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WAVELET SOLUTIONS AND STABILITY ANALYSIS OF PARABOLIC EQUATIONS

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Abstract. In this paper wavelet solutions of extended Sideways and non standard parabolic equations have been analyzed along with stabilization and errors estimation.

Key words and Phrases: Ill-posed Problem; Regularization; Meyer Wavelet; Inverse Heat Conduction; Sideways Heat Equation; Wavelet-Galerkin Method.

Abstrak. Di dalam paper ini dianalisa kestabilan dan estimasi kesalahan untuk penyelesaian wavelet dari persamaan parabolik non standar yang diperluas satu sisi.

Kata kunci: Masalah *ill-posed*; Regularisasi; Wavelet Meyer; Konduksi panas invers; Persamaan panas satu sisi; Metode Wavelet-Galerkin.

1. **Sideways Heat Equation: An Introduction**

Ill-posed problems have always been in the focus of industrial applications. An inverse ill-posed problem is one for which a small perturbation on the boundary specification (g) can amount to a big alteration on its solution, if it exists. That is, if the solution exists, it does not depend continuously on data (g). Meyer multiresolution analysis plays a key role in the solution of parabolic heat conduction problems.

Organization of paper is as follows. Wavelet regularized and Galerkin solutions of Standard Sideways heat equation (SHE) along with stability and errors estimation has been reviewed as first part. In the second part, wavelet regularization of extended SHE has been obtained and errors have been estimated. Third part introduces inequality based wavelet-Galerkin solution of extended SHE. In it

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numerical solution of non standard parabolic equations of extended SHE has been subjected to stability consideration. Fourth part is devoted to test problem while fifth part is the conclusion. At the end set of references are listed.

Consider the following one dimensional parabolic heat conduction problem in quarter plane $(x \geq 0, t \geq 0)$, assuming that body is large,

$$\begin{cases} u_{xx} = u_t, & x \geq 0, t \geq 0; \\ u(x,0) = 0, & x \geq 0; \\ u(1,t) = g(t), & t \geq 0, u|_{x\to\infty} \text{ bounded.} \end{cases} \tag{1}$$

Assumption $u|_{x\to\infty}$ bounded guarantee the uniqueness of solution. A solution $u(x,t) \in L^2(0,\infty)$ for $x \geq 0$ can be obtained from initial temperature data g. This equation called Sideways heat equation, is an inverse ill posed problem. This equation is a model of a situation where one wants to determine the surface heat flux (temperature) on the both sides of a heat conducting body from measured transient temperature at a fixed location inside the body. The simplest example can of lacquer coating to be applied on one side of a particle board. The surface under coating is inaccessible to measurement due to extreme heat condition. Inside temperature g_ε is measured at $x = 1$ indirectly through a thermocouple inserted into plate through other side. Based on internal measurements, surface $(x = 0)$ temperature is computed.

But even with precautions, measurement errors are bound to occur in g. Let $g_\varepsilon \in L^2(R)$ be the perturbed data such that data error

$$\|g - g_\varepsilon\| \leq \varepsilon \tag{2}$$

for some bound $\varepsilon > 0$. Impose *a priori* bound on the solution at $x = 0$, i.e.,

$$\|u(0,t)\| \leq M. \tag{3}$$

The problem 1 can now be moduled as

$$\begin{cases} u_{xx} = u_t, & x \geq 0, t \geq 0; \\ u(x,0) = 0, & x \geq 0; \\ \|u(1,t) - g_\varepsilon\| \leq \varepsilon & \\ \|u(0,t)\| \leq M & \end{cases} \tag{4}$$

The problem now becomes well posed, that is, stable in the sense that for any two solutions u_1 and u_2

$$\|u_1(x,t) - u_2(x,t)\| \leq 2M^{1-x}\varepsilon^x, 0 \leq x < 1.$$

This was proved by Levine in 1983. The problem 4 was approximated for the first time by [18] and [11] by multiscale analysis and wavelet techniques of measured data. In frequency space, $u(x,t), g(t), g_\varepsilon(t)$ extended to whole of t-axis by defining $u(x,t), g(t), f(t) = u(0,t) \in L^2(R)$ to be zero for $t < 0$.

Although we intend to recover $x > 0$ for $0 \leq x < 1$, the problem specification includes the heat equation for $x > 1$ together with boundedness at infinity. To obtain $u(x,t)$ for $x > 1, u_x(1,t)$ is determined also. Equation 4 together with $u_x(1,t)$ is a Cauchy problem.

Mattos and Lopes [14] gave another version of 1

$$\begin{cases} k(x)u_{xx}(x,t) = u_t(x,t), & t \geq 0, 0 \leq x < 1 \ [0 < \alpha \leq k(x) < \infty] \\ u(0,t) = g(t) \\ u_x(0,t) = 0 \end{cases}$$

where $k(x)$ is smooth. While Elden et al. [4, 3] gave yet another type:

$$\begin{cases} (k(x)u_x(x,t))_x = u_t(x,t), & t \geq 0, 0 \leq x < 1 \ [0 < \alpha \leq k(x) < \infty]; \\ u(0,t) = g(t) \\ u_x(0,t) = 0 \end{cases}$$

There are quite other methods available for solving various parabolic heat conduction equations as difference approximation, optimal filtering, optimal approximation, method of lines, dual least square, singular value analysis and spectral and Tikhonov regularizations. See, for details, [22], [13], [5, 6], [20], [8, 10] and [24].

Taking Fourier transform (FT) on both sides of 1 w.r.t. t, the frequency space solution $\hat{u}(x,\omega) \in L^2(R)$ is

$$\hat{u}(x,\omega) = e^{(1-x)\sqrt{i\omega}}\hat{g}(\omega) \tag{5}$$

Also $\hat{f}(\omega) = \hat{u}(0,\omega) = e^{\sqrt{i\omega}}\hat{g}(\omega)$. Since $\sqrt{i\omega}$ tends to infinity as $|\omega| \to \infty$, the problem thus is ill-posed. Further, the existence of the solution in $L^2(R)$ depends on fast decay of $\hat{g}_\varepsilon$ at high frequencies. The solution $u(x,t)$ to 1 is

$$u(x,t) = \int_{-\infty}^{\infty} e^{i\omega t} e^{(1-x)\sqrt{i\omega}}\hat{g}(\omega)d\omega. \tag{6}$$

By Parseval formula,

$$\|u(x,t)\|^2 = \|\hat{u}(x,\omega)\|^2 = \int_{-\infty}^{\infty} e^{(1-x)\sqrt{2|\omega|}}|\hat{g}(\omega)|^2 d\omega,$$

where $\frac{\sqrt{|\omega|}}{2}$ is real part of $\sqrt{i\omega}$. This shows that $\hat{g}(\omega)$ has to decay rapidly as $\omega \to \infty$. If the initial data g is noisy, the $\hat{g}(\omega)$ will have high frequency components and are to be cut by the Meyer multiresolution analysis.

Meyer wavelets have compact support in frequency domain (but not in time domain) and decay very fast. Orthogonal projection on to Meyer scaling spaces prevent high frequency noise from destroying the numerical solution i.e. perturbation. Regularize the problem by eliminating higher frequencies from the solution 6 by taking only $|\omega| < \omega_{\max}$. The regularized solution is

$$\tilde{u}(x,t) = \int_{-\infty}^{\infty} e^{i\omega t} e^{(1-x)\sqrt{i\omega}} \hat{g}_\varepsilon(\omega)\chi_{\max} d\omega, \tag{7}$$

where $\chi_{\max}$ is the characteristic function of the interval $[-\omega_{\max}, \omega_{\max}]$.

Here we present some error estimates from [2].

Theorem 1.1. *Suppose that we have two regularized solutions $\tilde{u}_1$ and $\tilde{u}_2$ defined by 7 with data g_1 and g_2 satisfying $\|g_1 - g_2\| < \varepsilon$. If we select $\omega_{max} = 2\left(\log\left(\frac{M}{\varepsilon}\right)\right)^2$, then we get the error bound*

$$\|\tilde{u}_1(x,t) - \tilde{u}_2(x,t)\| < M^{1-x}\varepsilon^x.$$

Theorem 1.2. *Let u and $\tilde{u}$ defined be the solutions of 6 and 7 with the same exact data g and let $\omega_{max} = 2\left(\log\left(\frac{M}{\varepsilon}\right)\right)^2$. Then*

$$\|u(x,t) - \tilde{u}(x,t)\| < M^{1-x}\varepsilon^x.$$

Theorem 1.3. *Suppose that u is given by 6 with exact data g and that $\tilde{u}$ given by 7 with measured data g_ε. Select $\omega_{max} = 2\left(\log\left(\frac{M}{\varepsilon}\right)\right)^2$, then we get the error bound*

$$\|u(x,t) - \tilde{u}(x,t)\| < 2M^{1-x}\varepsilon^x.$$

Meyer Multiresolution Analysis and Wavelet Regularization (MRA)

Let $\alpha_j = 2^j\alpha_0$, where $\alpha_0 = \frac{2}{3}\pi$, $j \in Z$. The FT of Meyer scaling is given by [1]

$$\hat{\varphi}(\omega) = \begin{cases} 1, & |\omega| \leq \alpha_0; \\ \cos\left[\frac{\pi}{2}\nu\left(\frac{|\omega|}{2\alpha} - 1\right)\right], & \alpha_0 \leq |\omega| \leq \alpha_1; \\ 0, & \text{otherwise.} \end{cases}$$

where ν is C^k differentiable function $(0 \leq k \leq \infty)$ satisfying

$$\nu(x) = \begin{cases} 1, & x \leq 0; \\ 0, & x \geq 0. \end{cases}$$

with additional condition $\nu(x) + \nu(1-x) = 1$.

Clearly $\hat{\varphi}$ is a C^k function. Corresponding wavelet is given by

$$\hat{\psi}(\omega) = \begin{cases} e^{i\omega/2}\sin\left[\frac{\pi}{2}\nu\left(\frac{|\omega|}{2\alpha} - 1\right)\right], & \alpha_0 \leq |\omega|; \\ e^{i\omega/2}\cos\left[\frac{\pi}{2}\nu\left(\frac{|\omega|}{2\alpha} - 1\right)\right], & \alpha_0 \leq |\omega| \leq \alpha_1; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{supp}\hat{\varphi} = [-\alpha_1, \alpha_1]$$

$$\text{supp}\hat{\psi} = [-\alpha_2, -\alpha_0] \cup [\alpha_0, \alpha_2]$$

MRA $\{V_j\}_{\in Z}$ of Meyer wavelet is generated by

$$V_j = \{\varphi_{j,k}, k \in Z\};\ \varphi_{j,k} = 2^{j/2}\varphi(2^j x - k), j, k \in Z$$
$$\hat{\varphi}_{j,k} = \int_R \varphi_{j,k} e^{-ix\omega} dx = 2^{-j/2} e^{-ik2^{-j}\omega} \hat{\varphi}(2^{-j}\omega).$$

Also $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), j, k \in Z$ constitutes an orthonormal basis of $W_j \in L^2(R)(V_{j+1} = V_j \oplus W_j)$.

$$\hat{\psi}_{j,k} = 2^{-j/2} e^{-ik2^{-j}\omega} \hat{\psi}(2^{-j}\omega)$$
$$\text{supp}\hat{\varphi}_{j,k} = [-\alpha_{j+1}, \alpha_{j+1}], k \in Z$$
$$\text{supp}\hat{\psi}_{j,k} = [-\alpha_{j+2}, -\alpha_j] \cup [\alpha_j, \alpha_{j+2}], k \in Z$$

Let Π_j and P_j (fixed $j \in N$) be the orthogonal projections of $L^2(R)$ onto V_j and W_j respectively. Then for $h, w \in L^2(R)$

$$h = \Pi_j h(t) = \sum_{k \in Z} \langle h, \varphi_{lk} \rangle \varphi_{lk}(t), l \le j$$
$$w = P_j w(t) = \sum_{k \in Z} \langle h, \psi_{lk} \rangle \psi_{lk}(t), l \le j$$

The corresponding orthogonal projections in frequency space follow as:

$$\hat{\Pi}_j : L^2(R) \to \hat{V}_j = \overline{span\{\hat{\varphi}_{jk}\}}_{k \in Z}$$
$$\hat{P}_j : L^2(R) \to \hat{W}_j = \overline{span\{\hat{\psi}_{jk}\}}_{k \in Z}$$

According to 5, for any function $g \in L^2(R)$ such that its FT $\hat{g}$ belongs to V_j, there exists a solution $\hat{u}$ with boundary condition $\hat{u}(1, \omega) = \hat{g}(\omega)(\hat{g}(\omega) = \hat{h}(\omega))$. The whole mechanism suggests for a Fourier regularization process involving a family of problems in the frequency space parameterized by $j \in Z$ defined by

$$\begin{cases} \hat{u}_{xx}(x, \omega) = i\omega \hat{u}(x, \omega), & \omega \in R, 0 \le x < \infty; \\ \hat{u}(1, \omega) = \Pi_j \hat{g}_\varepsilon(\omega), & t \ge 0, \hat{u}|_{x \to \infty} \text{ bounded.} \end{cases} \tag{8}$$

This has a unique solution since support of $\Pi_j \hat{g}_\varepsilon$ is compact. The solution is

$$\hat{u}(x, \omega) = e^{(1-x)\sqrt{i\omega}} \Pi_j \hat{g}_\varepsilon(\omega). \tag{9}$$

For wavelet regularization,

$$\begin{cases} \hat{u}_{xx}(x, \omega) = i\omega \hat{u}(x, \omega), & \omega \in R, 0 \le x < \infty; \\ \hat{u}(1, \omega) = P_j \hat{g}_\varepsilon(\omega), & t \ge 0, \hat{u}|_{x \to \infty} \text{ bounded.} \end{cases} \tag{10}$$

The unique solution $\hat{u}$ does not have any high frequency components as support of $P_j \hat{g}_\varepsilon$ is compact. The solution in this case is

$$\hat{u}(x,\omega) = e^{(1-x)\sqrt{\omega}} P_j \hat{g}_\varepsilon(\omega). \tag{11}$$

Notice that

$$\hat{g} = \hat{\Pi}_j \hat{g} + (1 - \hat{\Pi}_j)\hat{g} = \hat{\Pi}_j \hat{g} + \hat{P}_j \hat{g} = \sum_{k\in Z} \langle \hat{g}, \hat{\varphi}_{lk} \rangle \hat{\varphi}_{lk} + \sum_{l\geq j} \sum_{k \in Z} \langle \hat{g}, \hat{\psi}_{lk} \rangle \hat{\psi}_{lk}$$

This implies

$$\hat{\Pi}_j \hat{g} = \hat{g} \text{ for } |\omega| \leq \alpha_j \text{ or } |\omega| \geq \alpha_{j+2} \text{ since } \hat{\psi}_{j,k}(\omega) = 0 \ \forall l \leq j$$

$$\hat{P}_j \hat{g} = \hat{g} \text{ for } |\omega| \geq \alpha_{j+1}$$

$$\hat{P}_j \hat{g} = 0 \text{ for } l > j \text{ and } |\omega| \leq \alpha_{j+1}$$

P_j can be considered as a low pass filter as frequencies higher than α_{j+1} are filtered away.

Theorem 1.4. [18] *Let g_ε be the measured data satisfying 2. Let $u_{\varepsilon j}(x,t)$ denote the inverse Fourier transform of the solution of 8 with $g = g_\varepsilon$. If $j = j(\varepsilon)$ is such that*

$$\varepsilon e^{\sqrt{\alpha_{j+1}}} \leq M \text{ and } j(\varepsilon) \to \infty \text{ when } \varepsilon \to 0,$$

then for $0 \leq x \leq 1$

$$\|u(x,t) - u_{\varepsilon j}(x,t)\| \to 0 \text{ when } \varepsilon \to 0,$$

and, moreover,

$$\|u_{\varepsilon j}(x,t) - u(x,t)\|^2 \leq M^{2(1-x)} \left[\varepsilon^{2x} + \varepsilon_j^{2x}\right], \text{ where } \varepsilon_j = Me^{-\sqrt{\alpha_j + 1}} \tag{12}$$

Theorem 1.5. [18] *Let g_ε be the measured data satisfying 2. Let $v_{\varepsilon j}(x,t)$ denote the inverse Fourier transform of the solution of 12 with $g = g_\varepsilon$. If $j = j(\varepsilon)$ is such that*

$$\varepsilon e^{\sqrt{\alpha_{j+1}}} \leq M \text{ and } j(\varepsilon) \to \infty \text{ when } \varepsilon \to 0,$$

then for $x > \frac{\sqrt{2}-1}{\sqrt{2}}$

$$\|v_{\varepsilon j}(x,t) - v(x,t)\| \to 0 \text{ when } \varepsilon \to 0,$$

and, the following inequality holds:

$$\|u(x,t) - v_{\varepsilon j}(x,t)\|^2 \leq 2M^{2(1-x)} \left[2\varepsilon^{2x} + \varepsilon_j^{2x}\right] + 2C_j M^{2(1-\sqrt{2}/2-x)} \varepsilon_j^{2(x+\sqrt{2}/2-1)},$$

where $\varepsilon_j = Me^{-\sqrt{\alpha_j + 1}}$ and $\{C_j\}$ is a certain sequence converging to 0 as $j \to \infty$.

Theorem 1.6. [18] *Let g_ε be the measured data satisfying 2. Let $\{x_k\}$ be the decreasing sequence of knots that holds:*

$$1 > x_0 > \left(1 - 2^{-\frac{1}{2}}\right) x_1 > \cdots > x_{k-1} > x_k > \left(1 - 2^{-\frac{k+1}{2}}\right) x_{k-1}, \ k = 1, 2, \cdots$$

Let $v_{\varepsilon j}(x,t)$ be defined by the recurrence relation

$$\hat{v}_{\varepsilon j}(x,\omega) = \begin{cases} e^{(1-x)\sqrt{i\omega}} P_j \hat{g}_\varepsilon(\omega), & x \in [x_0, 1); \\ e^{(x_{k-1}-x)\sqrt{i\omega}} P_{j-k} \hat{v}_{\varepsilon j}(x_{k-1},\omega), & x \in [x_k, x_{k-1}),\ k = 1, 2, \cdots. \end{cases}$$

Then for any $x \in (0,1)$

$$\|v_{\varepsilon j}(x,t) - u(x,t)\| \to 0 \text{ when } \varepsilon \to 0.$$

Remarks. According to [9] interval (0,1) is to be replaced by $(e^*, 1)$, where $e^* = \lim_{k\to\infty} e_k, e^k = \left(1 - 2^{-\frac{1}{2}}\right)\left(1 - 2^{-\frac{2}{2}}\right)\cdots\left(1 - 2^{-\frac{k}{2}}\right)$ and $0.037513 < e^* < 0.037514$.

Galerkin Solution of 1 in Scaling Spaces V_j

Approximating solution of 1 in scaling spaces

$$\begin{aligned} \langle u_{xx} - u_t, \varphi_{jk}\rangle &= 0, \\ \langle u(0,t), \varphi_{jk}\rangle &= \langle P_j g, \varphi_{jk}\rangle, \\ \langle u_x(0,t), \varphi_{jk}\rangle &= \langle 0, \varphi_{jk}\rangle, k \in Z. \end{aligned} \tag{13}$$

where φ_{jk} is the orthonormal basis of V_j given by the scaling function φ. Letting the approximate solution $u_j(x,t) \in V_j$ be in the form of variable separable

$$u_j(x,t) = \sum_{l\in Z} W_l(x)\varphi_{jl}(t).$$

The equation 13 reduces to infinite dimensional differential equation **??**

$$\frac{d^2W}{dx^2} = D_j(x)W \text{ with } W(1) = \gamma, W'(1) = \nu, \tag{14}$$

where $\gamma = P_j g = \sum_{l\in Z} \gamma_l \varphi_{jl} = \sum_{l\in Z} \langle g, \varphi_{jl}\rangle\varphi_{jl}$. $\nu = P_j h = \sum_{l\in Z} \nu_l \varphi_{jl} = \sum_{l\in Z}\langle \nu, \varphi_{jl}\rangle\varphi_{jl}$. Here $\|D_j(x)\| \le \pi 2^{-j}$. Solution $W = \gamma e^{\left((1-x)\sqrt{D_j}\right)}$ for exact data γ.

Theorem 1.7. [19] *Let u_j and $\tilde{u}_j$ be solutions in V_j of the approximating problem 1 with $g = P_j g$ for the boundary specifications g and $\tilde{g}$ respectively. If $\|g - \tilde{g}\| < \varepsilon$, then*

$$\|u_j(x,t) - \tilde{u}_j(x,t)\| \le e^{\left((1-x)\sqrt{\frac{1}{2}2^{-j}\pi}\right)}.$$

For $j = j(\varepsilon)$ such that $2^{-j} \le \frac{2}{\pi}\left(\log\frac{M}{\varepsilon}\right)$, we have

$$\|u_j(x,t) - \tilde{u}_j(x,t)\| \le \varepsilon^x M^{1-x}.$$

Theorem 1.8. [19] *If u is a solution of problem 1, then*

$$\|u(x,t) - P_j u(x,t)\| \le M e^{\left\lfloor -x\sqrt{\frac{1}{3}\pi 2^{-j}}\right\rfloor},$$

For j such that $2^{-j} \le \frac{3}{\pi}\log\varepsilon^{-1}$, we have

$$\|u(x,t) - P_j u(x,t)\| \le M\varepsilon^{-x^2}.$$

2. Wavelet Regularization of Extended Sideways Heat Equation

Now we find wavelet regularization of the other version of sideways heat conduction problem, the extended SHE. Consider the following heat conduction problem [Mattos et al.]:

$$\begin{cases} k(x)u_{xx}(x,t) = u_t(x,t), \quad [0 < \alpha \le k(x) < \infty]; \\ u(0,t) = g(t), \\ u_x(0,t) = 0, \end{cases} \tag{15}$$

where $g \in L^2(R)$ is such that the measured data g_ε satisfies $\|g - g_\varepsilon\| < \varepsilon$ for some constant $\varepsilon > 0$, and $0 < \alpha \le k(x) < \infty$, k continuous. To find $u \in L^2(R)$ subject to *a priori* bound $\|F\| = \|u(1,t)\| \le M$, and $u|_{x\to\infty}$ bounded.

Define u, g, F to the whole t-axis by defining them to be zero for $t < 0$. Taking FT of 15 w.r.t. t,

$$k\int_0^\infty e^{-i\omega t}u_{xx}dt = \int_0^\infty e^{-i\omega t}u_t dt \text{ or } \hat{u}_{xx} = \frac{i\omega}{k(x)}\hat{u}.$$

This implies

$$\hat{u}(x,\omega) = Ae^{x\sqrt{i\omega/k}} + Be^{-x\sqrt{i\omega/k}} \tag{16}$$

But $\hat{u}(0,\omega) = \hat{g}(\omega)$, and so

$$\hat{g}(\omega) = A + B \tag{17}$$

Differentiating 16 w.r.t. x,

$$\hat{u}_x(x,\omega) = \frac{i\omega}{k}\left[Ae^{x\sqrt{i\omega/k}} - Be^{-x\sqrt{i\omega/k}}\right]. \tag{18}$$

Using $\hat{u}_x(0,\omega) = 0$ in 18,

$$0 = A - B. \tag{19}$$

From 17 and 19,

$$A = B = \frac{1}{2}\hat{g}(\omega).$$

So, the frequency space solution $\hat{u}(x,\omega) \in L^2(R)$ is

$$\begin{aligned} \hat{u}(x,\omega) &= \cosh\sqrt{\frac{i\omega}{k}}x\hat{g}(\omega) \\ \hat{F} = \hat{u}(1,\omega) &= \cosh\sqrt{\frac{i\omega}{k}}\hat{g}(\omega) \end{aligned} \tag{20}$$

Parseval's formula yields

$$\|u\|^2 = \|\hat{u}\|^2 = \int_{-\infty}^{\infty}\left|\cosh\sqrt{\frac{i\omega}{k}}\hat{g}(\omega)\right|^2 d\omega$$

showing the rapid decay of $\hat{g}(\omega)$ at high frequencies. Fourier regularized solution

$$\hat{u}(x,\omega) = \cosh\sqrt{\frac{i\omega}{k}}x\hat{\Pi}_j\hat{g}_\varepsilon(\omega).$$

Wavelet regularized solution of (22 is)

$$\hat{u}(x,\omega) = \cosh\sqrt{\frac{i\omega}{k}}x\hat{P}_j\hat{g}_\varepsilon(\omega).$$

Stability and Error Estimation

We state and approve the following theorems:

Theorem 2.1. *Let g be the true data of the problem 15 and g_ε the noisy measured data satisfying $\|g - g_\varepsilon\| \leq \varepsilon$for some $\varepsilon > 0$. Let there exist* a priory *bound* $\|F\| = \|\cosh\sqrt{\frac{i\omega}{k}}\hat{g}\| \leq M$. *Further, assume that $j = j(\varepsilon)$ be such* $\varepsilon\cosh\sqrt{\frac{i\alpha_{j+2}}{k}} \geq M$. *Then*

$$\|\hat{g} - \hat{P}_j g_\varepsilon\| = \|g - P_j g_\varepsilon\| \leq \frac{M}{\cosh\sqrt{\frac{i\alpha_{j+2}}{k}}} + \left[C_j\left(\varepsilon^2 + \frac{M^2}{\cosh\sqrt{\frac{i\alpha_{j+2}}{k}}}\right)\right]^{\frac{1}{2}}$$

where C_j is a certain sequence converging to zero as $j \to \infty$.

Lemma 2.2. $\|\hat{g} - \Pi_j\hat{g}_\varepsilon\|^2 \leq \frac{M^2}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}$

Proof.

$$\begin{aligned}
\|\hat{g} - \Pi_j\hat{g}_\varepsilon\|^2 &= \int_{-\infty}^{\infty} |\hat{g} - \Pi_j\hat{g}_\varepsilon|^2 d\omega \\
&= \int_{|\omega|\leq\alpha_{j+2}} |\hat{g} - \Pi_j\hat{g}_\varepsilon|^2 d\omega + \int_{|\omega|\geq\alpha_{j+2}} |\hat{g} - \Pi_j\hat{g}_\varepsilon|^2 d\omega \\
&= \int_{|\omega|\geq\alpha_{j+2}} |\hat{g}|^2 d\omega = \int_{|\omega|\geq\alpha_{j+2}} \frac{|\hat{F}|^2}{\cosh^2\sqrt{\frac{i\omega}{k}}} d\omega \leq \frac{M^2}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}.
\end{aligned}$$

□

Lemma 2.3. $\|\Pi_j\hat{g}_\varepsilon - P_j\hat{g}_\varepsilon\|^2 \leq C_j\left(\varepsilon^2 + \frac{M^2}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}\right)$

Proof.

$$
\begin{aligned}
\|\Pi_j \hat{g}_\varepsilon - P_j \hat{g}_\varepsilon\|^2 &= \int_{-\infty}^{\infty} |(\Pi_j - P_j)\hat{g}_\varepsilon|^2 d\omega \\
&= \int_{-\infty}^{\infty} \left|\sum \langle \hat{g}_\varepsilon, \hat{\psi}_{j+1,k}\rangle \Pi_j \hat{\psi}_{j+1,k}\right|^2 d\omega \\
&= \int_{|\omega|\le\alpha_{j+2}} \left|\sum \langle \hat{g}_\varepsilon, \hat{\psi}_{j+1,k}\rangle \Pi_j \hat{\psi}_{j+1,k}\right|^2 d\omega \\
&= \int_{|\omega|\le\alpha_{j+2}} \left|(P_{j+1} - P_j)\hat{g}_\varepsilon\right|^2 d\omega \\
&\le \int_{\alpha_{j+1}\le|\omega|\le\alpha_{j+2}} \left|(P_{j+1} - P_j)(\hat{g}\varepsilon - \hat{g})\right|^2 d\omega + \\
&\quad \int_{|\omega|\le\alpha_{j+2}} \left|(P_{j+1} - P_j)\hat{g}\right|^2 d\omega \\
&\le \varepsilon^2 + \int \left|P_{j+1} - P_j\right|^2 \frac{|\hat{F}|^2}{\cosh^2\sqrt{\frac{i\omega}{k}}} d\omega, |\omega| \in [\alpha_{j+1}, \alpha_{j+2}] \\
&\le \varepsilon^2 + \int \left|P_{j+1} - P_j\right|^2 \frac{|\hat{F}|^2}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}} d\omega, C_j \in \|P_{j+1} - P_j\|^2 \\
&= C_j\left(\varepsilon^2 + \frac{M^2}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}\right).
\end{aligned}
$$

□

Main Proof:

$$
\begin{aligned}
\|\hat{g} - P_j g_\varepsilon\| &\le \|\hat{g} - \Pi_j g_\varepsilon\| + \|\Pi_j \hat{g}_\varepsilon - P_j g_\varepsilon\| \\
&\le \frac{M}{\sqrt{\frac{i\alpha_{j+2}}{k}}} + \left[C_j\left(\varepsilon^2 + \frac{M^2}{\cosh\sqrt{\frac{i\alpha_{j+2}}{k}}}\right)\right]^{\frac{1}{2}},
\end{aligned}
$$

using Lemma 2.2 and Lemma 2.3. □

Theorem 2.4. *Let g be the true data of the problem 15 and g_ε the noisy measured data satisfying $\|g - g_\varepsilon\| \le \varepsilon$ for some $\varepsilon > 0$. Let there exist* a priory *bound* $\|F\| = \|\cosh\sqrt{\frac{i\omega}{k}}\hat{g}\| \le M$. *Further, assume that $j = j(\varepsilon)$ be such $\varepsilon \cosh\sqrt{\frac{i\alpha_{j+2}}{k}} \ge M$.*

Then for $0 \le x < 1$

$$\begin{aligned} \|\hat{u}-\hat{v}_{\varepsilon j}\| &= \|u-v_{\varepsilon j}\| \\ &\le \left[\varepsilon^2\cosh^2\sqrt{i\alpha_{j+2}}x + M^2\left(\frac{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}\right)^2\right]^{\frac{1}{2}} + \\ &\quad \cosh\sqrt{\frac{i\alpha_{j+2}}{k}}x\left[C_j\left(\varepsilon^2+\frac{M^2}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}\right)\right]^{\frac{1}{2}} \end{aligned}$$

where C_j *is a certain sequence converging to zero as* $j \to \infty$.

Lemma 2.5.

$$\|\hat{u}-\hat{u}_{\varepsilon j}\|^2 \le \varepsilon^2\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x + M^2\left(\frac{\cosh^2\sqrt{\frac{i\omega}{k}}x}{\cosh^2\sqrt{\frac{i\omega}{k}}}\right)$$

Proof.

$$\begin{aligned} \|\hat{u}-\hat{u}_{\varepsilon j}\|^2 &= \int|\hat{g}-\Pi_j\hat{g}_\varepsilon|^2\left|\cosh\sqrt{\frac{i\omega}{k}}x\right|^2 d\omega \\ &= \int_{|\omega|\le\alpha_{j+2}}|\hat{g}-\hat{g}_\varepsilon|^2\cosh^2\sqrt{\frac{i\omega}{k}}x d\omega + \int_{|\omega|\ge\alpha_{j+2}}|\hat{g}|^2\cosh^2\sqrt{\frac{i\omega}{k}}x d\omega \\ &\le \varepsilon^2\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x + \int_{|\omega|\ge\alpha_{j+2}}\frac{\|\hat{F}\|\cosh^2\sqrt{\frac{i\omega}{k}}x}{\cosh^2\sqrt{\frac{i\omega}{k}}}d\omega \\ &\le \varepsilon^2\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x + M^2\left(\frac{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}\right) \end{aligned}$$

□

Lemma 2.6.

$$\|\hat{u}_{\varepsilon j}-\hat{v}_{\varepsilon j}\|^2 \le \cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x\left(C_j\left(\varepsilon^2+\frac{M^2}{\cosh\sqrt{\frac{i\omega}{k}}}\right)\right)$$

Proof.

$$\begin{aligned} \|\hat{u}_{\varepsilon j}-\hat{v}_{\varepsilon j}\|^2 &= \int_{|\omega|\le\alpha_{j+2}}\cosh^2\sqrt{\frac{i\omega}{k}}x|(\Pi_j-P_j)\hat{g}_\varepsilon|^2 d\omega \\ &\le \cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x\left(C_j\left(\varepsilon^2+\frac{M^2}{\cosh\sqrt{\frac{i\omega}{k}}}\right)\right), \end{aligned}$$

using Lemma 2.3 □

Main Proof:

$$\begin{aligned}\|\hat{u}-\hat{v}_{\varepsilon j}\| &= \|\hat{u}-\hat{u}_{\varepsilon j}\| + \|\hat{u}_{\varepsilon j}-\hat{v}_{\varepsilon j}\| \\ &\leq \left[\varepsilon^2\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x + M^2\left(\frac{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}\right)^2\right]^{\frac{1}{2}} + \\ &\quad \cosh\sqrt{\frac{i\alpha_{j+2}}{k}}x\left[C_j\left(\varepsilon^2+\frac{M^2}{\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}}\right)\right]^{\frac{1}{2}}.\end{aligned}$$

□

Theorem 2.7. *Let g be the true data of the problem 15 and g_ε the noisy measured data satisfying $\|g-g_\varepsilon\|\leq\varepsilon$ for some $\varepsilon>0$. Let there exist* a priory *bound* $\|F\| = \|\cosh\sqrt{\frac{i\omega}{k}}\hat{g}\| \leq M$. *Further, assume that $j=j(\varepsilon)$ be such $\varepsilon\cosh\sqrt{\frac{i\alpha_{j+1}}{k}} \geq M$. Then*

$$\|\hat{u}-P_j\hat{u}\| \leq \varepsilon\cosh\sqrt{\frac{i\alpha_{j+1}}{k}}x.$$

Proof.

$$\|\hat{u}-\hat{u}_{\varepsilon j}\|^2 \leq \varepsilon^2\cosh^2\sqrt{\frac{i\alpha_{j+2}}{k}}x + M^2\left(\frac{\cosh^2\sqrt{\frac{i\omega}{k}}x}{\cosh^2\sqrt{\frac{i\omega}{k}}}\right)$$

$$\begin{aligned}\|\hat{u}-P_j\hat{u}\| &= \int|(\hat{g}-P_j\hat{g}_\varepsilon)|^2\cosh^2\sqrt{\frac{i\omega}{k}}x\,d\omega \\ &= \int_{|\omega|\leq\alpha_{j+1}}|\hat{g}-\hat{g}_\varepsilon|^2\cosh^2\sqrt{\frac{i\omega}{k}}x\,d\omega + \int_{|\omega|\geq\alpha_{j+1}}|\hat{g}|^2\cosh^2\sqrt{\frac{i\omega}{k}}x\,d\omega \\ &\leq \varepsilon^2\cosh^2\sqrt{\frac{i\alpha_{j+1}}{k}}x.\end{aligned}$$

□

3. Inequality Based Wavelet-Galerkin Solutions

A. Wavelet-Galerkin Solution of Sideways Heat Equation

In wavelet Galerkin approach, heat equation can be solved efficiently and in numerically stable way without introducing high frequency components. Data is projected on to Meyer scaling spaces. Weak formulation of approximating problem on scaling spaces V_j, where test function is also from V_j, converts the system in to infinite dimensional second order initial value ordinary differential equation with variable coefficients.

Lemma 3.1. *If* $\{\varphi_{jk}\}_{k\in Z}$ *is the orthogonal basis of scaling spaces* V_j *such that the matrix*

$$\lfloor (D_j)_{lk}(x)\rfloor_{l,k\in Z} = \left\lfloor \frac{1}{k(x)}\langle \varphi'_{jl}, \varphi_{jk}\rangle \right\rfloor_{l,k\in Z}.$$

The matrix D_j is skew symmetric and equal along diagonals. Moreover, $\|D_j(x)\| \leq \frac{\pi 2^{-j}}{k(x)}$.

Theorem 3.2. [14] *Let* u *and* v *be positive continuous functions,* $x \geq a$ *and* $c > 0$. *If*

$$u(x) = c + \int_a^x \int_a^s v(\tau)u(\tau)ds$$

then

$$u(x) \leq ce^{\int_a^x \int_a^s v(\tau)u(\tau)ds}.$$

Solution of 15 in Frequency Domain

$$\begin{cases} k(x)\hat{u}_{xx}(x,\omega) = i\omega\hat{u}(x,\omega), \quad \omega \in R, 0 \leq x < 1; \\ \hat{u}(0,\omega) = \hat{g}(\omega), \\ \hat{u}_x(0,\omega) = 0. \end{cases} \tag{21}$$

$$\hat{u}(0,\omega) = \hat{g}(\omega) + \int_0^x \int_0^s \frac{i\omega}{k(\tau)}\hat{u}(\tau,\omega)d\tau ds$$

Using Theorem 3.2,

$$|\hat{u}(0,\omega)| \leq |\hat{g}|e^{|\omega|\int_0^x \int_0^s \frac{1}{k(\tau)}d\tau ds}$$

Galerkin Solution of 15 in Scaling Spaces V_j

Approximating solution of 21 in scaling spaces

$$\begin{aligned} \langle k(x)u_{xx} - u_t, \varphi_{jk}\rangle &= 0 \\ \langle u(0,t), \varphi_{jk}\rangle &= \langle P_j g, \varphi_{jk}\rangle, \\ \langle u_x(0,t), \varphi_{jk}\rangle &= \langle 0, \varphi_{jk}\rangle, k \in Z. \end{aligned} \tag{22}$$

where φ_{jk} is the orthonormal basis of V_j given by the scaling function φ.

Letting the approximate solution $u_j(x,t) \in V_j$ be

$$u_j(x,t) = \sum_{l\in Z} W_l(x)\varphi_{jl}(t).$$

The equation 22 reduces to infinite dimensional differential equation [14]

$$\frac{d^2W}{dx^2} = D_j(x)W \text{ with } W(0) = \gamma, W'(0) = 0, \tag{23}$$

where $\gamma = P_j g = \sum_{z\in Z}\gamma_z\varphi_{jz} = \sum_{z\in Z}\langle g, \varphi_{jz}\rangle\varphi_{jz}$. Solution of 23 is analogous to the solution of 22

$$W(x) = \gamma + \int_0^x \int_0^s D_j(\tau)W(\tau)d\tau ds \text{ i.e.,} \tag{24}$$

By Theorem 3.2,

$$\|W(x)\| \leq \|\gamma\| \exp\left(2^{-j}\pi \int_0^s \frac{1}{k(\tau)} d\tau ds\right)$$

Theorem 3.3. [14] *Let u_j and $\tilde{u}_j$ be solutions in V_j of the approximating problem 15 with $g = P_j g$ for the boundary specifications g and $\tilde{g}$ respectively. If $\|g - \tilde{g}\| < \varepsilon$, then*

$$\|u_j(x,t) - \tilde{u}_j(x,t)\| \leq \varepsilon \exp\left(\frac{2^{-j}\pi}{2\alpha}\right) x^2,$$

For $j = j(\varepsilon)$ such that $2^{-j} \leq \frac{2\alpha}{\pi} \log \varepsilon^{-1}$, we have

$$\|u_j(x,t) - \tilde{u}_j(x,t)\| \leq \varepsilon^{1-x^2}.$$

Theorem 3.4. [14] *If u is a solution of problem 15 in V_j with $\|g\| \leq M$, then*

$$\|u(x,t) - P_j u(x,t)\| \leq M e^{\left(-\frac{1}{3}\frac{\pi}{\alpha} 2^{-j}\pi(1-x^2)\right)}$$

B. Wavelet-Galerkin Solution of Non Standard Parabolic Equation

We have stated and proved the following Theorem and found the solution of 27 in frequency domain as well as in scaling spaces. For details, refer to authors' paper [16].

Theorem 3.5. *Let $W(x)$ be continuous function, $x \geq 0$ and $\gamma = W(0) > 0$. $k'(x)$ is the derivative of $k(x)$. If*

$$W(x) = \gamma + \int_0^x \int_0^s \left[\frac{l}{k(\tau)} W(\tau) - \frac{k'(\tau)}{k(\tau)} W'(\tau)\right] d\tau ds \tag{25}$$

then

$$W(x) \leq \gamma e^{\int_0^x \int_0^s \left[\frac{l}{k(\tau)} + \frac{k'^2(\tau)}{4k^2(\tau)} W'(\tau)\right] d\tau ds} \tag{26}$$

Consider the following heat conduction problem

$$\begin{cases} k(x)u_{xx}(x,t) + k'u_x(x,t) - u_t(x,t) = 0, & t \geq 0, 0 \leq x < 1; \\ u(0,t) = g(t), \\ u_x(0,t) = 0. \end{cases} \tag{27}$$

$[0 < \alpha \leq k(x) \leq \beta < \infty, 0 < \delta \leq k'(x) \leq \vartheta < \infty]$.

Solution of 27 in Frequency Domain

Using Theorem 3.5

$$|\hat{u}(x,\omega)| \leq |\hat{g}(\omega)| e^{\int_0^x \int_0^s \left[\frac{1}{k(\tau)}|\omega| + \frac{k'^2(\tau)}{4k^2(\tau)}\right] d\tau ds} \tag{28}$$

Galerkin Solution of 27 in Scaling Spaces

By Theorem 3.5 and using Lemma 3.1,

$$\|W(x)\| \leq \|\gamma\| e^{\int_0^x \int_0^s \left[\frac{2^{-j}\pi}{k(\tau)} + \frac{k'^2(\tau)}{4k^2(\tau)}\right] d\tau ds} \tag{29}$$

Letting, the approximate solution, $u_j(x,t) \in V_j$

$$u_j(x,t) = \sum_{l \in Z} W_l(x) \varphi_{jl}(t).$$

Stability of Wavelet-Galerkin Method

We prove the following convergence theorems.

Theorem 3.6. *Let u_j and $\tilde{u}_j$ be solutions in V_j of the approximating problem 27 with $g = P_j g$ for the boundary specifications g and $\tilde{g}$ respectively. If $\|g - \tilde{g}\| < \varepsilon$, then*

$$\|u_j(x,t) - \tilde{u}_j(x,t)\| \leq \varepsilon \exp\left(\frac{2^{-j}\pi}{2\alpha} + \frac{\vartheta^2}{8\alpha^2}\right) x^2,$$

For $j = j(\varepsilon)$ such that $2^{-j} \leq \frac{2\alpha}{\pi} \log \varepsilon^{-1}$, we have

$$\|u_j(x,t) - \tilde{u}_j(x,t)\| \leq \varepsilon^{1-x^2} \exp\left[\frac{\vartheta^2}{8\alpha^2} x^2\right). \tag{30}$$

Proof. Let $u_j(x,t) = \sum_{l \in Z} W_l(x) \varphi_{jl}(t)$ and $\tilde{u}_j(x,t) = \sum_{l \in Z} \tilde{W}_l(x) \varphi_{jl}(t)$, where $W, \tilde{W}$ are solution 27

$$\begin{aligned}
\|u_j(x,t) - \tilde{u}_j(x,t)\| &= \|W(x) - \tilde{W}(x)\| \\
&\leq \|\gamma - \tilde{\gamma}\| \exp \int_0^x \int_0^s \left[\frac{2^{-j}\pi}{k(\tau)} + \frac{k'^2(\tau)}{4k^2(\tau)}\right] d\tau ds \text{ (using 29)} \\
&\leq \varepsilon \exp \int_0^x \int_0^s \left[\frac{2^{-j}\pi}{k(\tau)} + \frac{k'^2(\tau)}{4k^2(\tau)}\right] d\tau ds \\
&= \varepsilon \exp\left(\frac{2^{-j}\pi}{2\alpha} + \frac{\vartheta^2}{8\alpha^2}\right) x^2
\end{aligned}$$

If $j = j(\varepsilon)$ is such that $2^{-j} \leq \frac{2\alpha}{\pi} \log \varepsilon^{-1}$, then

$$\begin{aligned}
\|u_j(x,t) - \tilde{u}_j(x,t)\| &\leq \varepsilon e^{\log \varepsilon^{-x^2} + \frac{\vartheta^2}{8\alpha^2}} \\
&= \varepsilon e^{\log \varepsilon^{-x^2}} e^{\frac{\vartheta^2}{8\alpha^2}} \\
&= \varepsilon e^{1-x^2} e^{\frac{\vartheta^2}{8\alpha^2} x^2}.
\end{aligned}$$

□

Theorem 3.7. *If u is a solution of problem 27. Define $\hat{f} = \hat{g} \exp\left(\left[\frac{\pi}{3\alpha} 2^{-j} + \frac{\vartheta^2}{4\alpha^2}\right]\right) \in L^2(R)$. Then*

$$\|u(x,t) - P_j u(x,t)\| \leq M e^{\frac{2^{-j}\pi}{3\alpha} + \frac{\vartheta^2}{4\alpha^2}} x^2$$

For j such that $2^{-j} \leq \frac{3\alpha}{2\pi} \log \varepsilon^{-1}$, we have

$$\|u(x,t) - P_j u(x,t)\| \leq M \varepsilon^{1-x^2} e^{-\frac{\vartheta^2}{4\alpha^2}(1-x^2)} \tag{31}$$

Proof.

$$\begin{aligned}
\|u(x,t)-P_ju(x,t)\| &\leq \|\chi^+\hat{u}(x,\omega)\| = \left(\int_{|\omega|>\frac{2}{3}\pi 2^{-j}} |\hat{u}(x,\omega)|^2 d\omega\right)^{\frac{1}{2}} \\
&\leq \left(\int_{|\omega|>\frac{2}{3}\pi 2^{-j}} |\hat{g}(\omega)|^2 e^2 e^{\int_0^x \int_0^s \left[\frac{2^{-j}\pi}{k(\tau)}+\frac{k'^2(\tau)}{4k^2(\tau)}\right] d\tau ds}\right)^{\frac{1}{2}} d\omega \text{ from 28} \\
&\leq \left(\int |\hat{f}|^2 e^2 \left(-(1-x^2)\left[\frac{2\pi}{3\alpha}2^{-j}+\frac{\vartheta^2}{4\alpha^2}\right]\right)\right)^{\frac{1}{2}} d\omega \\
&\leq Me^{-(1-x^2)\left[\frac{2\pi}{3\alpha}2^{-j}+\frac{\vartheta^2}{4\alpha^2}\right]}
\end{aligned}$$

□

where $\hat{f} = \hat{g}e^{\frac{\pi}{3\alpha}2^{-j}+\frac{\vartheta^2}{4\alpha^2}}$, $\|\hat{f}\| \leq M$.
If $j = j(\varepsilon)$ such that $2^{-j} \leq \frac{3\alpha}{2\pi}\log\varepsilon^{-1}$, then

$$\begin{aligned}
\|u(x,t)-P_ju(x,t)\| &\leq Me^{\log\varepsilon^{1-x^2}} e^{-\frac{\vartheta^2}{4\alpha^2}(1-x^2)} \\
&= M\varepsilon^{1-x^2} e^{-\frac{\vartheta^2}{4\alpha^2}(1-x^2)}
\end{aligned}$$

4. **Numerical Example**

We consider the equation 27 with $k(x) = (x+a)^2$. Here

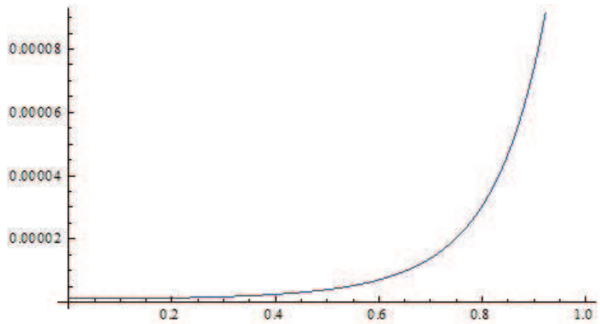
$$\begin{aligned}
\alpha &= \min_{0\leq x<1} k(x) = a^2 \\
\vartheta &= \max_{0\leq x<1} k'(x) < 2(1+a) \\
\frac{\vartheta}{2\alpha} &< \frac{1+a}{a^2} = \frac{1}{a}\left(1+\frac{1}{a}\right).
\end{aligned}$$

For $a \geq 1$, $\frac{\vartheta}{2\alpha} < 2$. Let $E = \frac{\vartheta}{2\alpha}$ so that equation 30 equivalents to

$$\|u_j(x,t)-\tilde{u}_j(x,t)\| \leq \varepsilon^{(1-x^2)} e^{\frac{E^2}{2}x^2} \tag{32}$$

for $j \geq \frac{\log[(\pi/2\alpha)(1/\log\varepsilon^{-1})]}{\log 2}$.

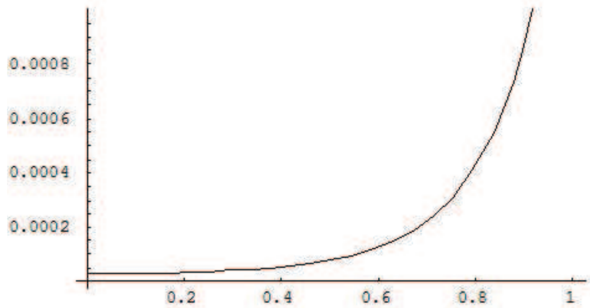
Graph of norm error versus space axis is



Equation 31 yields

$$\|u_j(x,t) - P_j u(x,t)\| \leq M\varepsilon^{(1-x^2)} e^{-E^2(1-x^2)} \tag{33}$$

for $j \geq \frac{\log[(2\pi/3\alpha)(1/\log\varepsilon^{-1})]}{\log 2}$
Graph of norm error error versus space axis is



In authors paper [16], inequality based wavelet-Galerkin solution of 27 has been found by taking $k(x)$ equal to $(x+a)^2$ and errors have been computed and compared at various values of a.

5. **Conclusion**

Wavelet regularization and Galerkin are important techniques to find the numerical solutions of partial differential parabolic equations. Gronwall based inequality approach to wavelet Galerkin method has additive advantage. Wavelet regularization solution of extended Sideways parabolic equation 15 and inequality based wavelet-Galerkin Solution of non standard parabolic equation 27 are stable.

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