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NEW INEQUALITIES ON THE HOMOGENEOUS FUNCTIONS

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Abstract. In this paper, we define the homogeneous functions $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$. Further we study properties, relations with means, applications to inequalities and partial derivatives. We also give some applications of these means related to Farey fractions and Ky Fan type inequalities.

1. INTRODUCTION

In [5], Imoru studied on the power mean and the logarithmic mean. Jia and Cao [6] obtained some inequalities for logarithmic mean. They also gave some applications. Mustonen [12] generalized the logarithmic mean by series expansions. He gave computational aspects of this mean. Mond et. al. [13] defined the logarithmic mean of two positive numbers is a mean. They also gave some applications related to operator theory.

For positive numbers a and b, Arithmetic mean, Geometric mean, Logarithmic mean, Heron mean, Power mean and Identric mean are as follows, cf: ([19], [11], [9], [5], [8], [6], [7], [10], [13], [12], [20]) see (1)-(6).

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Let

$$A = A(a,b) = \frac{a+b}{2}, \tag{1}$$

$$G = G(a,b) = \sqrt{ab}, \tag{2}$$

$$L = L(a,b) = \begin{cases} \frac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b. \end{cases}, \tag{3}$$

$$H = H(a,b) = \frac{a+\sqrt{ab}+b}{3}, \tag{4}$$

$$M_r = M_r(a,b) = \begin{cases} (\frac{a^r+b^r}{2})^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases}, \tag{5}$$

and

$$I = I(a,b) = \begin{cases} e^{(\frac{a\ln a - b\ln b}{a-b}-1)}, & a \neq b; \\ a, & a = b. \end{cases}. \tag{6}$$

In this paper, we introduced the new homogeneous functions $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$ more closely and get some interesting properties and inequalities which are contained in the heart of these functions. Also shown applications to mean inequalities and partial derivatives inrespect of these new functions are considered. Finally, we mentioned some remarks and applications related to Farey fractions.

2. SOME PROPERTIES

Definition 1. *For positive numbers a and b, let r be a positive real number and $\mu \in (-2, \infty)$. Then $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$ are defined as*

$$Gn_{\mu,r}(a,b) = \begin{cases} \frac{2}{\mu+2}A(a,b) + \frac{\mu}{\mu+2}M_r(a,b) = \frac{a+\mu M_r(a,b)+b}{\mu+2}, & r \neq 0; \\ \frac{2}{\mu+2}A(a,b) + \frac{\mu}{\mu+2}G(a,b) = \frac{a+\mu\sqrt{ab}+b}{\mu+2}, & r = 0. \end{cases}$$

and

$$gn_{\mu,r}(a,b) = \begin{cases} M_r^{\frac{\mu}{\mu+2}}(a,b)A^{\frac{2}{\mu+2}}(a,b), & r \neq 0; \\ G^{\frac{\mu}{\mu+2}}(a,b)A^{\frac{2}{\mu+2}}(a,b), & r = 0. \end{cases}.$$

According to Definition 1, we easily find the following characteristic properties for $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$.

Proposition 1. *Let $\mu \in (-2, \infty)$. Then we have*

1) $Gn_{\mu,r}(a,b) = Gn_{\mu,r}(b,a)$,

2) $gn_{\mu,r}(a,b) = gn_{\mu,r}(b,a)$, *3)* $Gn_{\mu,r}(a,b) = gn_{\mu,r}(a,b) \Leftrightarrow a = b$,

4) $Gn_{\mu,r}(ta,tb) = tGn_{\mu,r}(a,b)$,

5) $gn_{\mu,r}(ta,tb) = tgn_{\mu,r}(a,b)$,

6) $Gn_{0,r}(a,b) = gn_{0,r}(a,b) = A(a,b)$,

7) $Gn_{\infty,0}(a,b) = gn_{\infty,0}(a,b) = G(a,b)$,

8) $Gn_{2,0}(a,b) = M_{\frac{1}{2}}(a,b) = A(\sqrt{a},\sqrt{b})$,

9) $gn_{2,0}(a,b) = \sqrt{GA}$,

10) $Gn_{1,0}(a,b) = H(a,b)$,

and

11) $gn_{1,0}(a,b) = G^{\frac{1}{3}}A^{\frac{2}{3}}$.

Proposition 2. *Let $\mu \in (-2,\infty)$ and $r < 1$, then $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$ both are monotone decreasing functions with respect to μ for $0 < a < b$.*

Proposition 3. *Let $\mu \in (-2,\infty)$ and $r > 1$, then $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$ both are monotone increasing functions with respect to μ for $0 < a < b$.*

Proposition 4. *If $\mu \in (-2,\infty)$ then $Gn_{\mu,0}(a,b)$ and $gn_{\mu,0}(a,b)$ both are monotone decreasing functions with respect to μ for $0 < a < b$.*

3. SOME INEQUALITIES

By applying Taylor theorem and setting $a = x = t+1$ and $b = 1$ in (1)-(6), we have

$$L(x,1) = L(t+1,1) = 1 + \frac{t}{2} - \frac{1}{12}t^2 + \cdots, \tag{7}$$

$$I(x,1) = I(t+1,1) = 1 + \frac{t}{2} - \frac{1}{24}t^2 + \cdots, \tag{8}$$

$$M_r(x,1) = M_r(t+1,1) = 1 + \frac{t}{2} + \frac{r-1}{8}t^2 + \cdots, \tag{9}$$

$$Gn_{\mu_1,r}(x,1) = Gn_{\mu_1,r}(t+1,1) = 1 + \frac{t}{2} - \frac{(1-r)\mu_1}{(\mu_1+2)8}t^2 + \cdots, \tag{10}$$

and

$$gn_{\mu_2,r}(x,1) = gn_{\mu_2,r}(t+1,1) = 1 + \frac{t}{2} - \frac{(1-r)\mu_2}{(\mu_2+2)8}t^2 + \cdots. \tag{11}$$

(for detail see also [6]).

Theorem 1. *If $\mu \in (-2, \infty)$ then we have*

$$Gn_{\mu,r}(a,b) \geq gn_{\mu,r}(a,b)$$

with the equality holding if and only if $a = b$ or $\mu = 0 = 0$, $\mu = \infty$.

Proof. The proof follows from the definition of weighted Arithmetic-Geometric mean inequality.

Theorem 2. *Let $\mu_1, \mu_2 \in (-2, \infty)$, $r < 1$. If $\mu_1 \leq \frac{4}{1-r} \leq \mu_2$ then*

$$gn_{\mu_2,r}(a,b) \leq L(a,b) \leq Gn_{\mu_1,r}(a,b). \tag{12}$$

Furthermore $\mu_1 = \mu_2 = \frac{4}{1-r}$ is the best possibility for (12). Also for $r = 0$,

$$gn_{\mu_2,0}(a,b) \leq L(a,b) \leq Gn_{\mu_1,0}(a,b). \tag{13}$$

And $\mu_1 = \mu_2 = 4$ is the best possibility for (13).

Proof. Using proposition 2 we have

$$G \leq gn_{\mu_2,r}(a,b) \leq L \leq \cdots \leq A \text{ and } G \leq \cdots \leq L \leq Gn_{\mu_1,r}(a,b) \leq A. \tag{14}$$

is clear from (14) that there exist $\mu_1, \mu_2 \in (-2, \infty)$ such that (13) and (12) holds. Again by using (7), (10), and (11), we have

$$\begin{aligned} &gn_{\mu_2,r}(a,b) \leq L(a,b) \leq Gn_{\mu_1,r}(a,b) \\ &\qquad \Rightarrow -\frac{(1-r)\mu_2}{(\mu_2+2)8} \leq -\frac{1}{12} \leq -\frac{(1-r)\mu_1}{(\mu_1+2)8} \end{aligned}$$

with simple manipulation we have $\mu_1 \leq \frac{4}{1-r} \leq \mu_2$.

Hence the proof of (12) and (13).

With similar arguments, the following theorems can be proved:

Theorem 3. *For $\mu_1, \mu_2 \in (-2, \infty)$, $r \neq \frac{2}{3}$, $r < 1$ and if $\mu_1 \leq \frac{2}{2-3r} \leq \mu_2$ then*

$$gn_{\mu_2,r}(a,b) \leq I(a,b) \leq Gn_{\mu_1,r}(a,b). \tag{15}$$

Furthermore $\mu_1 = \mu_2 = \frac{2}{2-3r}$ is the best possibility for (15). Also for $r = 0$,

$$gn_{\mu_2,0}(a,b) \leq I(a,b) \leq Gn_{\mu_1,0}(a,b). \tag{16}$$

And $\mu_1 = \mu_2 = 1$ *is the best possibility for (16).*

Theorem 4. *For* $\mu_1, \mu_2 \in (-2, \infty)$, $r \neq 0$ *and if* $\mu_1 \leq \frac{2}{r} - 2 \leq \mu_2$ *then*

$$gn_{\mu_2,r}(a,b) \leq M_r(a,b) \leq Gn_{\mu_1,r}(a,b). \tag{17}$$

Furthermore $\mu_1 = \mu_2 = \frac{2}{r} - 2$ *is the best possibility for (17).*

4. APPLICATION TO MEAN INEQUALITIES

Definition 2. *Suppose* $a, b, c, d > 0$, *define the function* $f(\mu)$ *as* $f(\mu) = \frac{a+\mu\sqrt{ab}+b}{c+\mu\sqrt{cd}+d}$, *where* μ *is a real number at the interval* $(-\infty, \infty)$.

Observe that

1) If $f(-\infty) = \frac{G(a,b)}{G(c,d)} = f(\infty)$.

2) If $f(\mu) = \frac{Gn_{\mu,r}(a,b)}{Gn_{\mu,r}(c,d)}$ where $\mu \neq -2$ is a real number at the interval $(-\infty, \infty)$.

3) If $(a+b)\sqrt{cd} - (c+d)\sqrt{ab} \neq 0$, then $f(\mu) = \frac{Gn_{\mu,r}(a,b)}{Gn_{\mu,r}(c,d)}$ or $f(\mu) = \frac{a+\mu\sqrt{ab}+b}{c+\mu\sqrt{cd}+d}$.

Thus each of these represent a bilinear transformation provided μ is complex.

Proposition 5. *The function* $f(\mu)$ *is strictly decreasing if* $ad - bc > 0$, $a > b > c > d > 0$ *and for* $\mu_2 > \mu_1 > -2$.

Proposition 6. *The function* $f(\mu)$ *is strictly increasing if* $ad - bc < 0$, $a > b > c > d > 0$ *and for* $\mu_2 > \mu_1 > -2$.

Theorem 5. *Suppose* $a \geq b \geq c \geq d > 0$, $\mu \neq -2$. *If* $ad - bc < 0$, *then*

$$\frac{G(a,b)}{G(c,d)} < \frac{M_{\frac{1}{2}}(a,b)}{M_{\frac{1}{2}}(c,d)} < \frac{H(a,b)}{H(c,d)} < \frac{A(a,b)}{A(c,d)}.$$

If $ad - bc < 0$, *all inequalities reverse and if* $ad - bc = 0$, *all inequalities turn out to be equalities.*

Proof. Consider

$$\begin{aligned}\frac{H(a,b)}{H(c,d)}-\frac{A(a,b)}{A(c,d)}&=\frac{a+\mu\sqrt{ab}+b}{c+\mu\sqrt{cd}+d}-\frac{a+b}{c+d}\\&=\frac{\sqrt{cd}}{c+\sqrt{cd}+d}\left(\frac{\sqrt{ab}}{\sqrt{cd}}-\frac{a+b}{c+d}\right)\\&=\beta\left(\frac{\sqrt{ab}}{\sqrt{cd}}-\frac{a+b}{c+d}\right)\\&=\beta\left(\frac{G(a,b)}{G(c,d)}-\frac{A(a,b)}{A(c,d)}\right),\end{aligned}$$

$\beta>0$ where $\beta=\frac{\sqrt{cd}}{c+\sqrt{cd}+d}$. $\frac{G(a,b)}{G(c,d)}<\frac{A(a,b)}{A(c,d)}$, whenever $ad-bc>0$, see [8] $\Rightarrow$ $\frac{H(a,b)}{H(c,d)}<\frac{A(a,b)}{A(c,d)}$, if $ad-bc>0$.

Similarly we have $\frac{M_{\frac{1}{2}}(a,b)}{M_{\frac{1}{2}}(c,d)}<\frac{H(a,b)}{H(c,d)}$ and $\frac{G(a,b)}{G(c,d)}<\frac{M_{\frac{1}{2}}(a,b)}{M_{\frac{1}{2}}(c,d)}$ if $ad-bc>0$.

Corollary 1. *The following inequality was summarized and stated by J.-Ch. Kuang [7] and J. Rooin and M. Hassni [8]:*

$$G(a,b)\le L(a,b)\le M_{\frac{1}{3}}(a,b)\le M_{\frac{1}{2}}(a,b)\le H(a,b)\le M_{\frac{2}{3}}(a,b)\le A(a,b).$$

Using Theorem 5 and [8], we have $a\ge b\ge c\ge d>0$, $\mu\ne-2$ and if also $ad-bc>0$, then

$$\frac{G(a,b)}{G(c,d)}\le\frac{L(a,b)}{L(c,d)}\le\frac{M_{\frac{1}{3}}(a,b)}{M_{\frac{1}{3}}(c,d)}\le\frac{M_{\frac{1}{2}}(a,b)}{M_{\frac{1}{2}}(c,d)}\le\frac{H(a,b)}{H(c,d)}\le\frac{M_{\frac{2}{3}}(a,b)}{M_{\frac{2}{3}}(c,d)}\le\frac{A(a,b)}{A(c,d)}. \tag{18}$$

If $ad-bc<0$, all inequalities reverse and $ad-bc=0$, all inequalities turn out to be equalities.

5. PARTIAL DERIVATIVES $Gn_{\mu,r}(a,b)$ AND $gn_{\mu,r}(a,b)$

Now we shall find partial derivatives of $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$. They are listed below:

$$\frac{\partial}{\partial a}(Gn_{\mu,r}(c,c))+\frac{\partial}{\partial b}(Gn_{\mu,r}(c,c))=1,$$
$$\frac{\partial}{\partial a}(Gn_{\mu,r}(c,c))\ge 0\ (and)\ \frac{\partial}{\partial b}(Gn_{\mu,r}(c,c))\ge 0$$

and

$$0\le\frac{\partial}{\partial a}(Gn_{\mu,r}(c,c))\le 1\ (and)\ 0\le\frac{\partial}{\partial b}(Gn_{\mu,r}(c,c))\le 1.$$

Observe that this property does not hold for arbitrary points. Therefore we have

$$\frac{\partial}{\partial a}(Gn_{\mu,r}(c,c)) = \frac{\partial}{\partial b}(Gn_{\mu,r}(c,c)) = \frac{1}{2}.$$

From the above, we obtain

$$\frac{\partial^2}{\partial a^2}(Gn_{\mu,r}(c,c)) + 2\frac{\partial^2}{\partial a \partial b}(Gn_{\mu,r}(c,c)) + \frac{\partial^2}{\partial b^2}(Gn_{\mu,r}(c,c)) = 0,$$

and

$$\frac{\partial^2}{\partial a^2}(Gn_{\mu,r}(c,c)) = -\frac{\partial^2}{\partial a \partial b}(Gn_{\mu,r}(c,c)) = \frac{\partial^2}{\partial b^2}(Gn_{\mu,r}(c,c)).$$

Finally we get

$$\frac{\partial^2}{\partial a^2}(Gn_{\mu,r}(c,c)) = -\frac{(1-r)\mu}{4(\mu+2)}\frac{1}{c}.$$

The proofs are obtained by simple direct computations and also valid for $gn_{\mu,r}(a,b))$.

Put $r = 0$. Then the above results are valid for $Gn_{\mu,0}(a,b))$ and $gn_{\mu,0}(a,b))$.

6. SOME REMARKS AND APPLICATIONS

In [20], Zhang et. al. defined the generalized Heron mean $H_r(a,b;k)$ and its dual form. They also gave some results related to this mean.

Definition 3. *([20]) Suppose $a > 0$, $b > 0$, k is a natural number and r is a real number. Then the generalized power-type Heron mean and its dual form are defined as follows*

$$H_r(a,b;k) = \begin{cases} \left(\frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{(k-i)r}{k}} b^{\frac{ir}{k}}\right)^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases}$$

and

$$h_r(a,b;k) = \begin{cases} \left(\frac{1}{k}\sum_{i=0}^{k} a^{\frac{(k+1-i)r}{k}} b^{\frac{ir}{k+1}}\right)^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases}.$$

Theorem 6. *([20]) If k is a fixed natural number, then $H_r(a,b;k)$ dan $h_r(a,b;k)$ are monotone increasing with respect to a and b for fixed real numbers r, or with respect to r for fixed positive numbers a and b; and are logarithmical concave on $(0,+\infty)$, and logarithmical convex on $(-\infty,0)$ with respect to r.*

Theorem 7. *([20]) If $b_1 \geq b_2 > 0$ and $\frac{a_1}{b_1} \geq \frac{a_2}{b_2}$, then $\frac{H_r(a_1,a_2;k)}{H_r(b_1,b_2;k)}$ and $\frac{h_r(a_1,a_2;k)}{h_r(b_1,b_2;k)}$ are monotone increasing with respect to r in $\mathbb{R}$.*

Farey fractions are defined as follows cf. (see for detail [2], [16], [17], [18]):

The set of Farey fractions of order n, denoted by F_n, is the set of reduced fractions in the closed interval [0,1] with denominators $\leq n$, listed in increasing order of magnitude.

If $\frac{h}{k} < \frac{H}{K}$ are adjacent Farey fractions, then it is known that $hK - kH = -1$. The mediant of adjacent Farey fractions $\frac{h}{k} < \frac{H}{K}$ is $\frac{h+H}{k+K}$. It satisfies the inequality $\frac{h}{k} < \frac{h+H}{k+K} < \frac{H}{K}$. The following inequality can be obtained by repeating the calculation of mediants n-times, successively ([16], [17], [18]):

$$\frac{h}{k} < \frac{h+H}{k+K} < \cdots < \frac{h+nH}{k+nK} < \frac{H}{K}.$$

Theorem 8. *If $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are non-negative reduced adjacent Farey fractions and satisfy the conditions of Theorem 7, then $\frac{H_r(a_1,a_2;k)}{H_r(b_1,b_2;k)}$ and $\frac{h_r(a_1,a_2;k)}{h_r(b_1,b_2;k)}$ are monotone increasing with respect to r in $\mathbb{R}$.*

We now give some application to Ky Fan type inequalities as follows:

Through this section, given n arbitrary non negative real numbers $x_1, x_2, \cdots, x_n$ belonging to $\left(0, \frac{1}{2}\right]$, we denote by A_n and G_n, the unweighted arithmetic and geometric means of $x_1, x_2, \cdots, x_n$ respectively, ie

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, G_n = \frac{1}{n}\sum_{i=1}^{n} x_i^{\frac{1}{n}}$$

and by A'_n and G'_n, the unweighted arithmetic and geometric means of $1-x_1, 1-x_2, \cdots, 1-x_n$ respectively, ie

$$A'_n = \frac{1}{n}\sum_{i=1}^{n} 1 - x_i, G'_n = \frac{1}{n}\sum_{i=1}^{n}(1-x_i)^{\frac{1}{n}}$$

with the above notations, the Ky Fan's inequality [3] asserts that:

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n},$$

with the equality holding if and if $x_1 = x_2 = \cdots = x_n$.

In 1988, H.Alzer[1] obtained an additive analogue of Ky Fan's inequality as follows:

$$A'_n - G'_n \leq A_n - G_n, \tag{19}$$

with the equality holding if and if $x_1 = x_2 = \cdots = x_n$.

Also in 1995, J. E. Pecaric and H. Alzer [14], using the Dingas Identity in [4], proved that:

$$A_n^n - G_n^n \leq {A'_n}^n - {G'_n}^n, \tag{20}$$

in which if $n = 1, 2$, equality always holds in 19, and if $n \geq 3$, the equality valid if and if $x_1 = x_2 = \cdots = x_n$.

Theorem 9. *Suppose $x_1 = x_2 = \cdots = x_n$ belonging to $\left(0, \frac{1}{2}\right]$ not all equal, then*

(i)

$$\max\left\{\frac{{A'_n}^n - {G'_n}^n}{A_n^n - G_n^n}\left(\frac{A_nG_n}{A'_nG'_n}\right)^{\frac{n}{2}}, \frac{A'_n - G'_n}{A_n - G_n}\left(\frac{A_nG_n}{A'_nG'_n}\right)^{\frac{1}{2}}\right\}$$

$$< \frac{\ln\left(\frac{A'_n}{G'_n}\right)}{\ln\left(\frac{A_n}{G_n}\right)} < \frac{A'_n - G'_n}{A_n - G_n}\frac{A_n + \sqrt{A_nG_n} + G_n}{A'_n + \sqrt{A'_nG'_n} + G'_n}$$

$$< \min\left\{\frac{A'_n - G'_n}{A_n - G_n}, \frac{A_n + \sqrt{A_nG_n} + G_n}{A'_n + \sqrt{A'_nG'_n} + G'_n}\right\} < 1,$$

(ii)

$$\frac{A'_n}{G'_n} < \left(\frac{A_n}{G_n}\right)^{\frac{A'_n - G'_n}{A_n - G_n}\frac{A_n + \sqrt{A_nG_n} + G_n}{A'_n + \sqrt{A'_nG'_n} + G'_n}} < \min\left\{\left(\frac{A_n}{G_n}\right)^{\frac{A'_n - G'_n}{A_n - G_n}}, \left(\frac{A_n}{G_n}\right)^{\frac{A_n + \sqrt{A_nG_n} + G_n}{A'_n + \sqrt{A'_nG'_n} + G'_n}}\right\}$$

is the refinement of (20).

(iii)

$$\max\left\{\frac{{A'_n}^n - {G'_n}^n}{A_n^n - G_n^n}\left(\frac{A_nG_n}{A'_nG'_n}\right)^{\frac{n}{2}}, \frac{A'_n - G'_n}{A_n - G_n}\left(\frac{A_nG_n}{A'_nG'_n}\right)^{\frac{1}{2}}\right\} < \frac{\ln\left(\frac{A'_n}{G'_n}\right)}{\ln\left(\frac{A_n}{G_n}\right)}$$

$$< \frac{{A'_n}^{\frac{1}{2}} - {G'_n}^{\frac{1}{2}}}{{A_n}^{\frac{1}{2}} - {G_n}^{\frac{1}{2}}}\frac{{A_n}^{\frac{1}{2}} + {G_n}^{\frac{1}{2}}}{{A'_n}^{\frac{1}{2}} + {G'_n}^{\frac{1}{2}}} < \min\left\{\frac{{A'_n}^{\frac{1}{2}} - {G'_n}^{\frac{1}{2}}}{{A_n}^{\frac{1}{2}} - {G_n}^{\frac{1}{2}}}, \frac{{A_n}^{\frac{1}{2}} + {G_n}^{\frac{1}{2}}}{{A'_n}^{\frac{1}{2}} + {G'_n}^{\frac{1}{2}}}\right\}$$

(iv)

$$\frac{\ln\left(\frac{A'_n}{G'_n}\right)^{A'_n + \sqrt{A'_nG'_n} + G'_n}}{\ln\left(\frac{A_n}{G_n}\right)^{A_n + \sqrt{A_nG_n} + G_n}} < \frac{A'_n - G'_n}{A_n - G_n} < \frac{A'_n + \sqrt{A'_nG'_n} + G'_n}{A_n + \sqrt{A_nG_n} + G_n}$$

is the refinement of (19).

Proof. For proving the first half of (i), put $a = A'_n$, $b = G'_n$, $c = A_n$, and $d = G_n$ in (18) ie $\frac{L(a,b)}{L(c,d)} < \frac{G(a,b)}{G(c,d)}$ also with $a = {A'_n}^n$, $b = {G'_n}^n$, $c = A_n^n$, and $d = G_n^n$, for proving the second half of (i), put $a = A'_n$, $b = G'_n$, $c = A_n$, and $d = G_n$ in

$\frac{H(a,b)}{H(c,d)} < \frac{L(a,b)}{L(c,d)}$ of (18) and use $A_n + \sqrt{A_n G_n} + G_n < A'_n + \sqrt{A'_n G'_n} + G'_n$ with simple computation we get the inequality (i).

To prove (ii), put $a = A'_n$, $b = G'_n$, $c = A_n$, and $d = G_n$ use $\frac{H(a,b)}{H(c,d)} < \frac{L(a,b)}{L(c,d)}$ of (18) and $A_n + \sqrt{A_n G_n} + G_n < A'_n + \sqrt{A'_n G'_n} + G'_n$.

To prove the first half of (iii), put $a = A'_n$, $b = G'_n$, $c = A_n$, and $d = G_n$ in (18) ie $\frac{L(a,b)}{L(c,d)} < \frac{G(a,b)}{G(c,d)}$ with $a = {A'_n}^n$, $b = {G'_n}^n$, $c = A^n_n$, and $d = G^n_n$, the second half of (iii), put $a = A'_n$, $b = G'_n$, $c = A_n$, and $d = G_n$ in $\frac{M_{\frac{1}{2}}(a,b)}{M_{\frac{1}{2}}(c,d)} < \frac{L(a,b)}{L(c,d)}$ of (18) and use $A_n + \sqrt{A_n G_n} + G_n < A'_n + \sqrt{A'_n G'_n} + G'_n$.

To prove (iv), $a = A'_n$, $b = G'_n$, $c = A_n$, and $d = G_n$ use $\frac{H(a,b)}{H(c,d)} < \frac{L(a,b)}{L(c,d)}$ of (18) and (i) of Theorem 9.

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