11].

Lemma 2.1. [9] *Let X and Y are NQD, then*
(i) $EXY \leq EXEY$;
(ii) $P(X > x, Y > y) \leq P(X > x)P(Y > y)$, for all $x, y \in R$;
(iii) if $f(x), g(x)$ are non-decreasing (or non-increasing) functions, then $f(X), g(X)$ are also NQD.

Lemma 2.2. *Let $\{Y_i, i \geq 1\}$ be a sequence of centered pairwise NQD random variables and $E|Y_i|^2 < \infty$ for every $i \geq 1$. Then there exists C, such that*

$$E|\sum_{i=1}^{n} Y_i|^2 \leq \sum_{i=1}^{n} E|Y_i|^2$$

$$E \max_{1 \leq k \leq n} |\sum_{i=1}^{k} Y_i|^2 \leq C \log^2 n \sum_{i=1}^{n} E|Y_i|^2$$

.

Proof. By Lemma 2.1 and $EY_i = 0$, then

$$E|\sum_{i=1}^{n} Y_i|^2 \leq \sum_{i=1}^{n} E|Y_i|^2 + 2 \sum_{1 \leq i < j \leq n} EY_i EY_j \leq \sum_{i=1}^{n} E|Y_i|^2.$$

By $E|\sum_{i=1}^{n} Y_i|^2 \leq \sum_{i=1}^{n} E|Y_i|^2$ and Theorem 2.4.1 in Stout [12], we have

$$E \max_{1 \leq k \leq n} |\sum_{i=1}^{k} Y_i|^2 \leq C(\frac{\log(2n)}{\log 2})^2 \sum_{i=1}^{n} E|Y_i|^2 \leq C \log^2 n \sum_{i=1}^{n} E|Y_i|^2.$$

Now we give the concept of $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is h- integrable concerning the array of constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$.

Definition 2.3. [3] *Let* $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of random variables and* $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *an array of constants with* $\sum_{k=u_n}^{v_n} |a_{nk}| \leq C$ *for all* $n \in N$ *and some constant* $C > 0$. *Let moreover* $\{h(n), n \geq 1\}$ *be an increasing sequence of positive constants with* $h_n \uparrow \infty$ *as* $n \uparrow \infty$. *The array* $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *is said to be* h- *integrable concerning the array of constants* $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *if the following two conditions hold:*

$$\limsup_{n \to \infty} \sum_{k=u_n}^{v_n} |a_{nk}| E|X_{nk}| < \infty$$

and

$$\lim_{n \to \infty} \sum_{k=u_n}^{v_n} |a_{nk}| E|X_{nk}| I(|X_{nk}| > h(n)) = 0.$$

We inspired by the concept of $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is h- integrable concerning the array of constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ that we get the following Theorem.

Theorem 2.4 *Let* $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of rowwise pairwise NQD random variables and* $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of constants with* $\sum_{k=u_n}^{v_n} |a_{nk}| \leq C$ *for all* $n \in N$ *and some constant* $C > 0$. *Let moreover* $\{h(n), n \geq 1\}$ *be an increasing sequence of positive constants with* $h_n \uparrow \infty$ *as* $n \uparrow \infty$. *Suppose that*
(a) $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *is* h- *integrable concerning the array of constants* $\{a_{nk}\}$,
(b) $h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0$ *as* $n \to \infty$,
(c) $h(n) \geq C(\log n)^{1+\delta}$ *for some* $\delta > 0$.
Let $T_n = \sum_{k=u_n}^{v_n} a_{nk} X_{nk}$, *for all* $n \geq 1$ *and* $EX_{nk} = 0$, *for all* $n \geq 1, u_n \leq k \leq v_n$. *Then*

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} n^{-1} P(\max_{1 \leq j \leq n} |T_j| > \varepsilon) < \infty. \tag{2.1}$$

Proof. $\forall i \geq 1$, define $X_i^{(n)} = X_{ni} I(|a_{ni} X_{ni}| \leq h(n)) + h(n) I(|a_{ni} X_{ni}| > h(n)) - h(n) I(|a_{ni} X_i| < -h(n))$, $T_j^{(n)} = \sum_{i=1}^{j} (X_i^{(n)} - EX_i^{(n)})$,

then $\forall \varepsilon > 0$,

$$P(\max_{1 \le j \le n} |T_j| > \varepsilon) \le P(\max_{1 \le j \le n} |a_{nj} X_{nj}| > h(n)) + P(\max_{1 \le j \le n} |T_j^{(n)}| > \varepsilon - \max_{1 \le j \le n} |\sum_{i=1}^{j} EX_i^{(n)}|). \ (2.2)$$

First we show that

$$\max_{1 \le j \le n} |\sum_{i=1}^{j} EX_i^{(n)}| \to 0, \, as \, n \to \infty. \tag{2.3}$$

In fact, by (a) and (b), then

$$\max_{1 \le j \le n} |\sum_{i=1}^{j} EX_i^{(n)}|$$

$$= \max_{1 \le j \le n} |\sum_{i=1}^{j} E[X_{ni} I(|a_{ni} X_{ni}| \le h(n)) + h(n) I(|a_{ni} X_{ni}| > h(n)) - h(n) I(|a_{ni} X_{ni}| < -h(n))]|$$

$$\le \max_{1 \le j \le n} |\sum_{i=1}^{j} EX_{ni} I(|X_{ni}| > h(n))| + h(n) \sum_{j=1}^{n} P(|X_{nj}| > h(n)) \to 0, \, as \, n \to \infty. \tag{2.4}$$

From (2.4), hence (2.3) is true. From (2.2) and (2.3), it follows that for n large enough

$$P(\max_{1 \le j \le n} |T_j| > \varepsilon) \le P(\max_{1 \le j \le n} |a_{nj} X_{nj}| > h(n)) + P(\max_{1 \le j \le n} |T_j^{(n)}| > \frac{\varepsilon}{2}).$$

Hence we need only to prove that

$$I =: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|a_{nj} X_{nj}| > h(n)) < \infty,$$

$$II =: \sum_{n=1}^{\infty} n^{-1} P(\max_{1 \le j \le n} |T_j^{(n)}| > \frac{\varepsilon}{2}) < \infty. \tag{2.5}$$

By Markov inequality, (a), (b) and (c), it follows easily that

$$I = \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|a_{nj} X_{nj}| > h(n))$$

$$\le \sum_{n=1}^{\infty} n^{-1} \sum_{j=u_n}^{v_n} P(|a_{nj} X_{nj}| > h(n))$$

$$\le \sum_{n=1}^{\infty} n^{-1} \sum_{j=u_n}^{v_n} \frac{E|a_{nj} X_{nj}|}{h(n)}$$

$$\le C \sum_{n=1}^{\infty} n^{-1} (\log n)^{-1-\delta} < \infty. \tag{2.6}$$

By Markov inequality, Lemma 2.2, (a), (b) and (c), then

$$
\begin{aligned}
II \quad &= \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1 \le j \le n} |T_j^{(n)}| > \frac{\varepsilon}{2}\Big) \\
&\le \sum_{n=1}^{\infty} n^{-1} (\varepsilon/2)^{-2} E \max_{1 \le j \le n} |T_j^{(n)}|^2 \\
&\le C \sum_{n=1}^{\infty} n^{-1} (\log n)^2 \sum_{j=1}^{n} E|X_j^{(n)}|^2 \\
&\le C \sum_{n=1}^{\infty} n^{-1} (\log n)^2 \sum_{i=1}^{n} E|X_{ni} I(|a_{ni} X_{ni}| \le h(n)) + h(n) I(|a_{ni} X_{ni}| > h(n))|^2 \\
&< \infty.
\end{aligned}
\tag{2.7}
$$

Now we complete the proof of Theorem 2.4.

Corollary 2.5. *Under the conditions of Theorem* 2.1, *then*

$$
\lim_{n \to \infty} |T_n| = 0 \quad a.s.
$$

Proof. By (2.1), we have

$$
\begin{aligned}
\infty \quad &> \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1 \le j \le n} |T_j| > \varepsilon\Big) \\
&= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\Big(\max_{1 \le j \le n} |T_j| > \varepsilon\Big) \\
&\ge \frac{1}{2} \sum_{i=1}^{\infty} P\Big(\max_{1 \le j \le 2^i} |T_j| > \varepsilon\Big).
\end{aligned}
$$

By Borel-Cantelli Lemma, we have

$$
P\Big(\max_{1 \le j \le 2^i} |T_j| > \varepsilon \, i.o.\Big) = 0.
$$

Hence

$$
\lim_{i \to \infty} \max_{1 \le j \le 2^i} |T_j| = 0 \quad a.s.
$$

and using

$$
\max_{2^{i-1} \le n < 2^i} |T_n| \le \max_{1 \le j \le 2^i} |T_j|,
$$

We have

$$
\lim_{n \to \infty} |T_n| = 0 \quad a.s.
$$

Now we complete the proof of Corollary 2.5.

Remark Corollary 2.5. generalizes the results of Cabrera and Volodin [3].

3. CONCLUDING REMARKS

In this paper, some complete convergences and strong laws of large numbers for a weighted sums sequence of pairwise NQD random variables are obtained.

REFERENCES

1. L.E. BAUM AND M. KATZ, "Convergence rates in the law of large numbers", *Trans. Amer. Math. Soc.* **120** (1965), 108–123.
2. A. BOZORGNIA, R.F. PATTERSON AND R. L. TAYLOR, " On the strong laws of large numbers for arrays of rowwise independent random elements", *J. Math. and Math. Sci.* **16** (1993), 587–592.
3. M.O. CABRERA AND A.I. VOLODIN, "Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability", *J. Math. Anal. Appl.* **305** (2005), 644–658.
4. Y.S. CHOW AND T.L. LAI, "Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings", *Trans. Amer. Math. Soc.* **208** (1975), 51–72.
5. Y.S. CHOW AND T.L. LAI, "Paley-Type inequalities and convergence rates related to the law of large numbers and extended renewal theory", *Z. Wahrsch. Verw. Geb.* **45** (1978), 1–19.
6. P. ERDÖS, "On a theorem of Hsu-Robbins", *Ann. Math. Statist.* **20** (1949), 286–291.
7. P.L. HSU AND H.ROBBINS, "Complete convergence and the law of larege numbers", *Proc. Nat. Acad. Sci. (USA)*, **33**(2) 1947, 25–31.
8. D.K. JOAG AND F. PROSCHAN, "Negative associated of random variables with application", *Ann. Math. Statist.* **11** (1983), 286–295.
9. E.L. LEHMANN, "Some concepts of dependence", *Ann. Math. Statist.* **37** (1966), 1137–1153.
10. P. MATULA, "A note on the almost sure convergence of sums of negatively dependence random sequences", *Statist. Probab. Lett.* **15**(3) (1992), 209–213.
11. V.V. PETROV, *Limit theorems of probability theory sequences of independent random variables*, Oxford, Oxford Science Publications, 1995.

12. W. Stout, *Almost sure convergence*, New York, Academic Press, 1974.

13. Q.Y. Wu, "Convergence properties of pairwise NQD random sequences", *Acta Math. Sinica* **45** (2002), 617–624.

Guang-hui Cai : Department of Mathematics and Statistics, Zhejiang Gongshang University, Hangzhou 310035, P. R. China.
E-mail: cghzju@sohu.com