

STABILITY OF INTEGRODIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. Sufficient conditions have been derived for the asymptotic stability of the trivial solutions of a class of linear integrodifferential equations under impulsive perturbations. The sufficient conditions are formulated in terms of the parameters of the equations such that in the absence of the impulsive effects, these conditions are reduced to those of the non-impulsive equations.

1. INTRODUCTION

There are several dynamical systems which display characteristic corresponding to both continuous and discrete processes. These systems at certain instants of time are subjected to rapid changes represented by instantaneous jumps. Appropriate mathematical models for such systems are impulsive differential equations.

A number of biological neural networks and bursting rhythm models in physiology, optimal control models in economic dynamics, signal processing systems with frequency modulation and motion of flying objects, all incorporate abrupt and instantaneous changes in the state of the underlying dynamic system (see for instance [1, 6, 7, 8, 10]). These areas provide some examples of impulsive phenomena. Impulsive phenomena can also be found in the fields of electronics, population dynamics, economics, biology, automatic control systems, computer networking, robotics and telecommunications. It has thus become necessary to develop the study of a new class of dynamic systems to model phenomena which are subjected to sudden changes in the state of the systems.

The study of the stability of impulsive differential equations is much more difficult than that of ordinary differential equations because there is no “elegant

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description of properties of an impulsive system in terms of the eigenvalues of the matrix of the system as we have for a system of ordinary differential equations" ([12, p. 52]). Thus even for linear impulsive differential equations there is no likelihood of deriving necessary and sufficient conditions for stability of solutions.

In this paper we investigate the stability of a class of linear integrodifferential equations subjected to impulsive perturbations. We reduce the study of impulsive integrodifferential equations to one of a comparison equation with no impulses by using the variation of constants formula together with integral inequalities for piecewise continuous functions developed by Azbelev and Tsaliuk (see [13]). The resulting comparison equation is in the form of an integral equation whose solution we embed in the solution space of an associated differential equation. Thus our approach leads to the determination of sufficient conditions for the stability of equilibrium of the impulsive integrodifferential equations through a study of the non-impulsive integrodifferential equations. The sufficient conditions obtained are easy to verify since they are expressed in term of the parameters of the systems. In the absence of impulses, these conditions reduce to those of the non-impulsive systems.

2. PRELIMINARIES

Consider scalar impulsive system given by

$$\left. \begin{aligned} \frac{dx(t)}{dt} &= -ax(t), \quad t \neq \tau_j, \quad j = 1, 2, 3, \dots \\ x(0+) &= x_0 \\ x(\tau_j+) &= (1 + p_j)x(\tau_j-), \quad 0 \leq p_j \leq p, \quad j = 1, 2, 3, \dots \\ & \quad 0 < \tau_1 < \tau_2 < \dots < \tau_j < \dots, \\ & \quad \tau_j \rightarrow \infty \text{ as } j \rightarrow \infty \end{aligned} \right\} \quad (1)$$

in which a and $p = \max\{p_j\}$, $j = 1, 2, 3, \dots$ are real numbers; τ_j , $j = 1, 2, 3, \dots$ are the time instant at which the jump occurs; $x(\tau_j+)$ and $x(\tau_j-)$ are defined by

$$x(\tau_j+) = \lim_{h \rightarrow 0+} x(\tau_j + h), \quad x(\tau_j-) = \lim_{h \rightarrow 0+} x(\tau_j - h).$$

Furthermore we assume that the impulsive perturbation do not occur too often in the sense that there exists a positive number θ such that

$$\tau_{j+1} - \tau_j \geq \theta > 0, \quad j = 0, 1, 2, \dots$$

Let $i[t, t_0]$ denote the number of impulses occurring during the time interval $[t_0, t]$. Then it can be shown that

$$i[t, t_0] - 1 < \frac{t - t_0}{\theta}, \quad t > t_0. \quad (2)$$

Solutions of the system (1) is given by

$$x(t) = e^{-at} x_0 \prod_{k=1}^{k=j} (1 + p_k), \quad \tau_j < t \leq \tau_{j+1}, \quad j = 0, 1, 2, \dots \quad (3)$$

in which we use the notation $\prod_{k=m}^n = 1$ when $n < m$. From (3) we derive that

$$|x(t)| \leq e^{-at} |x_0| (1 + p)^{i[t,0]}, \quad \tau_j < t \leq \tau_{j+1}, \quad j = 0, 1, 2, \dots \quad (4)$$

where $i[t, 0]$ denotes the number of impulsive jumps in the state variable contained in the interval $[0, t]$. Using (2) with $t_0 = 0$ we obtain from (4) that

$$|x(t)| \leq e^{-\left(a - \frac{1}{\theta} \ln(1+p)\right)t} (1 + p) |x_0|, \quad t > 0. \quad (5)$$

If $W(t, s)$, $0 \leq s < t$ denotes the fundamental solution of the system (1), then we have the estimate

$$|W(t, s)| \leq (1 + p) e^{-q(t-s)}, \quad q = a - \frac{1}{\theta} \ln(1 + p), \quad t > s \geq 0. \quad (6)$$

Consider the initial value problem for the nonhomogeneous impulsive differential system

$$\left. \begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + f(t, x), \quad t \neq \tau_i, \quad i = 1, 2, 3, \dots \\ \Delta x|_{t=\tau_i} &= B_i x(\tau_i^-), \quad i = 1, 2, 3, \dots \\ x(t_0+) &= x_0 \end{aligned} \right\} \quad (7)$$

under the following assumptions:

- (i) $t_0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_k < \dots$ and $\lim_{k \rightarrow \infty} \tau_k = \infty$;
- (ii) A is an $n \times n$ constant matrix;
- (iii) $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous on $[\tau_{k-1}, \tau_k) \times \mathbb{R}^n$ and for every $x \in \mathbb{R}^n$, $k = 1, 2, 3, \dots$

$$\lim_{(t,y) \rightarrow (\tau_k, x)} f(t, y) \quad \text{exists for } t > \tau_k;$$

- (iv) for every $k = 1, 2, 3, \dots$, B_k is a constant $n \times n$ matrix such that $(I + B_k)$ is nonsingular.

Theorem 2.1. *Suppose the assumptions (i)–(iv) hold. Let $x(t)$ be any solution of the nonhomogeneous initial value problem (7) existing on $[t_0, \infty)$ and let $W(t, s)$ be the fundamental matrix solution of the homogeneous initial value problem where*

$$W(t, s) = X(t, t_0) X^{-1}(s, t_0). \quad (8)$$

Then $x(t)$ satisfies the integral equation

$$x(t) = W(t, t_0)x_0 + \int_{t_0}^t W(t, s)f(s, x(s)) ds, \quad t > t_0. \quad (9)$$

Proof. Let $X(t, t_0)$ be the fundamental matrix solution associated with

$$\left. \begin{aligned} \frac{dx(t)}{dt} &= Ax(t), \quad t \neq \tau_i, \quad i = 1, 2, 3, \dots \\ \Delta x|_{t=\tau_i} &= B_i x(\tau_i-), \quad i = 1, 2, 3, \dots \\ x(t_0+) &= x_0 \end{aligned} \right\}$$

where $t_0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_i < \dots$ and $t_i \rightarrow \infty$ as $i \rightarrow \infty$. We let

$$x(t) = X(t, t_0)y(t) \quad (10)$$

be a solution of the nonhomogeneous linear system (7).

We have by direct verification that

$$\frac{dx}{dt} = \frac{dX}{dt} y(t) + X(t, t_0) \frac{dy}{dt} = AX(t, t_0)y(t) + f(t, x(t)), \quad t \neq \tau_i$$

which leads to the system

$$\left. \begin{aligned} \frac{dy(t)}{dt} &= X^{-1}(t, t_0)f(t, x(t)) \\ \Delta y|_{t=\tau_i} &= X^{-1}(\tau_i, t_0)B_i x(\tau_i-), \quad i = 1, 2, 3, \dots \\ y(t_0+) &= X^{-1}(t_0, t_0)x(t_0) \end{aligned} \right\}$$

so that

$$y(t) = y(t_0) + \int_{t_0}^t X^{-1}(s, t_0)f(s, x(s)) ds$$

and hence using (8) and (10) we obtain

$$\begin{aligned} x(t) &= X(t, t_0)y(t_0) + \int_{t_0}^t X(t, t_0)X^{-1}(s, t_0)f(s, x(s)) ds \\ &= W(t, t_0)x(t_0) + \int_{t_0}^t W(t, s)f(s, x(s)) ds, \quad t > t_0. \end{aligned}$$

The proof is complete.

Our result in the next section is depend on the result of Azbelev and Tsaliuk (see [13]) which concerns with the comparison of solutions of integral inequalities containing piecewise continuous functions.

3. STABILITY ANALYSIS

In this section we shall obtain sufficient conditions for the asymptotic stability of a class of linear integrodifferential equations subjected to impulsive perturbations.

We consider the scalar impulsive integrodifferential system given by the following:

$$\left. \begin{aligned} \frac{dx(t)}{dt} &= -ax(t) + b\alpha \int_0^t e^{-\alpha(t-s)}x(s) ds, \quad t \neq \tau_i \\ x(0+) &= x_0 \\ \Delta x|_{t=\tau_i} &= x(\tau_i+) - x(\tau_i-) = p_i x(\tau_i-), \quad 0 \leq p_i \leq p, \quad i = 1, 2, 3, \dots \\ &0 < \tau_1 < \tau_2 < \dots < \tau_i < \dots, \quad \tau_i \rightarrow \infty \text{ as } i \rightarrow \infty \\ &\tau_{i+1} - \tau_i \geq \theta > 0, \quad i = 0, 1, 2, \dots \end{aligned} \right\} \quad (11)$$

in which α denotes a nonnegative real number; a , b and $p = \max\{p_i\}$, $i = 1, 2, 3, \dots$ are real numbers. The following result provides sufficient condition for the asymptotic stability of the trivial solution of the system (11).

Theorem 3.1. *Suppose the parameters a , b , p , and θ in the system (11) satisfy*

$$a - \frac{1}{\theta} \ln(1+p) > |b|(1+p), \quad (12)$$

then the trivial solution of the impulsive integrodifferential system (11) is asymptotically stable.

Proof. Consider the system (11) as a perturbation of the one in which $\alpha = 0$. Accordingly, we consider the unperturbed system

$$\left. \begin{aligned} \frac{du(t)}{dt} &= -au(t), \quad t \neq \tau_i \\ u(0+) &= x_0 \\ u(\tau_i+) &= (1+p_i)u(\tau_i-), \quad 0 \leq p_i \leq p, \quad i = 1, 2, 3, \dots \end{aligned} \right\}$$

From (4) we have

$$|u(t)| \leq e^{-at}|x_0|(1+p)^{i[t,0]}, \quad \tau_i < t \leq \tau_{i+1}, \quad i = 0, 1, 2, \dots$$

in which $i[t,0]$ denotes the number of impulsive jumps in the state variable contained in the interval $[0, t]$. Furthermore, from (5) we obtain that

$$|u(t)| \leq e^{-\{a - \frac{1}{\theta} \ln(1+p)\}t}(1+p)|x_0|, \quad t > 0.$$

If $W(t, s)$ denotes the fundamental matrix solution of this simplified system where $0 \leq s < t$, then following (6) we have the estimate

$$|W(t, s)| \leq (1+p)e^{-q(t-s)}, \quad q = a - \frac{1}{\theta} \ln(1+p), \quad t > s \geq 0.$$

By virtue of (9), any solution of the impulsive perturbed system is given by

$$x(t) = W(t, 0)x_0 + b\alpha \int_0^t W(t, s) \left(\int_0^s e^{-\alpha(s-u)} x(u) du \right) ds$$

which leads to the estimation

$$|x(t)| \leq (1+p)e^{-qt}x_0 + |b|\alpha \int_0^t (1+p)e^{-q(t-s)} \left(\int_0^s e^{-\alpha(s-u)} |x(u)| du \right) ds$$

or equivalently,

$$|x(t)|e^{qt} \leq (1+p)x_0 + |b|\alpha(1+p) \int_0^t \int_0^s e^{-\alpha(s-u)} e^{qs} e^{-qu} |x(u)| du ds.$$

We can now use theorem of Azbelev and Tsaliuk (see [13]) so as to set up a comparison integral equation. It is found that $|x(t)|e^{qt}$ is bounded above by $v(t)$ where $v(t)$ is a continuous solution of the integral equation

$$v(t) = (1+p)x_0 + b\alpha(1+p) \int_0^t \int_0^s e^{-(\alpha-q)(s-u)} v(u) du ds. \quad (13)$$

We now investigate the asymptotic behaviour of (t) . From the integral equation governing $v(t)$ in (13) we find that $v(t)$ belongs to the solution space of the differential system

$$\begin{aligned} \frac{dv(t)}{dt} &= |b|\alpha(1+p) \int_0^t e^{-(\alpha-q)(t-u)} v(u) du \\ \frac{d^2v(t)}{dt^2} &= |b|\alpha(1+p)v - (\alpha-q)|b|\alpha(1+p) \int_0^t e^{-(\alpha-q)(t-u)} v(u) du \\ &= |b|\alpha(1+p)v - (\alpha-q) \frac{dv(t)}{dt}. \end{aligned}$$

This system can be put into first order system

$$\begin{aligned} \frac{dv}{dt} &= u \\ \frac{du}{dt} &= |b|\alpha(1+p)v - (\alpha-q)u \end{aligned}$$

whose characteristic equation is given by

$$\mu^2 + (\alpha-q)\mu - |b|\alpha(1+p) = 0.$$

We note that any solution v of the integral equation (13) is of the form

$$v(t) = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t}, \quad t > 0.$$

The asymptotic behaviour of $x(t)$ is given by

$$\begin{aligned} x(t) &\leq c_1 e^{(\mu_1 - q)t} + c_2 e^{(\mu_2 - q)t} \\ &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \end{aligned}$$

where λ_1 and λ_2 are the roots of the equation

$$(\lambda + q)^2 + (\lambda + q)(\alpha - q) - |b|\alpha(1 + p) = 0$$

which is equivalent to

$$\lambda^2 + \lambda(\alpha + q) + \alpha\{q - |b|(1 + p)\} = 0.$$

Applying Routh-Hurwitz criteria, it follows that if

$$q - |b|(1 + p) > 0 \quad \text{or} \quad a - \frac{1}{\theta} \ln(1 + p) - |b|(1 + p) > 0,$$

which is the same condition as in (12), then the trivial solution of the impulsive integrodifferential system is asymptotically stable. This completes the proof.

In the above proof, we have reduced the integral equation into the differential system. This procedure is based on the work of [4] that contains a necessary and sufficient condition for the reducibility of a functional differential (or integrodifferential) equation to a system of ordinary differential equations. The method of [4] has been used by [3, 9, 11, 14] to reduce integrodifferential equations arising in population dynamics to systems of ordinary differential equations. Stability questions for ordinary differential equations are then easily answered using the Routh-Hurwitz criteria. The validity of such a method of reduction has been established by [2].

Next, we consider the impulsive integrodifferential system

$$\left. \begin{aligned} \frac{dx(t)}{dt} &= -ax(t) + b \int_0^\infty K(s)x(t-s) ds, \quad t \neq \tau_i \\ x(s) &= \varphi(s), \quad s \leq 0, \quad \varphi \text{ is bounded and continuous on } (-\infty, 0] \\ \Delta x|_{t=\tau_i} &= p_i x(\tau_i^-), \quad 0 \leq p_i \leq p, \quad i = 1, 2, 3, \dots \\ 0 &< \tau_1 < \tau_2 < \dots < \tau_i < \dots, \quad \tau_i \rightarrow \infty \quad \text{as } i \rightarrow \infty \\ \tau_{i+1} - \tau_i &\geq \theta > 0, \quad i = 0, 1, 2, \dots \end{aligned} \right\} \quad (14)$$

where a , b and $p = \max\{p_i\}$, $i = 1, 2, 3, \dots$ are real numbers. The kernel $K : [0, \infty) \rightarrow [0, \infty)$ is bounded and continuous on $[0, \infty)$ and is such that

$$\int_0^\infty K(s) ds < \infty \quad \text{and} \quad \int_0^\infty sK(s) ds < \infty. \quad (15)$$

The following theorem provides condition that guarantees the asymptotic stability of the system (14).

Theorem 3.2. *Suppose the delay kernel K , the parameters a , b , θ , and p are such that*

$$a - \frac{1}{\theta} \ln(1+p) > |b|(1+p) \int_0^\infty K(s) ds, \quad (16)$$

then the trivial solution of the impulsive system (14) is asymptotically stable.

Proof. From (9), any solution of the impulsive integrodifferential system (14) is a solution of the integral equation

$$x(t) = W(t,0)\varphi(0) + b \int_0^t W(t,s) \left(\int_0^\infty K(u)x(s-u) du \right) ds \quad (17)$$

where $W(t,s)$ denotes the fundamental solution associated with the unperturbed impulsive differential system. By using the estimates of the fundamental solution, we obtain from (17) that

$$|x(t)| \leq (1+p)e^{-qt}\varphi(0) + |b|(1+p) \int_0^t e^{-q(t-s)} \int_0^\infty K(u)|x(s-u)| du ds.$$

where

$$q = a - \frac{1}{\theta} \ln(1+p). \quad (18)$$

We observe that $|x(t)| \leq v(t)$ where by theorem of Azbelev and Tsaliuk (see [13]), $v(t)$ is a solution of

$$v(t) = (1+p)e^{-qt}\varphi(0) + |b|(1+p) \int_0^t e^{-q(t-s)} \int_0^\infty K(u)v(s-u) du ds$$

and such a $v(t)$ is also a solution of

$$\begin{aligned} \frac{dv(t)}{dt} &= -q(1+p)e^{-qt}\varphi(0) + |b|(1+p) \int_0^\infty K(u)v(t-u) du - \\ &\quad q|b|(1+p) \int_0^t e^{-q(t-s)} \int_0^\infty K(u)v(s-u) du ds \\ &= -qv + |b|(1+p) \int_0^\infty K(u)v(t-u) du. \end{aligned} \quad (19)$$

We now examine the asymptotic behaviour of the integrodifferential equation (19) by using a Lyapunov functional $V(t)$ defined by

$$V(t) = |v(t)| + |b|(1+p) \int_0^\infty K(s) \left(\int_{t-s}^t |v(u)| du \right) ds.$$

We note that $V(t) > 0$ for $t > 0$ and that

$$V(0) \leq |v(0)| + |b|(1+p) \int_0^\infty K(s) \left(s \sup_{u \in [t-s, t]} |v(u)| \right) ds. \quad (20)$$

From the hypothesis on $K(\cdot)$ given in (15), it will follow from (20) that $V(0)$ is well-defined and $V(0) < \infty$. We compute the upper right derivative $\frac{d^+V}{dt}$ along the solutions of the integrodifferential equation (19) to obtain

$$\begin{aligned} \frac{d^+V}{dt} &\leq -q|v(t)| + |b|(1+p) \int_0^\infty K(s) ds |v(t)| \\ &\leq - \left\{ q - |b|(1+p) \int_0^\infty K(s) ds \right\} |v(t)|. \end{aligned} \quad (21)$$

It follows from (16), (18), and (21) that $V(\cdot)$ is nonincreasing and we derive from (21) that

$$V(t) + \left\{ q - |b|(1+p) \int_0^\infty K(s) ds \right\} \int_0^t |v(u)| du \leq V(0). \quad (22)$$

It then follows that $v(t)$ is bounded. Also we have from (22) that $v(\cdot) \in L_1(0, \infty)$ which together with uniform continuity of v on $(0, \infty)$ implies by Barbalat's lemma (see [5]) that

$$\lim_{t \rightarrow \infty} v(t) = 0.$$

Since $|x(t)| \leq v(t)$, it follows that the trivial solution of the impulsive integrodifferential system is asymptotically stable. This completes the proof.

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