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NON AUTOMATICALLY EXERCISED (NAE) EUROPEAN CAPPED CALL PRICING THEORY

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Abstract. The objective of this paper is to present a methodology for deriving Black Scholes formulae via a simple lognormal distribution approach and introduce European capped non automatically exercise (NAE) call option pricing theory.

1. INTRODUCTION

Option or option contract is a security which gives its holder the right to buy or sell the underlying asset under the contracting conditions. Option pricing theory has advanced along many fronts since the invention by Black and Scholes in [1]. The valuation standard option pricing theory based on distribution approach has been done by many researchers such as Brooks in [2] with normal and lognormal distribution, Corrado in [3] with generalized lambda distribution, and Markose and Alentorn in [4] with generalized gamma distribution. The objective of this paper is to present a methodology for deriving Black Scholes formulae via a simple lognormal distribution approach and introduce European capped **non automatically exercise** (**NAE**) call option pricing theory. In this option, if the stock price at time of expiration is greater than the cap value L, we deal that L as the price of stock and of course the payoff is capped at $L-K$, conversely if the cap is not crossed then the payoff becomes the standard call, $\max(0, S_T - K)$. In this option we see that the payoff opportunities are more limited, so they are cheaper to buy than standard. The approach adopted here is based on the risk-adjusted discounting of expected future cash flows. In this model we have that stock price S_T is distributed lognormal.

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2. MAIN RESULTS

2.1 Lognormal and Brownian Motion

A log normal distribution is given by the following pdf function

$$f(S_T) = \frac{1}{S_T \sigma_l \sqrt{2\pi}} \exp\left\{ -\frac{[\ln(S_T) - \mu_l]^2}{2\sigma_l^2} \right\}, S_T > 0. \tag{1}$$

$$S_T \backsim \log n\left(\mu_l, \sigma_l^2\right)$$

μ_l and σ_l^2 are the expected value and variance of $\ln S_T$ respectively, and l denotes the underlying reference index having a lognormal distribution. Specifically $\ln S_T$ has a normal distribution.

$$\ln S_T \backsim N\left(\mu_l, \sigma_l^2\right). \tag{2}$$

Here we take that stock price follows the Samuelson model in [5], that is stock price is a random process called geometric Brownian motion with

$$S_T = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T} \quad , \quad S_0 > 0 \tag{3}$$

where S_0 is stock price at time 0, $r \geq 0$ is riskless interest rate, $\sigma > 0$ is the volatility, T is time of expiration, and W_T is a standard Brownian process with mean 0, and variance T respectively. Then

$$\ln S_T = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T. \tag{4}$$

Mean and variance of $\ln S_T$ are

$$\mu_l = E(\ln S_T) = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T \tag{5}$$

$$\sigma_l^2 = \sigma^2 T.$$

respectively.

For standard European call options, the payoff function is assumed to depend on the last value S_T and not on all the values $S_0, S_1..., S_T$. We define $C_{BS}(x)$ as the standard Black Scholes call option price with exercise price x. Thus for standard European call option with contract price K, the option price based on lognormal is given by

$$C_{Log}(K) = \exp\{-rT\} E[\max(0, S_T - K)]. \tag{6}$$

$$= \exp\{-rT\} \left(\int_K^\infty S_T f(S_T)\, dS_T - K \int_K^\infty f(S_T)\, dS_T \right). \tag{7}$$

Take a look and compute the first Integral in (7). By taking μ_l and σ_l in (5) and $Y = \ln S_T$ we have that

$$\int_K^\infty S_T f(S_T)\, dS_T = \int_{\ln K}^\infty \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}} \exp\left\{ y - \frac{\left[y - \ln S_0 - \left(r - \frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma\sqrt{T}} \right\} dy. \quad (8)$$

With a little bit algebraic manipulation the exponential part in (8) can be written as $-\frac{1}{2}\left(\frac{y-\ln S_0 - \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)^2 + \ln S_0 + rT$. Let $Z = \frac{Y-\ln S_0 - \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and then $Y = \ln S_0 + \left(r + \frac{1}{2}\sigma^2\right)T + Z\sigma\sqrt{T}$, $dy = \sigma\sqrt{T}dz$. The lower limit of integration becomes

$$z = -\frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = -d_1.$$

So we have solution for the first integral in (7) is

$$\begin{aligned}\int_K^\infty S_T f(S_T)\, dS_T &= \int_{-d_1}^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2 + \ln S_0 + rT\right\} dz \\ &= S_0 e^{rT}\left(N(d_1)\right),\end{aligned}$$

with $d_1 = \frac{\ln(S_0/K)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$,and $N(x)$ is the cumulative standard normal distribution function. Next take a look at the second integral in (7) as a probability function of S_T

$$\begin{aligned}\int_K^\infty f(S_T)\, dS_T &= \Pr\left[K < S_T < \infty\right] \quad (9)\\ &= \Pr\left[\ln K - \ln S_0 < \left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T < \infty\right].\end{aligned}$$

We know from (2) and (4) that $\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T \backsim N\left(\left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$. So we have that $Z = \frac{\left(r-\frac{1}{2}\sigma^2\right)T + \sigma W_T - \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{W_T}{\sqrt{T}} \backsim N(0,1)$. Then the value of (9) becomes

$$\begin{aligned}\int_K^\infty f(S_T)\, dS_T &= \Pr\left[-\frac{\ln(S_0/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} < Z < \infty\right] \\ &= N(d_2),\end{aligned}$$

with $d_2 = \frac{\ln(S_0/K)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$. From solution of two integration in (7), the European standard call option price based on lognormal distribution and Brownian motion is

$$C_{Log}(K) = S_0 N(d_1) - K\exp(-rT)N(d_2) \tag{10}$$
$$= C_{BS}(K). \tag{11}$$

This result is exactly the same as the Black Scholes standard [1].

2.2 NAE European Capped Option Pricing

Here, in this paper, we will introduce the **NAE** European Capped call option pricing. In this call option, if the underlying asset price at maturity time is greater than the cap value L, the payoff is capped at $L-K$. So we have the payoff function is $[\max(\min(S_T, L) - K, 0)]$, and the price of this call option is given by the formulae :

$$C_{cap} = \exp\{-rT\}E\left[\max\left(\min\left(S_T, L\right) - K, 0\right)\right]. \tag{12}$$

We calculate the **NAE** European Capped call option price formulae based on Black Scholes equation (10). Now

$$E\left[\max\left(\min\left(S_T, L\right) - K, 0\right)\right] = \begin{array}{l} \int\limits_0^L \left[\max\left(0, S_T - K\right)\right] f\left(S_T\right) dS_T\big|_{S_T<L} + \\ \int\limits_L^\infty \left[\max\left(0, L - K\right)\right] f\left(S_T\right) dS_T\big|_{S_T\geq L} \end{array} \tag{13}$$

The value of the integral in (12) can be found with algebraic manipulation and then using equation (10) leads to the analytical formulae for **NAE** Capped call option price

$$C_{cap} = \exp\{-rT\}\left(\int\limits_K^\infty (S_T - K) f\left(S_T\right) dS_T - \int\limits_L^\infty (S_T - L) f\left(S_T\right) dS_T\right) \tag{14}$$
$$= \left(S_0 N(d_1) - Ke^{-rT}N(d_2)\right) - \left(S_0 N(d_3) - Le^{-rT}N(d_4)\right) \tag{15}$$
$$= C_{BS}\left(K\right) - C_{BS}\left(L\right) \tag{16}$$

where $d_3 = \frac{\ln(S_0/L) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, d_4 = \frac{\ln(S_0/L) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$. So we have this call option price formulae is the Standard European call option price with contract price K minus Standard European call option price with contract price L. In general we can see that this price is cheaper than standard option.

2.3 Properties

In this section we will present some properties and analytical results of this option pricing model compared to standard option.

1. For $S_0 \geq K$, If $T \to 0$, then d_1 and $d_2 \to \infty$, but d_3 and $d_4 \to -\infty$. So we have $C_{BS}(K) \to S_0 - K, C_{BS}(L) \to 0$ and $C_{cap} \to (S_0 - K)$.
2. For $S_0 < K$, If $T \to 0$, we have that all d_1, d_2, d_3 and $d_4 \to -\infty$. So we have $C_{BS}(K) \to 0, C_{BS}(L) \to 0$ and $C_{cap} \to 0$.

3. (a) If $T \to \infty$, then all of $d_1, d_2, d_3, d_4 \to \infty$, and of course all $N(d_1), N(d_2), N(d_3), N(d_4) \to 1$. We have $C_{BS}(K) \to S_0, C_{BS}(L) \to S_0$ and on the other hand $C_{cap} \to 0$. Illustration of this result can be seen in figure 1(a). The solid curve represents the plot of standard option, while the dash-dot one represents **NAE** European Capped option price. Table 1 give an example of comparison between both options in different time of expiration. Notice that in standard option the price get more expensive tend to asset price as the time of expiration get longer. However in **NAE** European Capped option, the price go up and then go down tend to zero as the time of expiration get longer, see figure 2(a).

Table 1. Option price in different time of expiration.

	Input				
Stock Price (S)	8000	8000	8000	8000	8000
Strike Price (K)	8000	8000	8000	8000	8000
Cap (L)	8800	8800	8800	8800	8800
Time of Exp.(T)	1	2	5	15	50
Interest rate (r)	0.1	0.1	0.1	0.1	0.1
Volatility(σ)	0.35	0.35	0.35	0.35	0.35
NAE Capped	317.3	318.69	267.01	114.25	4.16
Standard BS	1481.56	2258.84	3887.81	6513.75	7982.61

(b) If $L \to \infty$, obviously $C_{BS}(L) \to 0$ and then $C_{cap} \to C_{BS}(K)$. This show that if the cap $L \to \infty$, the option is exactly the same as that standard, see also figure 1(b). The dash-dot curve represents standard option price.

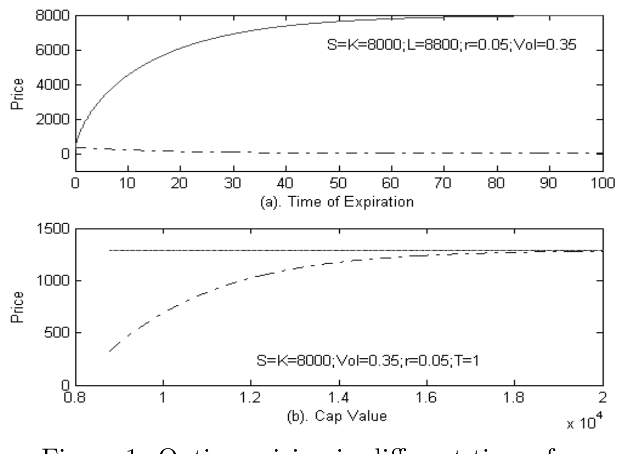

Figure 1. Option pricing in different time of expiration T (a), and cap value L (b)

4. If $L \to K$, then $C_{BS}(L) \to C_{BS}(K)$ and of course the option price $C_{cap} \to 0$. It means that if the value of the payoff function get smaller then the option price also get cheaper.

5. (a) If $\sigma_1 \leq \sigma_2$ then $C_{BS}^{\sigma_1}(K) \leq C_{BS}^{\sigma_2}(K)$ and $C_{BS}^{\sigma_1}(L) \leq C_{BS}^{\sigma_2}(L)$. But from algebra we know that $C_{BS}^{\sigma_1}(K) - C_{BS}^{\sigma_1}(L)$ not always $\leq C_{BS}^{\sigma_2}(K) - C_{BS}^{\sigma_2}(L)$ or $C_{cap}(\sigma_1)$ not always $\leq C_{cap}(\sigma_2)$. See figure 2(a) for more detail plot. We can search the volatility's value which maximizes the option price by differencing $\frac{dC_{cap}}{d\sigma} = S\sqrt{T}N'(d_1) - S\sqrt{T}N'(d_3) = 0$. This gives $\sigma = \sqrt{\frac{\ln(\frac{K.L}{S^2})}{T} - 2r}$ that maximizes the option price.

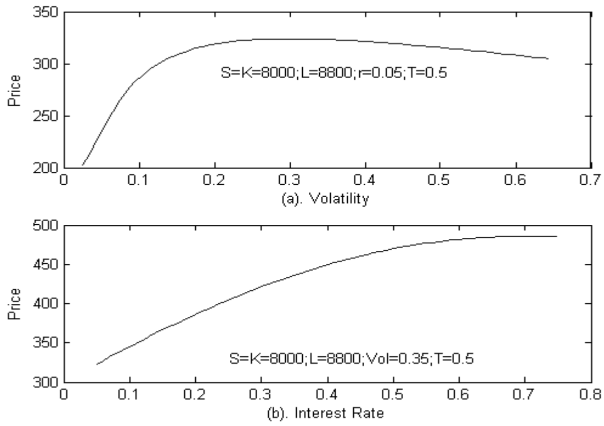

Figure 2. Option price in different volatility (a), and interest rate r (b)

(b) If $r_1 \leq r_2$ then clearly $C_{cap}(r_1) \leq C_{cap}(r_2)$. See figure 2(b).

3. CONCLUDING REMARKS

We have shown that Black Scholes formulae can be derived by lognormal distribution approach and more simple than the original Black Scholes. From the definition of **NAE** European capped, this option is cheaper than the real european capped option. In the real european capped option if the stock price reaches the cap value prior the expiration time, this option is automatically exercised. Both of them are cheaper than the standard.

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