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EQUIVALENCE OF n-NORMS ON THE SPACE OF p-SUMMABLE SEQUENCES

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Abstract. We study the relation between two known n-norms on ℓ^p, the space of p-summable sequences. One n-norm is derived from Gähler's formula [3], while the other is due to Gunawan [6]. We show in particular that the convergence in one n-norm implies that in the other. The key is to show that the convergence in each of these n-norms is equivalent to that in the usual norm on ℓ^p.
Key words: n-normed spaces, p-summable sequence spaces, n-norm equivalence.

Abstrak. Dalam makalah ini dipelajari kaitan antara dua norm-n di ℓ^p, ruang barisan *summable-p*. Norm-n pertama diperoleh dari rumus Gähler [3], sementara norm-n kedua diperkenalkan oleh Gunawan [6]. Ditunjukkan antara lain bahwa kekonvergenan dalam norm-n yang satu mengakibatkan kekonvergenan dalam norm-n lainnya. Kuncinya adalah bahwa kekonvergenan dalam masing-masing norm-n tersebut setara dengan kekonvergenan dalam norm biasa di ℓ^p.
Kata kunci: ruang norm-n, ruang barisan *summable-p*, kesetaraan norm-n

1. Introduction

In [6], Gunawan introduced an n-norm on ℓ^p $(1 \le p \le \infty)$, the space of p-summable sequences (of real numbers), given by the formula

$$\|x_1, \dots, x_n\|_p := \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \text{abs} \begin{vmatrix} x_{1j_1} & \cdots & x_{nj_1} \\ \vdots & \ddots & \vdots \\ x_{1j_n} & \cdots & x_{nj_n} \end{vmatrix}^p \right]^{1/p}$$

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for $1 \le p < \infty$, and

$$\|x_1, \dots, x_n\|_\infty = \sup_{j_1} \sup_{j_2} \cdots \sup_{j_n} \left\{ \text{abs} \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \right\},$$

where $x_i = (x_{ij}),\ i = 1, \dots, n$. For $p = 2$, the above formula may be rewritten as

$$\|x_1, \dots, x_n\|_2 = \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2},$$

where $\langle x_i, x_j \rangle$ denotes the usual inner product on ℓ^2. Here $\|x_1, \dots, x_n\|_2$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in ℓ^2.

In general, an n-norm on a real vector space X is a mapping $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ which satisfies the following four conditions:

(N1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(N2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, \dots, x_n\| = |\alpha|\, \|x_1, \dots, x_n\|$ for $\alpha \in \mathbb{R}$;

(N4) $\|x_1 + x_1', x_2, \dots, x_n\| \le \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$.

The theory of n-normed spaces was developed by Gähler in 1969 and 1970 [3, 4, 5]. The special case where $n = 2$ was studied earlier, also by Gähler, in 1964 [2]. Related work may be found in [1]. For more recent works, see [7, 8, 10].

If X is equipped with a norm $\|\cdot\|$, then according to Gähler, one may define an n-norm on X (assuming that X is at least n-dimensional) by the formula

$$\|x_1, \dots, x_n\|^* := \sup_{\substack{f_i \in X',\ \|f_i\| \le 1 \\ i = 1, \dots, n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}.$$

Here X' denotes the dual of X, which consists of bounded linear functionals on X.

For $X = \ell^p$ $(1 \le p < \infty)$, we know that $X' = \ell^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. In this case the above formula reduces to

$$\|x_1, \dots, x_n\|_p^* := \sup_{\substack{z_i \in \ell^{p'},\ \|z_i\|_{p'} \le 1 \\ i = 1, \dots, n}} \begin{vmatrix} \sum x_{1j} z_{1j} & \cdots & \sum x_{1j} z_{nj} \\ \vdots & \ddots & \vdots \\ \sum x_{nj} z_{1j} & \cdots & \sum x_{nj} z_{nj} \end{vmatrix},$$

where $\|\cdot\|_{p'}$ denotes the usual norm on $\ell^{p'}$ and each of the sums is taken over $j \in \mathbb{N}$. Thus, on ℓ^p, we have two definitions of n-norms, one is due to Gunawan and the other is derived from Gähler's formula. For $p = 2$, one may verify that the two n-norms are identical.

The purpose of this paper is to study the relation between the two n-norms on ℓ^p for $1 \le p < \infty$. In particular, we shall show that the two n-norms are weakly equivalent, that is, the convergence in one n-norm implies that in the other. Here a sequence $(x(m))$ in an n-normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to *converge* to $x \in X$ if $\|x(m) - x, x_2, \dots, x_n\| \to 0$ as $m \to \infty$, for every $x_2, \dots, x_n \in X$.

For convenience, we prove the result for $n = 2$ first, and then extend it to any $n \geq 2$.

2. Main Results

Recall that Gunawan's definition of 2-norm on ℓ^p $(1 \leq p \leq \infty)$ is given by

$$\|x, y\|_p = \left[\frac{1}{2} \sum_j \sum_k \text{abs} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix}^p \right]^{1/p}$$

if $1 \leq p < \infty$, and

$$\|x, y\|_\infty = \sup_j \sup_k \left\{ \text{abs} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \right\}.$$

Meanwhile, Gähler's definition is given by

$$\|x, y\|_p^* = \sup_{z, w \in \ell^{p'},\ \|z\|_{p'}, \|w\|_{p'} \leq 1} \begin{vmatrix} \sum x_j z_j & \sum x_j w_j \\ \sum y_j z_j & \sum y_j w_j \end{vmatrix}.$$

By the same trick as in [6], one may obtain

$$\|x, y\|_p^* = \sup_{z, w \in \ell^{p'},\ \|z\|_{p'}, \|w\|_{p'} \leq 1} \frac{1}{2} \sum_j \sum_k \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}.$$

From the last expression, we have the following fact.

Fact 2.1. The inequality $\|x, y\|_p^* \leq 2^{1/p} \|x, y\|_p$ holds for every $x, y \in \ell^p$.

Proof. By Hölder's inequality for $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\frac{1}{2} \sum_j \sum_k \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix} \leq \left[\frac{1}{2} \sum_j \sum_k \text{abs} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix}^p \right]^{1/p}$$

$$\times \left[\frac{1}{2} \sum_j \sum_k \text{abs} \begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}^{p'} \right]^{1/p'}$$

Now, observe that

$$\begin{aligned}\left[\sum_j\sum_k \text{abs}\begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}^{p'}\right]^{1/p'} &\le \left[\sum_j\sum_k \big[|z_jw_k|+|z_kw_j|\big]^{p'}\right]^{1/p'} \\ &\le \left[\sum_j\sum_k |z_jw_k|^{p'}\right]^{1/p'} + \left[\sum_j\sum_k |z_kw_j|^{p'}\right]^{1/p'} \\ &= 2\,\|z\|_{p'}\|w\|_{p'}.\end{aligned}$$

But for $\|z\|_{p'},\ \|w\|_{p'} \le 1$ we have

$$\left[\frac{1}{2}\sum_j\sum_k \text{abs}\begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}^{p'}\right]^{1/p'} \le 2^{1-(1/p')} = 2^{1/p}.$$

This proves the inequality.

Note that for $p = 1$, Hölder's inequality gives

$$\frac{1}{2}\sum_j\sum_k \left|\begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix}\right|\left|\begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}\right| \le \|x,y\|_1 \,.\, \|z,w\|_\infty \,.$$

But $\|z,w\|_\infty \le 2\,\|z\|_\infty\,\|w\|_\infty$ (see [6]), and so taking the supremum over $\|z\|_\infty$ and $\|w\|_\infty \le 1$, we get $\|x,y\|_1^* \le 2\|x,y\|_1$. □

Corollary 2.2 *If $(x(m))$ converges in $\|\cdot,\cdot\|_p$, then it also converges (to the same limit) in $\|\cdot,\cdot\|_p^*$.*

We shall show next that the convergence in $\|\cdot,\cdot\|_p^*$ also implies the convergence in $\|\cdot,\cdot\|_p$. We do so by showing that: (1) the convergence in $\|\cdot,\cdot\|_p^*$ implies that in $\|\cdot\|_p$, and (2) the convergence in $\|\cdot\|_p$ implies that in $\|\cdot,\cdot\|_p$.

The second implication is already proved in [6] (using the inequality $\|x,y\|_p \le 2^{1-(1/p)}\|x\|_p\|y\|_p$). Hence it remains only to show the first implication.

Theorem 2.3 *If $(x(m))$ converges in $\|\cdot,\cdot\|_p^*$, then it also converges (to the same limit) in $\|\cdot\|_p$.*

Proof. Let $(x(m))$ be a sequence in ℓ^p which converges to $x \in \ell^p$ in $\|\cdot,\cdot\|_p^*$. Then, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m \ge N$ we have

$$\frac{1}{2}\sum_j\sum_k \left|\begin{vmatrix} x_j(m)-x_j & x_k(m)-x_k \\ y_j & y_k \end{vmatrix}\right|\left|\begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}\right| < \epsilon$$

for every $y \in \ell^p$ and $z, w \in \ell^{p'}$ with $\|z\|_{p'},\ \|w\|_{p'} \le 1$. [Notice here that, for each m, we have $x(m) = (x_j(m)) \in \ell^p$.] In particular, if we take $y := (1, 0, 0, \dots),\ z = (z_j)$

with $z_j := \frac{\mathrm{sgn}(x_j(m)-x_j)|x_j(m)-x_j|^{p-1}}{\|x(m)-x\|_p^{p-1}}$ and $w := (1,0,0,\dots)$, then we have

$$\sum_{j=2}^{\infty} \frac{|x_j(m)-x_j|^p}{\|x(m)-x\|_p^{p-1}} < \epsilon.$$

[Here we are handling only the case where $\|x(m)-x\|_p \neq 0$.] Next, if we take $y := (0,1,0,\dots)$, $z = (z_1,0,0,\dots)$ with $z_1 := \frac{\mathrm{sgn}(x_1(m)-x_1)|x_1(m)-x_1|^{p-1}}{\|x(m)-x\|_p^{p-1}}$ and $w := (0,1,0,\dots)$, then we have

$$\frac{|x_1(m)-x_1|^p}{\|x(m)-x\|_p^{p-1}} < \epsilon.$$

Adding up, we get

$$\|x(m)-x\|_p = \sum_{j=1}^{\infty} \frac{|x_j(m)-x_j|^p}{\|x(m)-x\|_p^{p-1}} < 2\epsilon.$$

This shows that $(x(m))$ converges to x in $\|\cdot\|_p$. □

Corollary 2.4 *A sequence is convergent in* $\|\cdot,\cdot\|_p^*$ *if and only if it is convergent (to the same limit) in* $\|\cdot,\cdot\|_p$.

All these results can be extended to n-normed spaces for any $n \geq 2$. As an extension of Fact 2.1, we have:

Fact 2.5 The inequality $\|x_1,\dots,x_n\|_p^* \leq (n!)^{1/p}\|x_1,\dots,x_n\|_p$ holds for every $x_1,\dots,x_n \in \ell^p$.

Corollary 2.6 *If* $(x(m))$ *converges in* $\|\cdot,\dots,\cdot\|_p$*, then it converges (to the same limit) in* $\|\cdot,\dots,\cdot\|_p^*$.

Analogous to Theorem 2.3, we have:

Theorem 2.7 *If* $(x(m))$ *converges in* $\|\cdot,\dots,\cdot\|_p^*$*, then it also converges (to the same limit) in* $\|\cdot\|_p$.

Proof. Let $(x_1(m))$ be a sequence in ℓ^p which converges to $x_1 = (x_{11}, x_{12}, \dots) \in \ell^p$ in $\|\cdot,\dots,\cdot\|_p^*$. Then, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m \geq N$ we have

$$\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left|\begin{vmatrix} x_{1j_1}(m)-x_{1j_1} & \cdots & x_{1j_n}(m)-x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix}\begin{vmatrix} z_{1j_1} & \cdots & z_{1j_n} \\ \vdots & \ddots & \vdots \\ z_{nj_1} & \cdots & z_{nj_n} \end{vmatrix}\right| < \epsilon$$

for every $x_2,\dots,x_n \in \ell^p$ and $z_1,\dots,z_n \in \ell^p$ with $\|z_1\|,\dots,\|z_n\| \leq 1$. Now, take $x_k = z_k := (0,\dots,0,1,0,\dots)$ for every $k = 2,\dots,n$, where 1 is $(n+1-k)$-th term and $z_1 = (z_{11}, z_{12}, \dots) \in \ell^{p'}$ with $z_{1j} := \frac{\mathrm{sgn}(x_{1j}(m)-x_{1j})|x_{1j}(m)-x_{1j}|^{p-1}}{\|x_1(m)-x_1\|_p^{p-1}}$, then we have

$$\sum_{j_1=n}^{\infty} \frac{|x_{1j_1}(m) - x_{1j_1}|^p}{\|x_1(m) - x_1\|_p^{p-1}} < \epsilon.$$

Next, if we take $x_k = z_k := (0, \dots, 0, 1, 0, \dots)$ for every $k = 2, \dots, n$, where 1 is k-th term, and $z_1 := (z_{11}, 0, 0, \dots)$ with $z_{11} := \frac{\mathrm{sgn}(x_{11}(m)-x_{11})|x_{11}(m)-x_{11}|^{p-1}}{\|x_1(m)-x_1\|_p^{p-1}}$, then we have

$$\frac{|x_{11}(m) - x_{11}|^p}{\|x_1(m) - x_1\|_p^{p-1}} < \epsilon.$$

Similarly, if we alter the position of the entry 1 in x_k and z_k for $k = 2, \dots, n$, and change the nonzero entry of z_1 accordingly, then we can get

$$\frac{|x_{12}(m) - x_{12}|^p}{\|x_1(m) - x_1\|_p^{p-1}} < \epsilon$$

and so on until

$$\frac{\left|x_{1(n-1)}(m) - x_{1(n-1)}\right|^p}{\|x_1(m) - x_1\|_p^{p-1}} < \epsilon.$$

Adding up, we get

$$\|x_1(m) - x_1\|_p = \sum_{j_1=1}^{\infty} \frac{|x_{1j_1}(m) - x_{1j_1}|^p}{\|x_1(m) - x_1\|_p^{p-1}} < n\epsilon.$$

This shows that $(x(m))$ converges to x in $\|\cdot\|_p$. □

Corollary 2.8 *A sequence is convergent in $\|\cdot, \dots, \cdot\|_p^*$ if and only if it is convergent (to the same limit) in $\|\cdot, \dots, \cdot\|_p$.*

Related to the above results, one may also prove that a sequence is Cauchy in $\|\cdot, \dots, \cdot\|_p^*$ if and only if it is Cauchy in $\|\cdot, \dots, \cdot\|_p$. [A sequence $(x(m))$ in an n-normed space $(X, \|\cdot, \dots, \cdot\|)$ is Cauchy if given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|x(l) - x(m), x_2, \dots, x_n\| < \epsilon$ whenever $l, m \geq N$, for every $x_2, \dots, x_n \in X$.] Since $(\ell^p, \|\cdot, \dots, \cdot\|_p)$ is a Banach space [6], we conclude, by Theorem 2.7, that $(\ell^p, \|\cdot, \dots, \cdot\|_p^*)$ also forms an n-Banach space.

3. **Concluding Remarks**

As we have mentioned earlier, the case where $p = 2$ is of course special. Here, the two n-norms $\|\cdot, \dots, \cdot\|_2$ and $\|\cdot, \dots, \cdot\|_2^*$ are identical. Indeed, by using Cauchy-Schwarz inequality (see [9]), one may obtain

$$\|x_1, \dots, x_n\|_2^* = \sup_{\substack{z_i \in \ell^2,\ \|z_i\|_2 \leq 1 \\ i=1,\dots,n}} \begin{vmatrix} \langle x_1, z_1 \rangle & \cdots & \langle x_1, z_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1 \rangle & \cdots & \langle x_n, z_n \rangle \end{vmatrix} \leq \|x_1, \dots, x_n\|_2.$$

By taking $z_1, \dots, z_n$ to be the orthonormalized vectors obtained from $x_1, \dots, x_n$ through Gram-Schmidt process, one can show that the above upper bound is actually attained. Hence we have

$$\|x_1, \dots, x_n\|_2^* = \|x_1, \dots, x_n\|_2.$$

For $p \neq 2$, things are not so simple and we have difficulties in proving the strong equivalence between the two n-norms $\|\cdot, \dots, \cdot\|_p^*$ and $\|\cdot, \dots, \cdot\|_p$. The research on this problem, however, is still ongoing.

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