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CALCULATIONS ON THE SUPREMUM OF FUZZY NUMBERS VIA L_p METRICS

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Abstract. In this paper, it is proved that the supremum of a family of fuzzy numbers can be finitely approximated via L_p metrics and the concrete approaches are given. As a byproduct, it is proved that the L_1 metric d_1 defined via cut-set is equivalent to a metric which can be calculated directly via membership functions. Since the L_p metrics are analytic in nature, the results in this paper may have interesting applications in fuzzy analysis. For example, it may provide a method for the computation of various fuzzy-number-valued integrals.

1. INTRODUCTION AND PRELIMINARIES

Since the concept of fuzzy number was first introduced in the 1970's, it has been studied extensively from many different viewpoints. Fuzzy numbers has been used as a basic tool in different parts of fuzzy theory. In [4], it is shown that the endograph metric is approximative with respect to orders on E^1 and it is computable. From [3], we know that for uniformly supported bounded set of fuzzy numbers the L_p metrics and the endograph metric are equivalent. Thus, we can conclude that L_p metrics are also approximative. In [2], a method for calculating supremum and infimum of fuzzy sets via endograph metric is given which resembles the Riemann sum in calculus. In this paper, we will give out the concrete methods to approximate the supremum via L_p metrics.

First of all, we recall the basics of fuzzy numbers. Let R and I be the set of all real numbers and the unit interval respectively.

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Denote $E^1 = \{u|u : R \to I,\ u\ has\ the\ following\ properties\ (i) - (iv)\}$.
(i) u is normal, i.e., $u(x_0) = 1$ for some $x_0 \in R$;
(ii) u is quasiconvex(fuzzy convex), i.e., $u(rx + (1 - r)y) \geq min\{u(x), u(y)\}$ for all $x, y \in R$ and $r \in I$;
(iii) u is upper semicontinuous(u. s. c. for short);
(iv) The topological support u_0 of u is compact, where $u_0 = cl\{x|x \in R, u(x) > 0\}$.

Elements in E^1 are called fuzzy numbers. For $\alpha \in I$, let $u_\alpha = \{x \in R|u(x) \geq \alpha\}$ denote the $\alpha - cut$ of u, then all cuts of u are none-empty closed intervals. For each $\alpha \in I$, let $u_\alpha = [u_\alpha^-, u_\alpha^+]$. For any $u, v\imath E^1$, define:

$$d_p(u, v) = (\int_0^1 (max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\})^p d\alpha)^{\frac{1}{p}},$$

where $p \geq 1$ is an arbitrary real number. Then d_p is a separable but not complete metric on E^1, called L_p metrics. By the definition of L_p metrics, we have the following inequalities.

$$d_p(u, v) \leq (\int_0^1 (|u_\alpha^- - v_\alpha^-| + |u_\alpha^+ - v_\alpha^+|)^p d\alpha)^{\frac{1}{p}} \leq \|u^- - v^-\|_p + \|u^+ - v^+\|_p,$$

where$\| \cdot \|_p = (\int_0^1 | \cdot |^p d\alpha)^{\frac{1}{p}}$.

We note that in the literature, fuzzy numbers can also be equivalently defined as follows: $u = (l_u, r_u)$, where l_u and r_u are functions defined on certain closed intervals $[a_u, c_u]$ and $[c_u, b_u]$ with codomain I respectively, such that $l_u(a_u) = 0$, $l_u(c_u) = 1$, $r_u(c_u) = 1$, $r_u(b_u) = 0$; they are increasing and decreasing respectively, and both are upper u. s. c.. For our need we require that the domains of definition for l_u and r_u are the whole real line R, so that they can be extended uniquely to keep their monotonicity. That is, outside their domains, their values should be either 0 or 1 according to the monotonicity requirement.

For u, $v \in E^1$, define $u \leq v$ if only if $u_\alpha^- \leq v_\alpha^-$, $u_\alpha^+ \leq v_\alpha^+$ for all $\alpha \in I$, or equivalently, $l_u \geq l_v$ and $r_u \leq r_v$. Then $\leq$ is a partial order on E^1.

For other undefined notions, we refer to [1].

Remark 1 *By the definition of order relation on E^1 and the definition of L_p metrics, it is obvious that for $u, v, w \in E^1$, if $u \leq v \leq w$, then $d_p(u, v) \leq d_p(u, w)$, i.e., L_p metrics preserve the order on fuzzy numbers.*

2. CALCULATIONS VIA L_1 METRIC

In this section, we give some concrete calculation methods to approximate the supremum of fuzzy numbers via L_1 metric. Moreover, we prove that the L_1 metric defined via cut-set can be directly studied by a metric on E^1 which is defined via membership functions and give out the calculation approach on the supremum directly via membership function method.

Proposition 1[8] *Let $\{u_t|t \in T\}$ be a family of fuzzy numbers with $\alpha - cut$ representations $\{(u_t)_\alpha|\alpha \in [0, 1]\}$, $t \in T$. If the family is bounded above, and v is the* *supremum of the family, then the cut-set functions of v have the following representation:*

$$\begin{aligned} v_\alpha^- &= \bigvee_{t\in T}(u_t)_\alpha^-, \\ v_\alpha^+ &= \begin{cases} lim_{\alpha'\to\alpha^-}\bigvee_{t\in T}(u_t)_{\alpha'}^+ & \alpha\in Disc(v^+), \\ \bigvee_{t\in T}(u_t)_\alpha^+ & otherwise, \end{cases} \end{aligned}$$

for $\alpha\in(0,1]$, where $Disc(v^+)$ is the set of all discontinuous points of v^+, which is at most countable; while

$$v_0^- = \bigwedge_{\lambda>0}\bigvee_{t\in T}(u_t)_\lambda^-, \qquad v_0^+ = \bigvee_{t\in T}(u_t)_0^+.$$

We define a nested closed intervals $w=\{[w_\alpha^-,w_\alpha^+]|\alpha\in I\}$, where w^- and w^+ are functions defined on I, $w_\alpha^-=\bigvee_{t\in T}(u_t)_\alpha^-$, $w_\alpha^+=\bigvee_{t\in T}(u_t)_\alpha^+$ for $\alpha\in I$. It can be seen that w_α^- is increasing, w_α^+ is decreasing on I, $v_\alpha^-=w_\alpha^-$ for all $\alpha\in(0,1]$ and v_α^+ is the smallest u. s. c. function greater than w_α^+ and they may differ only at the discontinuous points of v_α^+, i.e., $w_\alpha^+=v_\alpha^+$ a.e. on I. It follows that $d_p(w,v)=0$, where $d_p(w,v)$ is defined by the following formula as if w is a fuzzy number:

$$d_p(w,v) = (\int_0^1 (max\{|w_\alpha^- - v_\alpha^-|, |w_\alpha^+ - v_\alpha^+|\})^p d\alpha)^{\frac{1}{p}}.$$

The above representation of supremum is based on the cut-set functions. There is also a representation of supremum based on the membership functions as follows:

Proposition 2[4] *Let $\{u_t|t\in T\}$ be a family of fuzzy numbers. If the family is bounded above, v the supremum of the set, then the membership function of v is given by the following formula:*

$$\begin{aligned} v(x) &= \begin{cases} \bigwedge_{t\in T} l_{u_t}(x), & x<v_1^-; \\ cl(\bigvee r_{u_t}(x)), & otherwise. \end{cases} \\ &= \begin{cases} \bigwedge_{t\in T} l_{u_t}(x), & x<v_1^-; \\ 1, & x=v_1^-; \\ \lim_{x'\to x^-}\bigvee_{t\in T} u_t(x'), & x\in Disc^+(v); \\ \bigvee_{t\in T} u_t(x), & otherwise. \end{cases} \end{aligned}$$

Where $Disc^+(v)$ is the set of all discontinuous points of v greater than v_1^-, which is at most countable since v is quasiconvex as a real function. The closure is taken in the induced fuzzy topological space $(I^R,\omega(\tau))$, where τ is the usual topology on R. (For related concepts on fuzzy topology, we refer to [6] and [7].)

Now, we consider the fuzzy set w' whose membership function is defined as follows:

$$w'(x) = \begin{cases} \bigwedge_{t\in T} l_{u_t}(x) & x\le v_1^-; \\ \bigvee_{t\in T} r_{u_t}(x) & otherwise. \end{cases}$$

Note that $w'(v_1^-) = 1$ and v is the smallest u. s. c. function greater than w'. w' is u. s. c. if only if $w' \in E^1$, i.e., $w' = v$. w' can also be equivalently defined as follows: $w' = (l_{w'}, r_{w'})$, the definitions of $l_{w'}$ and $r_{w'}$ are similar to the case of fuzzy numbers, that is $l_{w'}(x) = \bigwedge_{t \in T} l_{u_t}(x)$, $r_{w'}(x) = \bigvee_{t \in T} r_{u_t}(x)$. By the definition of v, we have $l_{w'} = l_v$ and $r_{w'}(x) \neq r_v(x)$ only if $x \in Disc^+(v)$. Thus $r_{w'}$ and r_v differ at most on a countable set.

Now, we proceed to consider the relation between w and w'. In fact, w does not necessarily correspond to the cut-set function of a fuzzy number, since w^+ may not be left continuous. In general, w may not even be a cut-set function of a fuzzy set. But we can define a fuzzy set w^* on R according to w as follows:

$$w^*(x) = sup\{\alpha \in I : x \in [w_\alpha^-, w_\alpha^+]\}.$$

It is obvious that $w^*(x) = v(x)$ for $x \in R$. In fact, $v(x) = sup\{\alpha \in I : x \in [v_\alpha^-, v_\alpha^+]\}$ and $[w_\alpha^-, w_\alpha^+] \subseteq [v_\alpha^-, v_\alpha^+] \subseteq [w_{\alpha-\varepsilon}^-, w_{\alpha-\varepsilon}^+]$ for $\alpha \in I$ and arbitrary $\varepsilon > 0$ such that $\alpha - \varepsilon \geq 0$. So $w_\alpha^- = (w^*)_\alpha^-$ for all $\alpha \in (0,1]$ and $w_\alpha^+ \neq (w^*)_\alpha^+$ only if $\alpha \in Disc(v^+)$. Since $v(x) \neq w'(x)$ if only $x \in Disc^+(v)$. So $w^*(x) \neq w'(x)$ only if $x \in Disc^+(v)$. The following example shows that $w^+ \neq v^+$ and $w' \neq v$ in general.

Example 1 Let $v = \chi_{[0,1]} + \frac{1}{2}\chi_{[1,2]} + \frac{1}{3}\chi_{[2,3]}$, where χ_A denotes the characteristic function of a set $A \subseteq R$. For $n = 3, 4, \cdots$, let

$$u_n(x) = \begin{cases} 1 & x \in [0, 1 - \frac{1}{n}], \\ \frac{1}{2} & x \in (1 - \frac{1}{n}, 2], \\ \frac{1}{3} - \frac{1}{n} & x \in (2, 3], \\ 0 & otherwise, \end{cases}$$

then, clearly, $v = \bigvee_{n=3}^{+\infty} u_n$,

$$w_\alpha^+ = \bigvee_{n=3}^{+} (u_n)_\alpha^+ = \begin{cases} 3 & \alpha \in [0, \frac{1}{3}), \\ 2 & \alpha \in [\frac{1}{3}, \frac{1}{2}], \\ 1 & \alpha \in (\frac{1}{2}, 1], \end{cases}$$

$$v_\alpha^+ = \begin{cases} & \alpha \in [0, \frac{1}{3}], \\ 2 & \alpha \in (\frac{1}{3}, \frac{1}{2}], \\ 1 & \alpha \in (\frac{1}{2}, 1], \end{cases}$$

and

$$w'(x) = \begin{cases} 1 & x \in [0, 1), \\ \frac{1}{2} & x \in [1, 2], \\ \frac{1}{3} & x \in (2, 3], \\ 0 & otherwise. \end{cases}$$

It follows that $w_{\frac{1}{3}}^+ \neq v_{\frac{1}{3}}^+$, $\frac{1}{3} \in Disc(v^+)$ and $w'(1) \neq v(1)$, $1 \in Disc^+(v)$.

Lemma 1 *Assume that $\{u_n\}$ and v are given as in Proposition 1, w is defined as above, and $u \in E^1$, $u \leq v$. Suppose that support$(v) \subseteq [a, b]$ and $a \leq w_0^- =$*

$\bigvee_{t\in T}(u_t)_0^-$. *If there exists* n *such that* $\frac{b-a}{n} < \frac{\varepsilon}{2}$,

$$w^-_{\frac{i}{n}} < u^-_{\frac{i}{n}} + \frac{\varepsilon}{2}, \quad i = 0, 1, \ldots, n-1; \tag{1}$$

and

$$w^+_{\frac{i}{n}} < u^+_{\frac{i}{n}} + \frac{\varepsilon}{2}, \quad i = 1, \ldots, n, \tag{2}$$

then $d_1(u, v) < 2\varepsilon$.

Proof Since $d_1(w, v) = 0$, so we only need to show that $d_1(u, w) < 2\varepsilon$. First, we show that $\|u^- - w^-\|_1 < \varepsilon$. Define two simple functions h_1 and h'_1 on $[0, 1]$ as follows: $h_1(\alpha) = w^-_{\frac{i}{n}}$, $h'_1(\alpha) = u^-_{\frac{i}{n}}$, for $\alpha \in [\frac{i}{n}, \frac{i+1}{n})$, $i = 0, 1, \ldots, n-1$; $h_1(1) = w^-_1$, $h'_1(1) = u^-_1$. Clearly, by (1) and the monotonicity of w^-_α and u^-_α , we have $h_1(\alpha) \le w^-_\alpha$, $h'_1(\alpha) \le u^-_\alpha$ and $0 \le h_1(\alpha) - h'_1(\alpha) < \frac{\varepsilon}{2}$ for all $\alpha \in [0, 1]$. Then $\int_0^1 (w^-_\alpha - h_1(\alpha))d\alpha = \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} (w^-_\alpha - h_1(\alpha))d\alpha \le \sum_{i=0}^{n-1} (w^-_{\frac{i+1}{n}} - w^-_{\frac{i}{n}})\frac{1}{n} = \frac{1}{n}(w^-_1 - w^-_0) < \frac{1}{n}(b - a) < \frac{\varepsilon}{2}$. Thus, we have

$$\begin{array}{rcl} \|w^- - u^-\|_1 & = & \int_0^1 (w^-_\alpha - h_1(\alpha))d\alpha + \int_0^1 (h_1(\alpha) - u^-_\alpha)d\alpha \\ & \le & \int_0^1 (w^-_\alpha - h_1(\alpha))d\alpha + \int_0^1 (h_1(\alpha) - h'_1(\alpha))d\alpha \\ & < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{array}$$

Second, we show that $\|w^+ - u^+\|_1 < \varepsilon$. Similar to the above case, we also define two simple functions h_2 and h'_2 on $[0, 1]$ as follows: $h_2(0) = w^+_0$, $h'_2(0) = u^+_0$; $h_2(\alpha) = w^+_{\frac{i}{n}}$, $h'_2(\alpha) = w^+_{\frac{i}{n}}$, for $\alpha \in (\frac{i-1}{n}, \frac{i}{n}]$, $i = 1, 2 \ldots, n$. By a similar argument as above, we have $\int_0^1 (w^+_\alpha - h_2(\alpha))d\alpha < \frac{\varepsilon}{2}$ and $\|u^+ - w^+\|_1 < \varepsilon$. Thus, $d_1(u, w) < 2\varepsilon$. So the proof is completed.

Based on the above discussion, we have the following algorithm of computing the supremum of fuzzy numbers via the L_1 metric d_1.

Theorem 1 *Under the hypothesis of Proposition 1.*
(i) Choose a, b *such that* $a \le w^-_0$ *and* $b > v^+_0 = \bigvee_{t\in T}(u_t)^+_0$*;*
(ii) Pick nature number n *such that* $\frac{b-a}{n} < \frac{\varepsilon}{2}$.
For $i = 0, 1, \cdots n-1$*, choose* $t_i \in T$ *such that*

$$w^-_{\frac{i}{n}} < (u_{t_i})^-_{\frac{i}{n}} + \frac{\varepsilon}{2}.$$

For $j = 1, 2, \cdots, n$*, choose* $t_{n+j-1} \in T$ *such that*

$$w^+_{\frac{j}{n}} < (u_{t_{n+j-1}})^+_{\frac{j}{n}} + \frac{\varepsilon}{2}.$$

Let $u = \bigvee_{i=0}^{2n-1} u_{t_i}$*, then* $d_1(u, v) < 2\varepsilon$.

In the following, we consider the case when the family of fuzzy numbers is given by their membership functions. It can be seen that the calculations of approximation with respect to supremum via L_1 metric can be carried out in a similar way as in the cut-set case. By the geometric meaning of integration and the integration variable transformation, we have the following lemma.

Lemma 2 *For $u, v \in E^1$, we have*

$$\|u^- - v^-\|_1 = \int_{min\{u_0^-,\ v_0^-\}}^{max\{u_1^-,\ v_1^-\}} |l_u(x) - l_v(x)| dx \tag{3}$$

and

$$\|u^+ - v^+\|_1 = \int_{min\{u_1^+,\ v_1^+\}}^{max\{u_0^+,\ v_0^+\}} |r_u(x) - r_v(x)| dx. \tag{4}$$

Proof Here, we only prove (3). The proof for (4) is similar. In order to prove this lemma, we resort to the Lebesgue measure of real two-dimensional space R^2. Let

$$A_1 = \{(x, \alpha) \in R^2 | min\{u_\alpha^-, v_\alpha^-\} \le x \le max\{u_\alpha^-, v_\alpha^-\} \text{ and } 0 \le \alpha \le 1\},$$

$$A_2 = \{(x, \alpha) \in R^2 | u_\alpha^- \le x < lim_{\alpha' \to \alpha^+} u_{\alpha'}^- \text{ and } \alpha \in Disc(u^-)\},$$

$$A_3 = \{(x, \alpha) \in R^2 | v_\alpha^- \le x < lim_{\alpha' \to \alpha^+} v_{\alpha'}^- \text{ and } \alpha \in Disc(v^-)\},$$

$$A = A_1 \bigcup A_2 \bigcup A_3$$

and

$$B = \left\{(x, \alpha) \in R^2 | \begin{array}{l} min\{l_u(x), l_v(x)\} \le \alpha \le max\{l_u(x), l_v(x)\} \text{ and} \\ min\{u_0^-, v_0^-\} \le x \le max\{u_1^-, v_1^-\}\} \end{array}\right. .$$

Here $Disc(u^-)$ and $Disc(v^-)$ are the sets of all discontinuous points of u^- and v^- respectively. Clearly, $A_i (i = 1, 2, 3)$ and B are measurable. The Lebesgue measure of $A_i (i = 1, 2, 3)$, A and B are denoted by $m(A_i)(i = 1, 2, 3)$, $m(A)$ and $m(B)$, respectively. By the meaning of integration and Lebesgue measure of R^2, we have $m(A_1) = \|u^- - v^-\|_1$ and $m(B) = \int_{min\{u_0^-,\ v_0^-\}}^{max\{u_1^-,\ v_1^-\}} |l_u(x) - l_v(x)| dx$.

Now, we show that $A = B$. First, for each $(x, \alpha) \in A$, there are three cases: Case 1. $(x, \alpha) \in A_1$, then $min\{u_\alpha^-, v_\alpha^-\} \le x \le max\{u_\alpha^-, v_\alpha^-\}$, so $min\{u_0^-, v_0^-\} \le x \le max\{u_1^-, v_1^-\}$. Without loss of generality, we can suppose that $u_\alpha^- \le v_\alpha^-$, then $u_\alpha^- \le x \le v_\alpha^-$, thus $u_\alpha^- \le x < u_1^-$ or $x \ge u_1^-$ and $l_v(x) = v(x) \le \alpha$. So $l_u(x) = u(x) \ge \alpha$ when $u_\alpha^- \le x \le u_1^-$ and $l_u(x) = 1$ when $x \ge u_1^-$. Thus, $l_v(x) \le \alpha \le l_u(x)$, i.e., $min\{l_u(x), l_v(x)\} \le \alpha \le max\{l_u(x), l_v(x)\}$. So $(x, \alpha) \in B$.
Case 2. $(x, \alpha) \in A_2$. Since $u_\alpha^- \le x < lim_{\alpha' \to \alpha^+} u_{\alpha'}^-$ and u_α^- is increasing, so $u_0^- \le x \le u_1^-$. As $l_u(x)$ is upper semicontinuous on R, $l_u(x) = \alpha$. Thus, $(x, \alpha) \in B$.
Case 3. $(x, \alpha) \in A_3$. Similarly to the case 2, we can show that $(x, \alpha) \in B$.

Thus, by the above discussion we have $A \subseteq B$.

Conversely, for each $(x,\alpha) \in B$, we will show that if $(x,\alpha) \notin A_2 \bigcup A_3$, then $(x,\alpha) \in A_1$. Since $(x,\alpha) \in B$, $min\{l_u(x), l_v(x)\} \le \alpha \le max\{l_u(x), l_v(x)\}$. Since $0 \le \alpha \le 1$, we have $min\{u_\alpha^-, v_\alpha^-\} \le x \le max\{u_\alpha^-, v_\alpha^-\}$. In fact, if $x < min\{u_\alpha^-, v_\alpha^-\}$, then $l_u(x) = u(x) < \alpha$ and $l_v(x) = v(x) < \alpha$ which contradict $\alpha \le max\{l_u(x), l_v(x)\}$; if $x > max\{u_\alpha^-, v_\alpha^-\}$, since $x \le max\{u_1^-, v_1^-\}$, so $0 \le \alpha < 1$. As $(x,\alpha) \notin A_2 \bigcup A_3$, i.e., $\alpha \notin Disc(u^-) \bigcup Disc(v^-)$, so $l_u(x) > \alpha$ and $l_v(x) > \alpha$ which contradict $\alpha > min\{l_u(x), l_v(x)\}$. Hence, for each $(x,\alpha) \in B$, $(x,\alpha) \in A$, i.e., $B \subseteq A$.

Thus, $A = B$. Since $Disc(u^-) \bigcup Disc(v^-)$ is at most countable, hence $m(A_2 \bigcup A_3) = 0$, thus $m(A_1) = m(B)$. The proof of (3) is thus completed.

Remark 2 From Lemma 2, it can be seen that the integral defined via cut-set can be represented by the integral defined via corresponding membership functions which is more direct in certain cases. By the properties of l_u, l_v and r_u, r_v, we can arbitrarily extend the integration interval of the formulas on the right hand side of (3) and (4), but their integration values remain the same:

$$\int_{-\infty}^{+\infty} |l_u(x) - l_v(x)| dx = \int_{min\{u_0^-, v_0^-\}}^{max\{u_1^-, v_1^-\}} |l_u(x) - l_v(x)| dx$$

and

$$\int_{-\infty}^{+\infty} |r_u(x) - r_v(x)| dx = \int_{min\{u_1^+, v_1^+\}}^{max\{u_0^+, v_0^+\}} |r_u(x) - r_v(x)| dx.$$

Now, we define another metric on E^1, which is based on the membership functions of fuzzy numbers. For any u, $v \in E^1$, define

$$d_1^*(u,v) = \int_{-\infty}^{+\infty} (|l_u(x) - l_v(x)| + |r_u(x) - r_v(x)|) dx.$$

Clearly, d_1^* is a metric on E^1.

By the definition of d_1, we have

$$\frac{1}{2}\int_0^1 (|u_\alpha^- - v_\alpha^-| + |u_\alpha^+ - v_\alpha^+|) d\alpha \le d_1(u,v) \le \int_0^1 (|u_\alpha^- - v_\alpha^-| + |u_\alpha^+ - v_\alpha^+|) d\alpha.$$

So by Lemma 2, Remark 2 and the definition of d_1^*, we have the following theorem.

Theorem 2 *Let $u, v \in E^1$, then*

$$\frac{1}{2} d_1^*(u,v) \le d_1(u,v) \le d_1^*(u,v).$$

Remark 3 The metric d_1^* on E^1 is based on the membership functions of fuzzy numbers. It is more direct than the L_1 metric d_1 in certain cases since d_1 is based on the cut-set functions. From Theorem 2, we can see that the metrics d_1 and d_1^* are uniformly equivalent. So they have the same topological properties. Hence, the metric d_1^* is also approximate to the supremum and it is computable.

Lemma 3 *Assume that $\{u_n\}$ and v are given as in Proposition 1, w is defined as above, and $u \in E^1$, $u \le v$. Let $a = u_0^-$ and $b = v_0^+$. Suppose that there exists natural number n such that $\frac{b-a}{n} < \frac{\varepsilon}{2}$,*

$$l_u(x_i) < l_{w'}(x_i) + \frac{\varepsilon}{2(b-a)}, \quad i = 1, 2, \cdots, n, \tag{5}$$

and

$$r_{w'}(x_i) < r_u(x_i) + \frac{\varepsilon}{2(b-a)}, \quad i = 1, 2, \cdots, n, \tag{6}$$

where $x_i = a + \frac{i(b-a)}{n}$. Then $d_1(u, v) < 2\varepsilon$.

Proof By Lemma 2 and Remark 2, we have

$$d_1(u, v) \le \int_a^b (l_u(x) - l_v(x))dx + \int_a^b (r_v(x) - r_u(x))dx.$$

Since $l_v = l_{w'}$ on $[a, b]$ and $r_v = r_{w'}$ a.e. on $[a, b]$, so

$$d_1(u, v) \le \int_a^b (l_u(x) - l_{w'}(x))dx + \int_a^b (r_{w'}(x) - r_u(x))dx.$$

First, we show that $\int_a^b (l_u(x) - l_{w'}(x))dx < \varepsilon$. Define two simple functions h_1, h_1' on $[a, b]$ as follows: $h_1(x_0) = l_{w'}(x_0), h_2(x_0) = l_u(x_0)$; $h_1(x) = l_{w'}(x_i), h_1'(x) = l_u(x_i)$, for $x \in (x_{i-1}, x_i], i = 1, 2, \cdots, n$. By (5), $h_1'(x) - h_1(x) < \frac{\varepsilon}{2(b-a)}$ for $x \in [a, b]$. And $\int_a^b (h_1(x) - l_{w'}(x))dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (h_1(x) - l_{w'}(x))dx \le \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (l_{w'}(x_i) - l_{w'}(x_{i-1}))dx = (l_w'(x_n) - l_{w'}(x_0))\frac{b-a}{n} < \frac{\varepsilon}{2}$. So

$$\begin{array}{rcl} \int_a^b (l_u(x) - l_{w'}(x))dx & \le & \int_a^b (h_1'(x) - l_{w'}(x))dx \\ & = & \int_a^b (h_1'(x) - h_1(x))dx + \int_a^b (h_1(x) - l_{w'}(x)dx \\ & < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{array}$$

Similarly, $\int_a^b (r_{w'}(x) - r_u(x))dx < \varepsilon$. The proof is thus completed.

By the discussion of Lemma 2, we have the following algorithm on the supremum of fuzzy numbers based on membership functions.

Theorem 3 *Assume that $\{u_t | t \in T\}$ and v are given as in Proposition 1, w is defined as above.*

(i) Choose $b > v_0^+ = \bigvee_{t\in T}(u_t)_0^+$*, pick* $t' \in T$ *and let* $(u_{t'})_0^- = a$*;*
(ii) Choose a natural number n *such that* $\frac{b-a}{n} < \frac{\varepsilon}{2}$*.*
For $i = 1, 2, \cdots, n,$ *choose* $t_i \in T$ *such that*

$$r_{w'}(a + \frac{i(b-a)}{n}) < r_{u_{t_i}}(a + \frac{i(b-a)}{n}) + \frac{\varepsilon}{2(b-a)}.$$

For $j = 1, 2, \cdots, n,$ *choose* $t_{n+j} \in T$ *such that*

$$l_{u_{t_{n+j}}}(a + \frac{j(b-a)}{n}) < l_{w'}(a + \frac{j(b-a)}{n}) + \frac{\varepsilon}{2(b-a)}.$$

Let $u_{t'} = u_{t_0}$*,* $u = \vee_{i=0}^{2n} u_{t_i}$*, then* $d_1(u, v) < 2\varepsilon$*.*

Remark 4 Note that in the proof of Lemma 3, we only used the values of u at the isolated points:

$$x_i = a + \frac{i(b-a)}{n}, \quad i = 1, 2, \cdots, n.$$

Thus, if we define $u' = (l_{u'}, r_{u'})$ such that $l_{u'}(x_i) = l_u(x_i)$, $r_{u'}(x_i) = r_u(x_i)$, $i = 1, 2, \cdots, n$, and let u' be linear between x_i and x_{i+1}, $i = 1, 2, \cdots, n-1$, then we also have $d_1(u', v) < 2\varepsilon$. Note that $l_{u'}(x_i) = l_{u_{t_i}}(x_i)$, $r_{u'}(x_i) = l_{u_{t_{i+n}}}(x_i)$ for some $t_i, t_{i+n} \in T$, $i = 1, 2, \cdots, n$, and u' is piecewise linear and hence continuous, thus we have the following:

Theorem 4 *Under the hypothesis of Proposition 2, for every* $\epsilon > 0$ *there is a piecewise linear fuzzy number* u' *which is determined by a finite number of values of a finite number of fuzzy numbers of* $\{u_t | t \in T\}$ *such that* $d_1(u', v) < 2\varepsilon$*.*

3. CALCULATIONS VIA $L_p (p \geq 1)$ METRICS

In this section, we discuss the finite approximate algorithm for supremum with respect to the general L_p metrics. The situation will be much more complicated than the case when $p = 1$.

Definition 1 Let f be a monotone left continuous function from [0,1] to R. Suppose that $a \leq f(x) \leq b$ for all $x \in [0, 1]$. For each $\varepsilon > 0$, pick natural number n such that $\frac{b-a}{n} < \frac{\varepsilon}{2}$. Let $E_i = \{x \in [0, 1] | a + \frac{i}{n}(b-a) \leq f(x) \leq a + \frac{i+1}{n}(b-a)\}$, $i = 0, 1, \cdots, n-1$; $E'_j = \{x \in [0, 1] | f(x) = a + \frac{j}{n}(b-a)\}, j = 0, 1, \cdots, n$. Now we define the partition points x_j as follows:

Case 1. E'_j is an interval (the interval is right closed since f is left continuous), take $x_j = sup E'_j$.

Case 2. E'_j is an empty set. If f is increasing, take $x_j = sup\{\bigcup_{i\leq j} E_i\}$; if f is decreasing, take $x_j = inf\{\bigcup_{i\leq j} E_i\}$.

Clearly, if f is increasing, then $0 = x_0 \le x_1 \le \cdots \le x_n = 1$; if f is decreasing, then $1 = x_0 \ge x_1 \ge \cdots \ge x_n = 0$. So they form a set of partitioning points for $[0, 1]$. It may happen that $\{x_i | i = 0, 1, \cdots n\}$ have repeated points. For our need we can rename these partition points and make them have no repeated points. So $0 = x_0 < x_1 < \cdots < x_n = 1$ when f is increasing and $1 = x_0 > x_1 > \cdots > x_n = 0$ when f is decreasing.

Remark 5 Based on the above partition and by the monotonicity of the function f, we have $0 \le f(x_{i+1}) - f(x) < \frac{\varepsilon}{2}$ for $x \in (x_i, x_{i+1})$ when f is increasing and $0 \le f(x) - f(x_i) < \frac{\varepsilon}{2}$ for $x \in (x_{i+1}, x_i)$ when f is decreasing.

Lemma 4 *Assume that w and and v are given as in Proposition 1, u is a fuzzy number such that $u \le v$. Let $u_0^- = a$, $w_1^- < b$. For the two functions w^+ and w^-, pick partitioning points $1 = x_0 > x_1 > \cdots > x_n = 0$ and $0 = y_0 < y_1 < \cdots < y_m = 1$ of $[0, 1]$ respectively defined in Definition 1. If the following conditions are satisfied, then $d_p(u, v) < (1 + 2^{\frac{1}{p}})\varepsilon$.*
(i) $w_{x_i}^+ < u_{x_i}^+ + \frac{\varepsilon}{2}$, $\quad i = 0, 1, \cdots, n-1$.
(ii) For every $y_j, j = 0, 1, \cdots, m-1$, there exists $y_j' \in (y_j, y_{j+1})$, $y_j' - y_j < \frac{1}{m}(\frac{\varepsilon}{b-a})^p$ and $w_{y_j'}^- < u_{y_j'}^- + \frac{\varepsilon}{2}$.

Proof First, we prove that $\|v^+ - w^+\|_p < \varepsilon$. For each $x \in [0, 1] \setminus \{x_i | i = 1, \cdots, n\}$, there exists a partitioning interval (x_{i+1}, x_i) such that $x \in (x_{i+1}, x_i)$. Since w^+ is decreasing on $[0, 1]$, so $lim_{x \to x_{i+1}^+} w_x^+$ exists, and it is denoted by $w_{x_{i+1}^+}^+$. So $w_x^+ \le w_{x_{i+1}^+}^+$. By the left continuity of w^+, we have $0 \le w_{x_{i+1}^+}^+ - w_{x_i}^+ < \frac{\varepsilon}{2}$. Hence, by (i), $w_x^+ - u_x^+ \le (w_{x_{i+1}^+}^+ - w_{x_i}^+) + (w_{x_i}^+ - u_{x_i}^+) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus for each $x \in [0, 1]$, $0 \le w_x^+ - u_x^+ < \varepsilon$. Since $w_x^+ \ne v_x^+$ only if $x \in disc(v^+)$ which is at most countable. So we have $0 \le v_x^+ - u_x^+ < \varepsilon$ a.e. on $[0, 1]$. Thus $\|v^+ - u^+\|_p < \varepsilon$.

Second, we prove that $\|u^- - w^-\|_p < 2^{\frac{1}{p}}\varepsilon$. For each $y \in [y_j', y_{j+1}]$, by left continuity of w^-, we have $0 \le w_{y_{j+1}}^- - w_{y_j'}^- < \frac{\varepsilon}{2}$. So $w_y^- - u_y^- \le (w_{y_{j+1}}^- - w_{y_j'}^-) + (w_{y_j'}^- - u_{y_j'}^-) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. And for each $y \in [y_j, y_j']$, $w_y^- - u_y^- \le w_1^- - u_0^- < b - a$. So we have

$$\begin{aligned} \|u^- - w^-\|_p &= (\textstyle\sum_{j=0}^{m-1} \int_{y_j}^{y_j'} (w_y^- - u_y^-)^p dy + \sum_{j=0}^{m-1} \int_{y_j'}^{y_j} (w_y^- - u_y^-)^p dy)^{\frac{1}{p}} \\ &< (m(b-a)^p \cdot \tfrac{1}{m}(\tfrac{\varepsilon}{b-a})^p + \varepsilon^p)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}}\varepsilon. \end{aligned}$$

So $\|u^- - v^-\|_p = \|u^- - w^-\|_p < 2^{\frac{1}{p}}\varepsilon$. Thus $d_p(u, v) < (1 + 2^{\frac{1}{p}})\varepsilon$. The proof is completed.

Based on the above discussion, we have the following algorithm on the supremum of fuzzy numbers.

Theorem 5 *Assume that* $\{u_t|t \in T\}$ *and* v *are given as in Proposition 1,* w *is defined as above.*
(i) Choose $b > v_1^-$, *pick* $t' \in T$, *let* $(u_{t'})_0^- = a$.
(ii) For w^+, *pick its partition points* $1 = x_0 > x_1 > \cdots > x_n = 0$ *defined as Definition 1. For* $i = 0, 1, \cdots, n-1$, *choose* $t_i \in T$ *such that*

$$w_{x_i}^+ < (u_{t_i})_{x_i}^+ + \frac{\varepsilon}{2}.$$

(iii) For w^-, *pick its partition points* $0 = y_0 < y_1 < \cdots < y_m = 1$ *defined as Definition 1. Let* $y_j' = y_j + \frac{1}{m}(\frac{\varepsilon}{b-a})^p \bigwedge \wedge\{y_{j+1} - y_j | j = 0, 1, \cdots, m-1\}, j = 0, 1, \cdots, m-1$. *For each* $j = 0, 1, \cdots, m-1$, *choose* $t_{n+j} \in T$ *such that*

$$w_{y_j'}^- < (u_{t_{n+j}})_{y_j'}^- + \frac{\varepsilon}{2}.$$

Let $t_{n+m} = t'$ *and* $u = \bigvee_{i=0}^{n+m} u_{t_i}$, *then* $d_p(u, v) < (1 + 2^{\frac{1}{p}})\varepsilon$.

Remark 6 Similar to Theorem 4, we can find a fuzzy number $u' \in E^1$ whose cut-set functions are both linear and determined by a finite number of cut-set values of a finite number of fuzzy numbers of $\{u_t|t \in T\}$.

Remark 7 All results in this paper are dually true for infimum.

In this paper, we have shown that the supremum of a family of fuzzy numbers can be finitely approximated via L_p metrics and give out the concrete approach to approximate the supremum. As our computation method is finite, it might be executed by computers. As a byproduct, it is proved that the L_1 metric d_1 defined via cut-set is equivalent to a metric which can be calculated directly via membership functions. The results in this paper also show that L_p metrics are useful metrics on fuzzy number spaces. Because the approximation to the supremum via L_p metrics are feasible and computable, moreover, the L_p metrics are analytic in nature, so our result may have applications in fuzzy analysis. For example, it could provide a method for the computation of various fuzzy-number-valued integrals.

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