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THE RAMSEY NUMBERS FOR DISJOINT UNION OF STARS

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Abstract. The Ramsey number for a graph G versus a graph H, denoted by $R(G,H)$, is the smallest positive integer n such that for any graph F of order n, either F contains G as a subgraph or $\overline{F}$ contains H as a subgraph. In this paper, we investigate the Ramsey numbers for union of stars versus small cycle and small wheel. We show that if $n_i \geq 3$ for $i = 1, 2, \ldots, k$ and $n_i \geq n_{i+1} \geq \sqrt{n_i} - 2$, then $R(\bigcup_{i=1}^{k} S_{1+n_i}, C_4) = \sum_{i=1}^{k} n_i + k + 1$ for $k \geq 2$. Furthermore, we show that if n_i is odd and $2n_{i+1} \geq n_i$ for every i, then $R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = R(S_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i$ for $k \geq 1$.

Key words and Phrases: Ramsey number, cycle, wheel.

Abstrak. Bilangan Ramsey untuk graf G terhadap graf H, dinotasikan oleh $R(G,H)$, yakni bilangan bulat terkecil n sehingga sembarang graf F berorde n, memenuhi F memuat G sebagai suatu subgraf atau $\overline{F}$ memuat H sebagai suatu subgraf. Dalam makalah ini, dikaji bilangan Ramsey untuk gabungan saling lepas bintang terhadap siklus berorde empat dan roda berorde lima. Kita akan menunjukkan bahwa apabila $n_i \geq 3$, $i = 1, 2, \ldots, k$ dan $n_i \geq n_{i+1} \geq \sqrt{n_i} - 2$, maka $R(\bigcup_{i=1}^{k} S_{1+n_i}, C_4) = \sum_{i=1}^{k} n_i + k + 1$ untuk $k \geq 2$. Selanjutnya, ditunjukkan bahwa jika n_i ganjil dan $2n_{i+1} \geq n_i$ untuk setiap i, maka $R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = R(S_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i$ dengan $k \geq 1$.

Kata kunci: Bilangan Ramsey, siklus, roda.

1. Introduction

For given graphs G and H, the *Ramsey number* $R(G,H)$ is defined as the smallest positive integer n such that for any graph F of order n, either F contains

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G or $\overline{F}$ contains H, where $\overline{F}$ is the complement of F. Chvátal and Harary [4] established a useful lower bound for finding the exact Ramsey numbers $R(G,H)$, namely $R(G,H) \geq (\chi(G)-1)(C(H)-1)+1$, where $\chi(G)$ is the chromatic number of G and $C(H)$ is the number of vertices of the largest component of H. We present some notations used in this note. Let G be any graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* of G, denoted by $|G|$, is the number of its vertices. The graph $\overline{G}$, the *complement* of G, is obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of G. A graph $F = (V', E')$ is a *subgraph* of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$. For $S \subseteq V(G)$, $G[S]$ represents the *subgraph induced* by S in G. If G is a graph and H is a subgraph of G, then denote $G[V(G)\backslash V(H)]$ by $G\backslash H$. For $v \in V(G)$ and $S \subset V(G)$, the *neighborhood* $N_S(v)$ is the set of vertices in S which are adjacent to v. Furthermore, we define $N_S[v] = N_S(v) \cup \{v\}$. If $S = V(G)$, then we use $N(v)$ and $N[v]$ instead of $N_{V(G)}(v)$ and $N_{V(G)}[v]$, respectively. The *degree* of a vertex v in G is denoted by $d_G(v)$. Let S_n be a *star* on n vertices and C_m be a *cycle* on m vertices. We denote the *complete bipartite* whose partite sets are of order n and p by $K_{n,p}$.

Since then the Ramsey numbers $R(G,H)$ for many combinations of graphs G and H have been extensively studied by various authours, see a nice survey paper [8]. In particular, the Ramsey numbers for combinations involving union of stars have also been investigated. Let S_n be a star of n vertices and W_m a wheel with m spokes.

For a combination of stars with wheels, Surahmat et al. [9] determined the Ramsey numbers for large stars versus small wheels. Their result is as follows.

Theorem A. (Surahmat and E. T. Baskoro, [9]) *For* $n \geq 3$,

$$R(S_n, W_4) = \begin{cases} 2n+1, & \text{if } n \text{ is even,} \\ 2n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Parsons in [7] considered about the Ramsey numbers for stars versus cycles as presented in Theorem .

Theorem B. (Parsons's upper bound, [7]) *For* $p \geq 2$, $R(S_{1+p}, C_4) \leq p + \sqrt{p} + 1$.

Hasmawati et al. in [6] and [5] proved that $R(S_6, C_4) = 8$, and $R(S_6, K_{2,m}) = 13$ for $m = 5$ or 6 respectively.

Let G be a graph. The number of vertices in a maximum independent set of G denoted by $\alpha_0(G)$, and the union of s *vertices-disjoint* copies of G denoted by sG . S. A. Burr et al. in [3], showed that if the graph G has n_1 vertices and the graph H has n_2 vertices, then

$$n_1s + n_2t - D \leq R(sG, tH) \leq n_1s + n_2t - D + k,$$

where $D = \min\{s\alpha_0(G), t\alpha_0(H)\}$ and k is a constant depending only on G and H. Recently, Baskoro et al. in [1] determined the Ramsey numbers for multiple copies of a star versus a wheel. Their results are given in the next theorem.

Theorem C. [1] *For $n \geq 3$,*

$$R(kS_n, W_4) = \begin{cases} (k+1)n, & \text{if } n \text{ is even and } k \geq 2, \\ (k+1)n - 1, & \text{if } n \text{ is odd and } k \geq 1. \end{cases}$$

2. **Main Results**

In this paper, we study the Ramsey numbers for disjoint union of stars versus small cycle and small wheel. The results are presented in the next two theorems. Before present these theorems let us present the lemma as follow.

Lemma 2.1. *For $k \geq 2$ and $n_i \geq 3$ for every i,*

$$R(\bigcup_{i=1}^{k} S_{n_i+1}, C_4) \geq \sum_{i=1}^{k} n_i + k + 1.$$

Proof. Let $n_i \geq 3$ for every i and $k \geq 2$. Consider $F \cong K_{\sum_{i=1}^{k}(n_i+1)-1} \cup K_1$. Graph F has $\sum_{i=1}^{k} n_i + k$ vertices, however it contains no $\bigcup_{i=1}^{k} S_{1+n_i}$. It is easy to see that $\overline{F}$ is isomorphic with $K_{1,\sum_{i=1}^{k}(n_i+1)-1}$. So, $\overline{F}$ contains no C_4. Hence, $R(\bigcup_{i=1}^{k} S_{1+n_i}, C_4) \geq \sum_{i=1}^{k}(n_i+1) + 1$. $\square$

Theorem 2.2. *Let $n_i \geq 3$ for $i = 1, 2, \ldots, k$. If $n_i \geq n_{i+1} \geq \sqrt{n_i} - 2$, then $R(\bigcup_{i=1}^{k} S_{1+n_i}, C_4) = \sum_{i=1}^{k} n_i + k + 1$ for $k \geq 2$.*

Proof. For $k = 2$, we show that $R(S_{1+n_2} \cup S_{1+n_2}, C_4) = n_1 + n_2 + 3$. Let F_1 be a graph of order $n_1 + n_2 + 3$ for $n_1, n_2 \geq 3$. Suppose $\overline{F_1}$ contains no C_4. Since $n_2 + 2 \geq \sqrt{n_1}$, then $|F_1| \geq n_1 + \sqrt{n_1} + 1$. By Parsons's upper bound, we have $|F_1| \geq R(S_{1+n_1}, C_4)$ for $n_1 \geq 3$. Thus $F_1 \supseteq S_{1+n_1}$.
Let $V(S_{1+n_1}) = \{v_0, \ldots, v_{n_1}\}$ with center v_0. Write $T = F_1 \backslash S_{1+n_1}$. Thus $|T| = n_2 + 2$. If there exists $v \in T$ with $d_T(v) \geq n_2$, then T contains S_{1+n_2}. Hence F_1 contains $S_{1+n_2} \cup S_{1+n_2}$. Therefore, we assume that for every vertex $v \in T$, $d_T(v) \leq (n_2 - 1)$.

Let u be any vertex in T. Write $Q = T \backslash N_T[u]$. Clearly, $|Q| \geq 2$. Observe that if there exists $s \in F_1$ where $s \neq u$ which is not adjacent to at least two vertices in Q, then $\overline{F_1}[\{s, u\} \cup Q]$ will contains C_4, a contradiction. Hence, for the remaining of the proof we will use the following assumption.

Assumption 1. Every vertex $s \in F_1$, $s \neq u$ is not adjacent to at most one vertex in Q.

Let u be adjacent to at least $n_2 - |N_T(u)|$ vertices in $S_{1+n_1} - v_0$, call them $v_1, \ldots, v_{n_2-|N_T(u)|}$. Observe that $n_2 - |N_T(u)| = |Q| - 1$. By Assumption 1, vertex v_0 is adjacent to at least $|Q| - 1$ vertices in Q, namely $q_1, \ldots, q_{n_2-|N_T(u)|}$. Then we have two new complete bipartite graphs, namely S'_{1+n_1} and S_{1+n_2}, where

$$V(S'_{1+n_1}) = (S_{1+n_1} \backslash \{v_1, \ldots, v_{n_2-|N_T(u)|}\}) \cup \{q_1, \ldots, q_{n_2-|N_T(u)|}\}$$

with v_0 as the center and

$$V(S_{1+n_2}) = N_T[u] \cup \{v_1, \ldots, v_{n_2-|N_T(u)|}\}$$

with u as the center. Hence, we have $F_1 \supseteq S_{1+n_2} \cup S_{1+n_2}$.

Now, we assume that u is adjacent to at most $n_2 - |N_T(u)| - 1$ vertices in $S_{1+n_1} - v_0$. This means u is not adjacent to at least $|N_T(u)| + 1$ vertices in $S_{1+n_1} - v_0$. Let $Y = \{y \in S_{1+n_1} - v_0 : yu \notin E(F_1)\}$. Then $|Y| \geq |N_T(u)| + 1 \geq 1$.

Suppose for every $y \in Y$, there exists $r \in N_T(u)$ such that $yr \notin E(F_1)$. Since $|N_T(u)| < |Y|$, then there exists $r_0 \in N_T(u)$ so that r_0 is not adjacent to at least two vertices in Y, say y_1 and y_2. This implies, $\overline{F_1}[u, r_0, y_1, y_2]$ forms a C_4, a contradiction. Hence, there exists $y' \in Y$ so that y' is adjacent to all vertices in $N_T(u)$. Furthermore, by Assumption 1 we have that $|N_T(y')| \geq |N_T(u)| + |Q| - 1 = |T| - 2 = n_2$.

Let q' be the vertex in Q which is not adjacent with y'. If $v_0u \notin E(F_1)$, then v_0 must be adjacent to q'. (Otherwise $\overline{F}$ would contain C_4 formed by $\{v_0, y', q', u\}$). Now we have two new stars, namely $S^1_{1+n_1}$ and $S^2_{1+n_2}$, where $V(S^1_{1+n_1}) = N_T[y']$ with y' as the center and $V(S^2_{1+n_2}) = (S_{1+n_1} \backslash \{y'\}) \cup \{q'\}$. If $v_0u \in E(F_1)$, then we also have two new stars. The first one is $S^1_{1+n_1}$ as in the previous case and the second one is $S^3_{1+n_2}$ where $V(S^3_{1+n_2}) = (S_{1+n_1} \backslash \{y'\}) \cup \{u\}$ with v_0 as the center. In case that y' is adjacent with all vertices in Q and $v_0u \notin E(F_1)$, then the second star is $S^4_{1+n_2}$ where $V(S^4_{1+n_2}) = (S_{1+n_1} \backslash \{y'\}) \cup \{q\}$, $q \in Q$ with v_0 as the center. The fact that $v_0q \in E(F_1)$ is guaranteed by Assumption 1.

Therefore, we have $F_1 \supseteq S_{1+n_2} \cup S_{1+n_2}$. Thus $R(S_{1+n_2} \cup S_{1+n_2}, C_4) \leq n_1 + n_2 + 3$. By Lemma 2.1 we have $R(S_{1+n_2} \cup S_{1+n_2}, C_4) = n_1 + n_2 + 3$.

We assume the theorem holds for every $2 \leq r < k$. Let F_2 be a graph of order $\sum_{i=1}^{k} n_i + k + 1$. Suppose $\overline{F}_2$ contains no C_4. We will show that $F_2 \supseteq \bigcup_{i=1}^{k} S_{1+n_i}$. By induction hypothesis, $F_2 \supseteq \bigcup_{i=1}^{k-1} S_{1+n_i}$. Write $B = F_2 \backslash \bigcup_{i=1}^{k-2} S_{1+n_i}$ and $T' = F_2[B]$. Thus $|T'| = n_{k-1} + n_k + 3$. Since $\overline{T}'$ contains no C_4 and its follows from the case $k = 2$ that T' contains $S_{1+n_{k-1}} \cup S_{1+n_k}$. Hence F_2 contains $\bigcup_{i=1}^{k} S_{1+n_i}$.

Thus we have $R(\bigcup_{i=1}^{k} S_{1+n_i}, C_4) \leq \bigcup_{i=1}^{k} n_i + k + 1$. On the other hand, we have $R(\bigcup_{i=1}^{k} S_{1+n_i}, C_4) \geq \sum_{i=1}^{k} n_i + k + 1$ (by Lemma 2.1). The assertion follows. $\square$

Theorem 2.3. *Let n_i be natural number for $i = 1, 2, \ldots, k$ and $n_i \geq n_{i+1} \geq 3$ for every i. If n_i is odd and $2n_{i+1} \geq n_i$ for every i, then $R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = R(S_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i$ for $k \geq 1$.*

PROOF. Let n_i be odd and $2n_{i+1} \geq n_i$ for every i. Consider $F \cong K_{-1+\sum_{i=1}^{k} n_i} \cup K_{n_k - 1}$. Clearly, the graph F has order $-2 + 2n_k + \sum_{i=1}^{k-1} n_i$, without containing $\sum_{i=1}^{k} S_{n_i}$ and $\overline{F}$ contains no W_4. Hence,

$$R(\bigcup_{i=1}^{k} S_{n_i}, W_4) \geq -1 + 2n_k + \sum_{i=1}^{k-1} n_i.$$

To obtain the Ramsey number we use an induction on k. For $k = 1$, we have $R(S_{n_1}, W_4) = 2n_1 - 1$ (by Theorem 1). For $k = 2$, we show that $R(S_{n_1} \cup S_{n_2}, W_4) = 2n_2 - 1 + n_1 = R(S_{n_2}, W_4) + n_1$.

Let F_1 be a graph with $|F_1| = 2n_2 - 1 + n_1 = 2n_1 - 1 + 2n_2 - n_1$. Assume that $\overline{F}_1$ contains no W_4. We show that F_1 contains $S_{n_1} \cup S_{n_2}$. Since $2n_2 \geq n_1$, then $|F_1| \geq 2n_1 - 1$. By Theorem 1, F_1 contains S_{n_1}. Write $L = F_1 \backslash S_{n_1}$. Thus $|L| = 2n_2 - 1$, such that L contains S_{n_2}. Hence, F_1 contains $S_{n_1} \cup S_{n_2}$. Therefore, $R(S_{n_1} \cup S_{n_2}, W_4) \leq 2n_2 - 1 + n_1$.

Suppose the theorem holds for every $r < k$. Let F_2 be a graph of order $-1 + 2n_k + \sum_{i=1}^{k-1} n_i$. Suppose $\overline{F_2}$ contains no W_4. By the assumption, F_2 contains $\bigcup_{i=1}^{k-1} S_{n_i}$. Let $L' = F_2 \backslash \bigcup_{i=1}^{k-1} S_{n_i}$. Thus $|L'| = 2n_k - 1$. Since $\overline{L'}$ contains no W_4, then by Theorem 1, $L' \supset S_{n_k}$. Hence, F_2 contains $\bigcup_{i=1}^{k} S_{n_i}$. Therefore, we have

$$R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = -1 + 2n_k + \sum_{i=1}^{k-1} n_i = R(S_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i.$$

$\square$

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