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THE RAMSEY NUMBERS OF LINEAR FOREST VERSUS $3K_3 \cup 2K_4$

I W. Sudarsana, E. T. Baskoro, H. Assiyatun, and S. Uttunggadewa

Abstract. For given two graphs G and H, a graph F is called a (G, H)-*good graph* if F contains no G and $\overline{F}$ contains no H. Furthermore, any (G, H)-good graph on n vertices will be denoted by (G, H, n)-good graph. The *Ramsey number* $R(G, H)$ is defined as the smallest natural number n such that no (G, H, n)-good graph exists. In this paper, we determine the Ramsey numbers $R(G, H)$ for disconnected graphs G and H. In particular, $G = \bigcup_{i=1}^{k} P_{n_i}$ and $H = 3K_3 \cup 2K_4$.

1. INTRODUCTION

We consider finite undirected graphs without loops and multiple edges. Let $G(V, E)$ be a graph, the notation $V(G)$ and $E(G)$ (in short V and E) stand for the vertex set and the edge set of the graph G, respectively. A graph $H(V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. For $A \subseteq V$, $G[A]$ represents the *subgraph induced* by A in G.

For given two graphs G and H, a graph F is called a (G, H)-*good graph* if F contains no G and $\overline{F}$ contains no H. Furthermore, any (G, H)-good graph on n vertices will be denoted by (G, H, n)-good graph. The *Ramsey number* $R(G, H)$ is defined as the smallest natural number n such that no (G, H, n)-good graph exists. The Ramsey numbers $R(G, H)$ for connected graphs G and H have been intensively studied since Chvátal and Harary [4] established the general lower bound $R(G, H) \geq (c(G) - 1)(h - 1) + 1$, where h is the chromatic number of H and $c(G)$ is the number of vertices of the largest component of G. A connected graph G is

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called H-*good* if $R(G, H) = (|G| - 1)(h - 1) + s$, where s is the chromatic surplus of H. The chromatic surplus of H is the minimum cardinality of a color class taken over all proper h colorings of H.

Let G_i be a graph with the vertex set V_i and the edge set E_i, $i = 1, 2, ..., k$. The *union* $G = \bigcup_{i=1}^{k} G_i$ has the vertex set $V = \bigcup_{i=1}^{k} V_i$ and the edge set $E = \bigcup_{i=1}^{k} E_i$. We denote the union by kF when $G_i = F$ for every i. If G_i is isomorphic to a tree for every i then the union is called a *forest*. A forest is called linear forest, if all the components are a path.

Some recent results on the Ramsey number for a combination of disconnected (union) and connected graphs can be found in ([1], [2], [6], [7], [9], [10], [11]). Other results concerning graph Ramsey numbers can be seen in [8]. In this note, we determine the Ramsey numbers $R(\bigcup_{i=1}^{k} P_{n_i}, 3K_3 \cup 2K_4)$.

2. MAIN RESULTS

Let us note firstly the previous theorems and lemmas used in the proof of our results.

Theorem 1 *(Chvátal [5]). Let T_n and K_m be a tree of order $n \geq 1$ and a clique of order $m \geq 1$, respectively. Then, $R(T_n, K_m) = (n - 1)(m - 1) + 1$.*

Theorem 2 *(Sudarsana et al. [11]). Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 6$ be integers. If $G = \bigcup_{i=1}^{k} P_{n_i}$ then*

$$R(G, 2K_4) = \max_{1 \leq i \leq k} \left\{ 2n_i + \sum_{j=i}^{k} n_j \right\} - 1. \tag{1}$$

Lemma 1 *(Sudarsana et al. [11]). Let $k, t \geq 1$ be integers. Let H be a connected graph with the chromatic number $h \geq 2$ and the chromatic surplus $s \geq 1$. Let G_i be connected graphs and satisfies $|G_k| \geq |G_{k-1}| \geq ... \geq |G_1| \geq \left(\frac{|H|-s}{h-1}\right)st + 1$. If $G = \bigcup_{i=1}^{k} G_i$ and G_i is a H-good for every $i = 1, 2, ..., k$, then*

$$R(G, tH) \geq \max_{1 \leq i \leq k} \left\{ (|G_i| - 1)(h - 2) + \sum_{j=i}^{k} |G_j| \right\} + st - 1. \tag{2}$$

Lemma 2 *(Sudarsana et al. [11]). If P_n is a path of order $n \geq 4$ then $R(P_n, 2K_3) = 2n$.*

Now, we are ready to prove our main results in the following.

Lemma 3 *If P_n is a path of order $n \geq 7$ then $R(P_n, 3K_3) = 2n + 1$.*

Proof. The inequality $R(P_n, 3K_3) \geq 2n+1$ is derived from Lemma 1. We will show the reverse inequality $R(P_n, 3K_3) \leq 2n+1$ by the following reason. Take an arbitrary graph G on $2n+1$ vertices and contains no P_n. We will show that $\overline{G}$ contains $3K_3$. Since $|G| > 2n$ then by Lemma 2 we obtain $\overline{G} \supseteq 2K_3$. Let $B = \{a_1, b_1, c_1\} \cup \{a_2, b_2, c_2\}$ be the vertex set of $2K_3$ in $\overline{G}$. We let $D = V(G) \backslash B$ and $T = G[D]$. Clearly, $|T| = 2(n-2) - 1$. Theorem 1 gives that $T \supseteq P_{n-2}$ or $\overline{T} \supseteq K_3$. If $\overline{T} \supseteq K_3$ then we finish the proof. Now, consider T contains P_{n-2} and call $P_{n-2} = (p_1 p_2 ... p_{n-3} p_{n-2})$. Let $K = V(T) \backslash V(P_{n-2})$ and hence $|K| = n-3$. Now, we consider the connection of the end vertices of P_{n-2} and the vertex set K. Since G contains no P_n then we obtain the following facts.

Fact 1. *The vertex p_1 or p_{n-2} is adjacent to at least two vertices in K.*

We let p_1 adjacent to y_1 and y_2 in K. Since G does not contain P_n then $\{p_{n-2}, y_1, y_2\}$ is an independent set in G. Therefore, the vertex set $\{p_{n-2}, y_1, y_2\}$ forms a K_3 in $\overline{T}$ and together with B we have $\overline{G} \supseteq 3K_3$.

Fact 2. *The vertex p_1 or p_{n-2} is adjacent to exactly one vertex in K.*

We let p_1 adjacent to x in K. Since G contains no P_n then p_{n-2} must not adjacent to any vertex in K except the vertex x. If $K \backslash x$ contains two independent vertices, call x_1 and x_2, in T then the vertex p_{n-2} together with x_1 and x_2 induce a K_3 in $\overline{T}$ and hence $\overline{G} \supseteq 3K_3$. Therefore, the vertex set $K \backslash x$ forms a K_{n-4} in T. Now, if there exists one vertex, say y, in $K \backslash x$ that is not adjacent to one vertex in B then the vertex set $\{p_{n-2}, x, y\} \cup B$ induces a $3K_3$ in $\overline{G}$. Since otherwise we will get that every vertex in $K \backslash x$ is adjacent to every vertex in B, which is impossible since G does not contain P_n with $n \geq 7$. Thus, $\overline{G}$ contains $3K_3$.

Fact 3. *The vertex p_1 and p_{n-2} do not adjacent to any vertex in K.*

If K contains two independent vertices, call x_1 and x_2, in T then the vertex set $\{p_{n-2}, x_1, x_2\}$ induce a K_3 in $\overline{T}$. Therefore, we finish the proof since we have $\overline{G} \supseteq 3K_3$. Now, consider K shapes a K_{n-3} in T. Thus without loss of generality, one of the following conditions holds:

(i). *The vertex p_1 or p_{n-2} is adjacent to every vertices in B.*

We let p_1 adjacent to every vertex in B. Since G does not contain P_n then $\{p_{n-2}\} \cup B$ is an independent set in T which each element does not adjacent to any vertex in K. Therefore, it can be verified that the set $\{k_1, k_2, p_{n-2}\} \cup B$ induce a $3K_3$ in $\overline{T}$, for any k_1, k_2 in K. So, $\overline{G} \supseteq 3K_3$.

(ii). *The vertex p_1 or p_{n-2} is adjacent to five vertices in B.*

We let p_1 adjacent to every vertex in $B\backslash a_2$. Since G contains no P_n then it is not difficult to verify that the sets $\{p_{n-2}, a_2, c_1\}$, $\{a_1, c_2, k_2\}$ and $\{b_1, b_2, k_1\}$ form a $3K_3$ in $\overline{G}$, for any k_1, k_2 in K.

(iii). *The vertex p_1 or p_{n-2} is adjacent to four vertices in B.*

We let p_1 adjacent to every vertex in $B\backslash\{a_2, c_2\}$. Again, since G contains no P_n then it can be verified that the sets $\{p_{n-2}, a_2, c_2\}$, $\{a_1, c_1, k_2\}$ and $\{b_1, b_2, k_1\}$ form a $3K_3$ in $\overline{G}$, for any k_1, k_2 in K.

(iv). *The vertex p_1 or p_{n-2} is adjacent to three vertices in B.*

Without loss of generality, we distinguish the following two cases.

Case 1. The vertex p_1 is adjacent to a_1, b_1 and c_1 in B

Thus the set $\{p_1, a_2, b_2, c_2\}$ is an independent set in T. Since G contain no P_n then it is easy verify that the sets $\{p_{n-2}, a_1, c_2\}$, $\{a_2, b_2, p_1\}$ and $\{b_1, c_1, k\}$ form a $3K_3$ in $\overline{G}$, for every $k \in K$.

Case 2. The vertex p_1 is adjacent to a_1, b_1 and c_2 in B

Therefore, the set $\{p_1, a_2, b_2\}$ is an independent set in T. Since G contain no P_n then it is easy verify that the sets $\{p_1, a_2, b_2\}$, $\{a_1, c_1, p_{n-2}\}$ and $\{b_1, c_2, k\}$ form a $3K_3$ in $\overline{G}$, for every $k \in K$.

(v). *The vertex p_1 or p_{n-2} is adjacent to two vertices in B.*

Without loss of generality, we distinguish the following two cases.

Case 1. The vertex p_1 is adjacent to a_1 and b_1 in B

Thus the set $\{p_1, a_2, b_2, c_2\}$ is an independent set in T. Now, if there exists one vertex, call y, in K that is not adjacent to one vertex, say c_2, in $B\backslash\{a_1, b_1\}$ then the vertex sets $\{p_1, a_2, b_2\}$, $\{b_1, c_2, y\}$ and $\{a_1, c_1, p_{n-2}\}$ form a $3K_3$ in $\overline{G}$. Since otherwise we get that vertex set $K\cup\{a_2, b_2, c_2, c_1\}$ induces a graph $K_{n-3}+(\overline{K}_3\cup K_1)$ in T, which is impossible since G does not contain P_n with $n \geq 7$. Thus, $\overline{G}$ contains $3K_3$.

Case 2. The vertex p_1 is adjacent to a_1 and a_2 in B

Since G contains no P_n then it is easy verify that the sets $\{p_{n-2}, b_1, c_1\}$, $\{p_1, b_2, c_2\}$ and $\{a_1, a_2, k\}$ form a $3K_3$ in $\overline{G}$, for every $k \in K$.

(vi). *The vertex p_1 or p_{n-2} is adjacent to one vertex in B.*

We let p_1 adjacent to a_1 in B. Now, consider the vertex set $\{b_1, c_1, a_2, b_2, c_2\}$. If there exists a vertex, say k_1, in K that is not adjacent to one vertex in $\{b_1, c_1, a_2, b_2, c_2\}$ then the vertex set $\{p_1, p_{n-2}, k_1\} \cup B$ induces a $3K_3$ in $\overline{G}$. Since otherwise we obtain that the vertex set $K \cup \{b_1, c_1, a_2, b_2, c_2\}$ induces a graph $K_{n-3} + (\overline{K}_3 \cup \overline{K}_2)$ in T, which is impossible since G does not contain P_n with $n \geq 7$. Thus, $\overline{G}$ contains $3K_3$.

(vii). *The vertex p_1 or p_{n-2} does not adjacent to any vertices in B.*

If there exists a vertex, say k, in K that is not adjacent to one vertex in B then the vertex set $\{p_1, p_{n-2}, k\} \cup B$ induces a $3K_3$ in $\overline{G}$. Since otherwise we derive that the vertex set $K \cup \{b_1, c_1, a_2, b_2, c_2\}$ induces a graph $K_{n-3} + (2\overline{K}_3)$ in T, which is impossible since G does not contain P_n with $n \geq 7$. Thus $\overline{G} \supseteq 3K_3$. This completes the proof. □

Theorem 3 *Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 7$ be integers. If $G = \bigcup_{i=1}^{k} P_{n_i}$ then*

$$R(G, 3K_3) = \max_{1 \leq i \leq k} \left\{ n_i + \sum_{j=i}^{k} n_j \right\} + 1. \tag{3}$$

Proof. For $1 \leq i \leq k$, let $G = \bigcup_{i=1}^{k} P_{n_i}$ and $G_i = \bigcup_{j=i}^{k} P_{n_j}$. Obviously, $G = G_1$. Suppose that the maximum of the right side of the equation (3) is achieved for i_0. Write $t_0 = \sum_{j=i_0}^{k} n_j$ and $t = n_{i_0} + t_0$. The lower bound $R(G, 3K_3) \geq t + 1$ can be obtained by using Lemma 1. We will prove $R(G, 3K_3) \leq t + 1$.

Let F be a graph of order $t + 1$ and suppose that $\overline{F}$ contains no $3K_3$. We shall show that F contains G. We prove this by induction on i. For $i = k$, we get $G = P_{n_k}$. Since $t + 1 \geq 2n_k + 1$ and $\overline{F} \not\supseteq 3K_3$ then the theorem holds by Lemma 3. Let us state the inductive hypothesis: F contains G_{i+1} for some $1 \leq i \leq k$. We will show that F contains G_i for any $i \geq 1$. By induction hypothesis, we have $F \supseteq G_{i+1}$. Clearly, $|G_{i+1}| = \sum_{j=i+1}^{k} n_j$. Let $A = V(F) \backslash V(G_{i+1})$ and $W = F[A]$, then $|W| = (t + 1) - \sum_{j=i+1}^{k} n_j$. By definition of t, we get $t \geq n_i + \sum_{j=i}^{k} n_j$ for every $i = 1, 2, ..., k$. Therefore, $|W| \geq 2n_i + 1$. Since $\overline{W} \not\supseteq 3K_3$ then Lemma 3 guarantees that W contains P_{n_i}. Therefore, $F \supseteq G_i$ for any $i \geq 1$. Thus $F \supseteq G_1$. □

Theorem 4 *Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 9$ be integers. Let $G = \bigcup_{i=1}^{k} P_{n_i}$ and $H = 3K_3 \cup 2K_4$. If $R(G, 2K_4) - R(G, 3K_3) \geq 9$ then*

$$R(G, H) = R(G, 2K_4). \tag{4}$$

Proof. By Theorem 2, we let $R(G, 2K_4) = l - 1$. Since $2K_4 \subset H$ then $R(G, H) \geq l - 1$. Now, we will show that $R(G, H) \leq l - 1$. Let U be a graph of order $l - 1$ and contains no G. We shall show that $\overline{U}$ contains H. Theorem 2 provides

$\overline{U} \supseteq 2K_4$. Let $L = V(U) \backslash V(2K_4)$ and $Q = U[L]$. Clearly, $|Q| = l - 9$. By Theorem 3, we let $R(G, 3K_3) = l'$. Thus, $|Q| = l - 9 = l' + (l - l') - 9 \geq l'$ when $l - l' \geq 9$. Since $Q \not\supseteq G$ then $\overline{Q} \supseteq 3K_3$. This concludes that $\overline{U}$ contains H. $\square$

Remark. If $n_i = n$ for every $i = 1, 2, ..., k$, then the union G is isomorphic to kP_n. Therefore, by Theorem 3 we obtain $R(kP_n, 3K_3) = (k+1)n + 1$ when $n \geq 7$. Meanwhile, Theorem 2 gives $R(kP_n, 2K_4) = (k+2)n - 1$ when $n \geq 6$ and Theorem 4 also provides $R(kP_n, 3K_3 \cup 2K_4) = (k+2)n - 1$ when $n \geq 9$. Furthermore, if $G = \bigcup_{i=1}^{k} l_i P_{n_i}$ and l_i is the number of the paths of order n_i in G then the following corollaries hold.

Corollary 1 *Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 7$ be integers. If $G = \bigcup_{i=1}^{k} l_i P_{n_i}$ then*

$$R(G, 3K_3) = \max_{1 \leq i \leq k} \left\{ n_i + \sum_{j=i}^{k} l_j n_j \right\} + 1. \tag{5}$$

Corollary 2 *Let $k \geq 1$ and $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 9$ be integers. Let $G = \bigcup_{i=1}^{k} l_i P_{n_i}$ and $H = 3K_3 \cup 2K_4$. If $R(G, 2K_4) - R(G, 3K_3) \geq 9$ then*

$$R(G, H) = \max_{1 \leq i \leq k} \left\{ 2n_i + \sum_{j=i}^{k} l_j n_j \right\} - 1. \tag{6}$$

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I W. Sudarsana: Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesha No. 10 Bandung 40132, Indonesia dan Jurusan Matematika, FMIPA, Universitas Tadulako (UNTAD), Jl. Soekarno-Hatta Km. 8 Palu 94118, Indonesia. E-mail: sudarsanaiwayan@yahoo.co.id

E. T. Baskoro: Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesha No. 10 Bandung 40132, Indonesia.
E-mail: ebaskoro@math.itb.ac.id

H. Assiyatun: Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesha No. 10 Bandung 40132, Indonesia.
E-mail: hilda@math.itb.ac.id

S. Uttunggadewa: Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesha No. 10 Bandung 40132, Indonesia.
E-mail: s_uttunggadewa }@math.itb.ac.id