

ON CERTAIN CLASSES OF P -VALENT FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATION AND DIFFERENTIAL OPERATOR

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Abstract. In this paper, we discuss the p -valent functions that satisfy the differential subordinations $\frac{z(I_p(r, \lambda)f(z))^{(j+1)}}{(p-j)(I_p(r, \lambda)f(z))^{(j)}} \prec \frac{a+(aB+(A-B)\beta)z}{a(1+Bz)}$. We also obtain coefficient inequalities, extreme points, integral representation and arithmetic mean. Further we investigate some interesting properties of operators defined on $A_p(r, j, \beta, a, A, B)$.

1. INTRODUCTION

Denote by A the class of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ analytic in $D = \{z \in C : |z| < 1\}$. Also denote A_p the class of all analytic functions of the form

$$f(z) = kz^p + \sum_{n=p-1}^{2p-1} t_{n-p+1} z^{n-p+1} - {}_2F_1(a, b; c; z), \quad |z| < 1 \quad (1)$$

where

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} z^n \\ (a, n) &= \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1, n-1), \quad c > b > 0, c > a+b \quad \text{and} \\ t_{n-p+1} &= \frac{(a, n-p+1)(b, n-p+1)}{(c, n-p+1)(n-p+1)!}, \quad k > 0. \end{aligned}$$

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These functions are analytic in the unit disk D (For details see [1], [5]).

Definition 1.1. A function $f \in A_p$ is said to be in the class $S_p^*(\alpha)$, p -valently starlike functions of order α , if it satisfies $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha$, $(0 \leq \alpha < p, z \in D)$. We note that $S_p^*(0) = S_p^*$, the class of p -valently starlike functions in D . A function $f \in A_p$ is said to be in the class $C_p(\alpha)$ of p -valently convex of order α , if it satisfies $\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha$, $(0 \leq \alpha < p, z \in D)$.

Let $h(z)$ be analytic and $h(0) = 1$. A function $f \in A_p$ is in the class $S_p^*(h)$ if

$$\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in \Delta. \quad (2)$$

The class $S_p^*(h)$ and a corresponding convex class $C_p(h)$ are defined by Ma and Minda [6]. But results about the convex class can be obtained easily from the corresponding result of functions in $S_p^*(h)$. If

$$h(z) = \frac{1+z}{1-z}, \quad (3)$$

then the classes reduce to the usual classes of starlike and convex functions. If $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leq \alpha < p$, then the classes reduce to the usual classes of starlike and convex functions of order α . If $h(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, then the classes reduce to the class of Janowski starlike function $S_p^*[A, B]$ defined by

$$S_p^*[A, B] = \left\{ f \in A_p : \frac{zf'}{f} \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1, z \in D \right\}. \quad (4)$$

If $h(z) = \left(\frac{1+z}{1-z} \right)^\alpha$, $0 < \alpha \leq 1$, then the classes reduce to the classes of strongly starlike and convex function of order α that consists of univalent functions $f \in A$ satisfying

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{z}, \quad 0 < \alpha \leq 1, z \in \Delta$$

or equivalently we have

$$SS^*(\alpha) = \left\{ f \in A_p : \frac{zf'}{f} \prec \left(\frac{1+z}{1-z} \right)^\alpha, \quad 0 < \alpha \leq 1, z \in \Delta \right\}. \quad (5)$$

Obradović and Owa [8], Silverman [11], Obradović and Tuneski [9] and Tuneski [12] have studied the properties of classes of functions defined in terms of the ratio of $1 + \frac{zf''(z)}{f'(z)}$ and $\frac{zf'(z)}{f(z)}$.

Definition 1.2. A function $f \in A_p$ is said to be p -valent Bazilevic of type η and order α if there exists a function $g \in S_p^*$ such that

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f^{1-\eta}(z)g^\eta(z)} \right\} > \alpha, \quad (z \in \Delta) \quad (6)$$

for some $\eta(\eta \geq 0)$ and $\alpha(0 \leq \alpha < p)$. We denote by $B_p(\eta, \alpha)$, the subclass of A_p consisting of all such functions. In particular, a function in $B_p(1, \alpha) = B_p(\alpha)$ is said to be p -valently close-to-convex of order α in Δ .

Definition 1.3. [7] For two functions f and g , analytic in Δ we say f is subordinate to g denoted by $f \prec g$ if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \Delta$. In particular, if the function g is univalent in Δ , the above subordination is equivalent to $f(0) = g(0)$, $f(\Delta) \subset g(\Delta)$. Also, we say that g is superordinate to f .

Using the techniques of Cho and Srivastava[4], Cho and Kim[3] and Uralegadi and Somanatha[12] we define the following transformation.

Definition 1.4. We define the multiplier transformation operator $I_p(r, \lambda)$ on the infinite series $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$ as

$$I_p(r, \lambda)f(z) = z^p - \sum_{n=1+p}^{\infty} \left(\frac{n+\lambda}{p+\lambda} \right)^r a_n z^n, \quad (\lambda \geq 0). \quad (7)$$

We note that Sălăgean derivative operators [9] is closely related to the operators $I_p(r, \lambda)$ when the coefficient of $f(z)$ is positive. Also note that the class $I_1(r, 1) = I_r$ [12], $I_1(r, \lambda) = I_r^\lambda$ the classes studied in [4] and [3].

Definition 1.5. For each $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$ we have

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} - \sum_{n=p+1}^{\infty} \frac{n!}{(n-j)!} a_n z^{n-j} \quad (8)$$

where $n, p \in N$, $p > j$, and $j \in N_0 = \{0\} \cup N$. For $j = 0$ we have $f^{(0)}(z) = f(z)$.

Definition 1.6. A function $f \in A_p$ is said to be in the class $A_p(r, j; h)$ if it satisfies

$$\frac{z(I_p(r, \lambda)f(z))^{(j+1)}}{(p-j)(I_p(r, \lambda)f(z))^{(j)}} \prec h(z) \quad (9)$$

where

$$h(z) = 1 + \frac{A-B}{a} \frac{\beta z}{1+Bz}, \quad z \in \Delta,$$

and $-1 \leq B < A \leq 1, 0 < \beta < p, a > 0$, we denote $A_p(r, j; h) = A_p(r, j, \beta, a, A, B)$.

We say that $f(z)$ is superordinate to $h(z)$ if $f(z)$ satisfies the following

$$h(z) \prec \frac{z(I_p(r, \lambda)f(z))^{(j+1)}}{(p-j)(I_p(r, \lambda)f(z))^{(j)}}$$

where $h(z)$ is analytic in Δ and $h(0) = 1$.

We note that if

$$\frac{z(I_p(r, \lambda)f(z))^{(j+1)}}{(I_p(r, \lambda)f(z))^{(j)}} \prec \frac{(p-j)[a + (aB + (A-B)\beta)z]}{a(1+Bz)}$$

so $h(0) = 1$ by choosing $j = r = 0, p = 1$, then $f(z) \in S_p^*(h)$. Also $a = A = \beta = 1, B = -1, f(z) \in S_p^*(1)$. But if $a = \beta = 1$ and $-1 \leq B < A \leq 1$ then $f(z) \in S^*[A, B]$, class of Janowski starlike function. If we put $p = a = \beta = A = 1, B = -1$ then $f(z) \in SS^*(1)$ classes of strongly starlike.

By Definition 1.2., if $g(z) \in S^*$, univalent starlike and $j = r = 0$ and $p = a = A = \beta = 1, B = -1$ and if $Re \left\{ \frac{zf'(z)}{f(z)^{-1}g^2(z)} \right\} > 1$, then $f(z) \in B(2, 1)$ class Bazilevic function of type $\eta = 2$ and order $\alpha = 1$.

2. MAIN RESULTS

In this section we obtain sharp coefficient estimates for functions in $A_p(r, j, \beta, a, A, B)$.

Theorem 2.1. *Let $f(z)$ be of the form (1). Then $f \in A_p(r, j, \beta, a, A, B)$ if and only if*

$$\sum_{m=p+1}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] k_m < 1 \quad (10)$$

where

$$\gamma^r(m, p) = \left(\frac{m+\lambda}{p+\lambda} \right)^r, \quad \lambda \geq 0, \quad p, r \in N$$

and

$$\delta(m, j) = \frac{m!}{(m-j)!}, \quad -1 \leq B < A \leq 1, \quad 0 < \beta < p, \quad j < p.$$

Proof. The function $f(z)$ of the form (1) can be expressed as $f(z) = \kappa z^p - \sum_{n=2p}^{\infty} k_{n-p+1} z^{n-p+1}$, or

$$f(z) = \kappa z^p - \sum_{m=p+1}^{\infty} k_m z^m \quad (11)$$

where $m = n - p + 1$ and $k_m = \frac{(a,m)(b,m)}{(c,m)m!}$, and also we have for all $r, j \in N_0$

$$\begin{aligned} (I_p(r, \lambda) f(z))^{(j)} &= \frac{\kappa p!}{(p-j)!} z^{p-j} - \sum_{m=p+1}^{\infty} \left(\frac{m+\lambda}{p+\lambda} \right)^r \frac{m!}{(m-j)!} k_m z^{m-j} \\ &= \kappa \delta(p, j) z^{p-j} - \sum_{m=p+1}^{\infty} \gamma^r(m, p) \delta(m, j) k_m z^{m-j}. \end{aligned} \quad (12)$$

Let $f(z) \in A_p(r, j, a, \beta, A, B)$ then

$$\left| \frac{az(I_p(r, \lambda) f(z))^{(j+1)} - a(p-j)(I_p(r, \lambda) f(z))^{(j)}}{R(p-j)(I_p(r, \lambda) f(z))^{(j)} - Baz(I_p(r, \lambda) f(z))^{(j+1)}} \right| < 1 \quad (13)$$

where $R = aB + (A - B)\beta$. Now, we can write

$$Re \left\{ \frac{a \sum_{m=p+1}^{\infty} \gamma^r(m, p) (\delta(m, j)(p-j) - \delta(m, j+1)) k_m z^{m-j}}{\beta(A-B)\delta(p, j+1) \kappa z^{p-j} - \sum_{m=p+1}^{\infty} \gamma^r(m, p) (\delta(m, j)(p-j)R - \delta(m, j+1)Ba) k_m z^{m-j}} \right\} < 1.$$

We choose the values of z on the real axis and letting $z \rightarrow 1^-$ then we have

$$\frac{a \sum_{m=p+1}^{\infty} \gamma^r(m, p) (\delta(m, j)(p-j) - \delta(m, j+1)) k_m}{\beta \kappa (A-B)\delta(p, j+1) - \sum_{m=p+1}^{\infty} \gamma^r(m, p) (aB(\delta(m, j)(p-j) - \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)) k_m} < 1,$$

and

$$\begin{aligned} &\sum_{m=p+1}^{\infty} \gamma^r(m, p) [a(1+B)(\delta(m, j)(p-j) \\ &- \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)] k_m < \beta \kappa (A-B)\delta(p, j+1) \end{aligned}$$

Conversely, we assume that the condition (10) holds true. Hence it is sufficient to show that $f \in A_p(r, j, \beta, a, A, B)$, that is to prove that

$$\left| \frac{az(I_p(r, \lambda) f(z))^{(j+1)} - a(p-j)(I_p(r, \lambda) f(z))^{(j)}}{(p-j)R(I_p(r, \lambda) f(z))^{(j)} - Baz(I_p(r, \lambda) f(z))^{(j+1)}} \right| < 1.$$

But we have

$$\begin{aligned}
& \left| \frac{az(I_p(r, \lambda)f(z))^{(j+1)} - a(p-j)(I_p(r, \lambda)f(z))^{(j)}}{(p-j)R(I_p(r, \lambda)f(z))^{(j)} - Baz(I_p(r, \lambda)f(z))^{(j+1)}} \right| \\
&= \left| [a \sum_{m=p+1}^{\infty} \gamma^r(m, p)(\delta(m, j)(p-j) - \delta(m, j+1))k_m z^{m-j}] / \right. \\
& \quad [\beta(A-B)\delta(p, j+1)\kappa z^{p-j} \\
& \quad - \sum_{m=p+1}^{\infty} \gamma^r(m, p)(aB(\delta(m, j)(p-j) - \delta(m, j+1)) - \\
& \quad \delta(m, j)(A-B)\beta(p-j))k_m] \left. \right| \\
&< \{ [a \sum_{m=p+1}^{\infty} \gamma^r(m, p)(\delta(m, j)(p-j) - \delta(m, j+1))k_m] / \\
& \quad [\beta\kappa(A-B)\delta(p, j+1) \\
& \quad - \sum_{m=p+1}^{\infty} \gamma^r(m, p)(aB(\delta(m, j)(p-j) - \delta(m, j+1)) - \\
& \quad \delta(m, j)(A-B)\beta(p-j))k_m] \} < 1
\end{aligned}$$

and so proof is complete.

The inequality (10) is sharp for the function

$$f(z) = z^p - \frac{\beta\kappa(A-B)\delta(p, j+1)}{\gamma^r(q, p)[a(1+B)(\delta(q, j)(p-j) - \delta(q, j+1)) - \delta(q, j)(A-B)\beta(p-j)]} z^q,$$

with $q \geq 1+p$.

Corollary 2.2. *Let $f \in A_p(r, j, \beta, a, A, B)$ then we have*

$$k_m < \frac{\kappa\beta(A-B)\delta(p, j+1)}{\gamma^r(m, p)[a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)]}, \quad (14)$$

with $m \geq p+1$

In the next theorem we prove that the class $A_p(r, j, \beta, \alpha, A, B)$ is closed under linear combination.

Theorem 2.3. *Let $f_q(z) = \kappa z^p - \sum_{m=p+1}^{\infty} k_{m,q} z^m$ ($q = 1, 2, \dots, t$) be in $A_p(r, j, \beta, a, A, B)$.*

Then the function $F(z) = \sum_{q=1}^t d_q f_q(z)$ where $\sum_{q=1}^t d_q = 1$, is also in $A_p(r, j, \beta, a, A, B)$.

Proof. We have

$$\begin{aligned} F(z) &= \sum_{k=1}^q d_k \left(\kappa z^p - \sum_{m=1+p}^{\infty} k_{m,q} z^m \right) = \kappa z^p - \sum_{q=1}^t d_q \left(\sum_{m=1+p}^{\infty} k_{m,q} z^m \right) \\ &= \kappa z^p - \sum_{m=1+p}^{\infty} \left(\sum_{q=1}^t d_q k_{m,q} \right) z^m. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &\sum_{m=1+p}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] \left(\sum_{q=1}^t d_q d_{m,q} \right) \\ &= \sum_{q=1}^t \left(\sum_{m=1+p}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] k_{m,q} \right) d_q \\ &< \sum_{q=1}^t d_q = 1. \end{aligned}$$

Now we prove that the class $A_p(r, j, \beta, a, A, B)$ is closed under arithmetic mean.

Theorem 2.4. Let $f_j(z) = \kappa z^p - \sum_{m=1+p}^{\infty} k_{m,q} z^m$ ($q = 1, 2, \dots, s$) are in $A_p(r, j, \beta, a, A, B)$.

Then the function $F(z) = \kappa z^p - \sum_{m=1+p}^{\infty} b_m z^m$ where $b_m = \frac{1}{s} \sum_{q=1}^s k_{m,q}$, is also in $A_p(r, j, \beta, a, A, B)$.

Proof. Since $f_j(z) \in A_p(r, j, \beta, a, A, B)$, then by (10) we have

$$\begin{aligned} &\sum_{m=1+p}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] \\ &k_{m,q} < 1, q = 1, 2, 3, \dots, s. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{m=1+p}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] b_m \\ &= \sum_{m=1+p}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] \left(\frac{1}{s} \sum_{q=1}^s k_{m,q} \right) \\ &\leq \frac{1}{s} \sum_{q=1}^s \left(\sum_{m=1+p}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] k_{m,q} \right) \\ &\leq \sum_{q=1}^s \frac{1}{s} = 1 \end{aligned}$$

and this completes the proof.

Theorem 2.5. (Extreme Points) Let $f_p(z) = z^p$ and for $m \geq 1+p$

$$f_m(z) = \kappa z^p - \frac{\beta\kappa(A-B)\delta(p, j+1)}{\gamma^r(m, p)[a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)]} z^m,$$

Then the function $f(z) \in A_p(r, j, \beta, a, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{m=p}^{\infty} \mu_m f_m(z) \quad (15)$$

where $\mu_m \geq 0$ and $\sum_{m=p}^{\infty} \mu_m = 1$.

Proof. Suppose that f can be expressed in the form (15) then we have

$$\begin{aligned} f(z) &= \sum_{m=p}^{\infty} \mu_m f_m(z) \\ &= \mu_p f_p(z) + \sum_{m=p+1}^{\infty} \mu_m f_m(z) \\ &= \mu_p \kappa z^p + \sum_{m=p+1}^{\infty} \mu_m (\kappa z^p - \frac{\beta \kappa (A-B) \delta(p, j+1)}{\gamma^r(m, p) [a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)]} z^m) \\ &= \kappa z^p - \sum_{m=1+p}^{\infty} \frac{\beta \kappa (A-B) \delta(p, j+1)}{\gamma^r(m, p) [a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)]} z^m \mu_m \end{aligned}$$

Consequently

$$\begin{aligned} &\sum_{m=p+1}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta \kappa (A-B) \delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa \delta(p, j+1)} \right] \\ &\quad \frac{\beta \kappa (A-B) \delta(p, j+1)}{\gamma^r(m, p) [a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)]} \mu_m \\ &= \sum_{m=1+p}^{\infty} \mu_m = 1 - \mu_p < 1. \end{aligned}$$

Therefore we conclude the result.

Conversely, let $f \in A_p(r, j, \beta, a, A, B)$ since by (10) we may set

$$\mu_m = \gamma^r(m, p) k_m \frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)}{\beta \kappa (A-B) \delta(p, j+1)},$$

with $m \geq 1+p$. Therefore $\mu_m \geq 0$ and if we set $\mu_p = 1 - \sum_{m=1+p}^{\infty} \mu_m$ then we can write

$$\begin{aligned} f(z) &= \kappa z^p - \sum_{m=1+p}^{\infty} k_m z^m \\ &= \kappa z^p - \sum_{m=1+p}^{\infty} \frac{\beta \kappa (A-B) \delta(p, j+1)}{\gamma^r(m, p) [a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1)) - \delta(m, j)(A-B)\beta(p-j)]} \mu_m z^m \end{aligned}$$

$$\begin{aligned}
&= \kappa z^p - \sum_{m=1+p}^{\infty} \mu_m (\kappa z^p - f_m(z)) \\
&= \kappa z^p (1 - \sum_{m=1+p}^{\infty} \mu_m) - \sum_{m=1+p}^{\infty} \mu_m f_m(z) \\
&= \sum_{m=p}^{\infty} \mu_m f_m(z).
\end{aligned}$$

Remark 2.6. The extreme points of the class $A_p(r, j, \beta, a, A, B)$ are the function $f_p(z), f_{m+p}(z), m \geq 1+p$ as in Theorem 2.1.

In the following theorem, we obtain the integral representation for $A_p(r, j, \beta, a, A, B)$.

Theorem 2.7. Let $f(z) \in A_p(r, j, \beta, a, A, B)$ then

$$f(z) = \exp \left[\int_0^z \frac{p(\psi(t)R + a)}{t(1 + B\psi(t))} dt \right]$$

where $|\psi(z)| < 1, z \in U$ and $R = aB + (A - B)\beta$. Also

$$f(z) = z^p \exp \left[\int_X \log(1 - Bxz) \frac{p(A-B)\beta}{aB} d\mu_{(x)} \right]$$

where $\mu_{(x)}$ is the probability measure on $X = \{x : |x| = 1\}$.

Proof. Set

$$\frac{z(I_p(r, \lambda)f(z))^{(j+1)}}{(p-j)(I_p(r, \lambda)f(z))^{(j)}} = Q(z)$$

Since $f(z) \in A_p(r, j, \beta, a, A, B)$ so $\left| \frac{a(Q(z)-1)}{R-aBQ(z)} \right| < 1$ where $R = aB + (A - B)\beta$.

Consequently we put $\frac{a(Q(z)-1)}{R-aBQ(z)} = \psi(z), |\psi(z)| < 1$.

Finally we can write $Q(z) = \frac{\psi(z)R+a}{a+aB\psi(z)}$ or

$$\frac{(I_p(r, \lambda)f(z))^{(j+1)}}{(I_p(r, \lambda)f(z))^{(j)}} = \frac{(p-j)(\psi(z)R + a)}{az(1 + B\psi(z))}$$

Then we have

$$\begin{aligned}
\log(I_p(r, \lambda)f(z))^{(j)} &= \int_0^z \frac{(p-j)(\psi(t)R + a)}{t(1 + B\psi(t))} dt \\
(I_p(r, \lambda)f(z))^{(j)} &= \exp \left[\int_0^z \frac{(p-j)(\psi(t)R + a)}{t(1 + B\psi(t))} dt \right]
\end{aligned}$$

for $r = j = 0$ we have

$$f(z) = \exp \left[\int_0^z \frac{p(\psi(t)R + a)}{t(1 + B\psi(t))} dt \right].$$

For obtaining the second representation let $X = \{x : |x| = 1\}$ then we have $\frac{a(Q(z)-1)}{R-aBQ(z)} = xz, z \in \Delta$ and then we conclude that

$$\begin{aligned} \frac{(I_p(r, \lambda)(f(z))^{(j+1)})}{(I_p(r, \lambda)f(z))^{(j)}} &= \frac{(p-j)(Rzx + a)}{az(1 + Bxz)} = \frac{(p-j)}{z} + \frac{x(p-j)(A-B)\beta}{a(1 + Bxz)} \\ &= (p-j) \left(\frac{1}{z} + \frac{x(A-B)\beta}{a(1 + Bxz)} \right) \end{aligned}$$

$$\log(I_p(r, \lambda)f(z))^{(j)} = (p-j) \left(\log z + \frac{(A-B)\beta}{aB} \log(1 + Bxz) \right)$$

$$\log \frac{(I_p(r, \lambda)f(z))^{(j)}}{z^{p-j}} = \frac{(p-j)(A-B)\beta}{aB} \log(1 + Bxz)$$

$$(I_p(r, \lambda)f(z))^{(j)} = z^{p-j} \exp \left[\int_X \log(1 - Bxz) \frac{(p-j)(A-B)\beta}{aB} d\mu_{(x)} \right]$$

where $\mu_{(x)}$ is probability measure on X . For $j = r = 0$ we have

$$f(z) = z^p \exp \left[\int_X \log(1 - Bxz) \frac{p(A-B)\beta}{aB} d\mu_{(x)} \right].$$

Now, we introduce an integral operator due to Bernardi [2]

$$L_c(f(z)) = \frac{p+c}{z^c} \int_0^z f(t)t^{c-1}dt, \quad (c > -p)$$

and we study the effect of this operator on class $A_p(r, j, \beta, a, A, B)$.

Theorem 2.8. *If $f \in A_p(r, j, \beta, a, A, B)$ then $L_c(f(z))$ is also in $A_p(r, j, \beta, a, A, B)$.*

Proof. If $f(z) = \kappa z^p - \sum_{m=1+p}^{\infty} k_m z^m$ then

$$\begin{aligned} L_c(f(z)) &= \frac{p+c}{z^c} \int_0^z \left(\kappa t^p - \sum_{m=1+p}^{\infty} k_m t^m \right) t^{c-1} dt \\ &= \kappa z^p - \sum_{m=1+p}^{\infty} \frac{p+c}{m+c} k_m z^m. \end{aligned}$$

Since $m > p$ then $\frac{p+c}{m+c} \leq 1$ so we have

$$\begin{aligned} & \sum_{m=1+p}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] \left(\frac{p+c}{m+c} \right) k_m \\ & \leq \sum_{m=1+p}^{\infty} \gamma^r(m, p) \left[\frac{a(1+B)(\delta(m, j)(p-j) - \delta(m, j+1))}{\beta\kappa(A-B)\delta(p, j+1)} - \frac{\delta(m, j)(p-j)}{\kappa\delta(p, j+1)} \right] k_m < 1. \end{aligned}$$

Thus $L_c(f(z)) \in A_p(r, j, \beta, a, A, B)$.

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