

## TOTALLY MAGIC INJECTIONS

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**Abstract.** A labelling is a mapping whose domain is some set of graph elements — the set of vertices, for example, or the set of all vertices and edges — and whose range is a set of positive integers. In particular, if the labels associated with any edge — the label on the edge itself, and those on its endpoints — always add to the same sum, the labeling, and the graph possessing it, is called *magic*. A related concept, a *vertex-magic total labeling*, is one in which the sum of the label on any vertex with the labels on the edges containing it is always constant. A labeling which has both the vertex-magic and edge-magic properties (usually with two different constants) is called *totally magic*, as is a graph possessing such a labeling. In this paper we survey what is known about totally magic graphs and an important generalization.

*Key words:* Magic, vertex-magic total labeling, totally magic.

**Abstrak.** Suatu pelabelan adalah suatu pemetaan yang memiliki domain berupa elemen-elemen graf dan range adalah himpunan bilangan bulat positif. Khususnya, jika label diasosiasikan dengan suatu sisi — label dari sisi itu sendiri dan label dari titik-titik ujung sisi tersebut — dengan penjumlahan mereka selalu sama, maka pelabelan dan graf yang memiliki sifat tersebut disebut *ajaib*. Sebuah konsep yang berkaitan, *pelabelan total titik ajaib*, adalah sebuah konsep dengan jumlahan dari label pada suatu titik dengan label-label dari sisi yang terkait dengan titik tersebut selalu konstan. Suatu pelabelan yang mempunyai kedua sifat titik ajaib dan sisi ajaib (biasanya dengan dua konstanta yang berbeda) disebut *ajaib secara total*, demikian juga dengan graf yang memiliki sifat tersebut. Pada paper ini kami mensurvey apa yang sudah diketahui tentang graf-graf ajaib secara ajaib dan sebuah perumusan yang penting.

*Kata kunci:* Ajaib, pelabelan total ajaib titik, ajaib secara total.

## 1. Basic Ideas

A *labeling* or *valuation* of a graph is a map that carries graph elements to numbers (usually to the positive or non-negative integers). The most common choices of domain are the set of all vertices and edges (such labelings are called *total* labelings), the vertex-set alone (*vertex*-labelings), or the edge-set alone (*edge*-labelings). Other domains are possible.

In many cases, it is interesting to consider the sum of all labels associated with a graph label. This will be called the *weight* of the element. For example, the weight of vertex  $x$  under labeling  $\lambda$  is

$$wt(x) = \lambda(x) + \sum_{y \sim x} \lambda(xy),$$

while

$$wt(xy) = \lambda(x) + \lambda(xy) + \lambda(y).$$

If necessary, the labeling can be specified by a subscript, as in  $wt_\lambda(x)$ .

Various authors have introduced labelings that generalize the idea of a magic square. Sedláček [18] defined a graph to be magic if it had an edge-labeling, with range the real numbers, such that the sum of the labels around any vertex equals some constant, independent of the choice of vertex. These labelings have been studied by Stewart (see, for example, [19]), who called a labeling *supermagic* if the labels are consecutive integers, starting from 1. Several others have studied these labelings; a recent reference is [8]. Some writers simply use the name “magic” instead of “supermagic” (see, for example, [10]).

Kotzig and Rosa [11] defined a magic labeling to be a total labeling in which the labels are the integers from 1 to  $|V(G)| + |E(G)|$ . The sum of labels on an edge and its two endpoints is constant. In 1996 Ringel and Llado [17] redefined this type of labeling (and called the labelings *edge-magic*, causing some confusion with papers that have followed the terminology of [12], mentioned below); see also [9]. Recently Enomoto *et al* [5] have introduced the name *super edge-magic* for magic labelings in the sense of Kotzig and Rosa, with the added property that the  $v$  vertices receive the smaller labels,  $\{1, 2, \dots, v\}$ .

In 1983, Lih [13] introduced magic labelings of planar graphs where labels extended to faces as well as edges and vertices, an idea which he traced back to 13th century Chinese roots. Bača (see, for example, [1, 2]) has written extensively on these labelings. A somewhat related sort of magic labeling was defined by Dickson and Rogers in [4].

Lee, Seah and Tan [12] introduced a weaker concept, which they called *edge-magic*, in 1992. The edges are labeled and the sums at the vertices are required to be congruent modulo the number of vertices.

Total labelings have also been studied in which the sum of the labels of all edges adjacent to the vertex  $x$ , plus the label of  $x$  itself, is constant. A paper on these labelings is [14].

In order to clarify the terminological confusion defined above, we shall restrict a *labeling* of a graph to be a one-to-one map that carries graph elements onto

the appropriate set of consecutive integers starting from 1. The most common choices of domain are the set of all vertices and edges (such labelings are called *total* labellings), the vertex-set alone (*vertex*-labelings), or the edge-set alone (*edge*-labellings); other domains are possible. For example, a total labeling of a graph with vertex-set  $V$  and edge-set  $E$  will be a one-to-one map from  $V \cup E$  onto  $\{1, 2, \dots, |V| + |E|\}$ . We shall use the words *valuation* or *injection* to describe mappings with more general ranges.

We then define the two most-studied labelings as follows:

An *edge-magic total labeling* or EMTL on  $G$  is a one-to-one map  $\lambda$  from  $V(G) \cup E(G)$  onto the integers  $1, 2, \dots, v + e$ , where  $v = |V(G)|$  and  $e = |E(G)|$ , with the property that, given any edge  $(xy)$ ,  $wt(xy) = k$  for any choice of edge  $xy$ , for some constant  $k$ .

A one-to-one map  $\lambda$  from  $E \cup V$  onto the integers  $\{1, 2, \dots, e + v\}$  is a *vertex-magic total labeling* if there is a constant  $h$  so that for every vertex  $x$ ,  $wt(x) = h$ .

A graph having an edge magic total labelling is called *edge-magic* and one with a VMTL is called *vertex-magic*. Some people throw in the word “total” after this, but we’ll stick with the grammatically correct version.

The most complete recent survey of graph labellings is [7]; see also [20],

## 2. Totally magic labelings

An interesting question is whether the same labeling could be both vertex-magic and edge-magic for a given graph (not necessarily with the same constant). In that case the labeling, and the graph, will be called *totally magic*. The constant row weight is called the *magic sum* and denoted  $k$ ; the vertex weight  $h$  is called the *magic constant*.

**2.1. Examples.** One quickly constructs three small examples of connected totally magic graphs. An obvious trivial example is the single vertex graph  $K_1$ . There are four totally magic labellings of the 3-vertex cycle  $C_3$ ; the three-vertex path  $P_3$  has two labellings. Among disconnected graphs, only one small example is known: there is exactly one totally magic labelling of  $K_1 \cup P_3$ . The four graphs are shown in Figure 1.

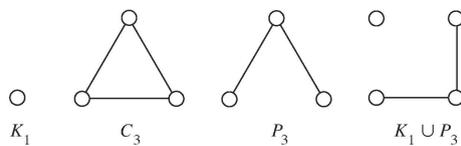


FIGURE 1. Small totally magic graphs

**2.2. Isolates, stars, and forbidden configurations.** It is easy to see that a graph with two isolated vertices cannot have a vertex-magic total labelling, as both isolates would have to receive label  $h$ . Similarly, if a vertex-magic graph contained an isolated edge  $xy$ , then  $\lambda(x) = h - \lambda(xy) = \lambda(y)$ , another contradiction. *A fortiori* we have

**Lemma 2.1.** *No totally magic graph has two isolated vertices or an isolated edge.*

Moreover, if  $K_1 \cup G$  is totally magic, the isolated vertex must necessarily receive the largest possible label, so the remaining labels form a totally magic labelling of  $G$ :

**Lemma 2.2.** *If a graph with an isolated vertex is totally magic, then the graph  $G$  resulting from the deletion of the isolate has a totally magic labelling with magic constant  $|V(G)| + |E(G)| + 1$ .*

It was shown in [14] that  $K_{m,n}$  is never vertex-magic when  $|m - n| > 1$ . Whence we have:

**Lemma 2.3.** *No star larger than  $K_{1,2}$  is totally magic.*

**Theorem 2.4.** [6] *Suppose the totally magic graph  $G$  has a leaf (vertex of degree 1)  $x$ . Then the component of  $G$  containing  $x$  is a star.*

The proof, and proofs of Theorems 5 through 9, can be found in [6].

**Corollary 2.5.** *The only connected totally magic graph containing a vertex of degree 1 is  $P_3$ .*

Every non-trivial tree has at least two vertices of degree 1, so the only totally magic trees are  $K_1$  and  $P_3$ .

A totally magic graph cannot have two stars as components, because their centers would each receive label  $k - h$ . It follows that the components of a totally magic graph can include at most one  $K_1$  and at most one star, and all other components have minimum degree at least 2, and consequently have as many edges as vertices.

**Corollary 2.6.** *The only totally magic proper forest is  $K_1 \cup P_3$ .*

**Theorem 2.7.** [6] *The only totally magic graphs with a component  $K_1$  are  $K_1 \cup P_3$  and  $K_1$  itself.*

**Theorem 2.8.** [6] *If a totally magic graph  $G$  contains two adjacent vertices of degree 2, then the component containing them is a cycle of length 3.*

**Corollary 2.9.** *No totally magic graph contains as a component a path other than  $P_3$  or a cycle other than  $K_3$ .*

(Lemma 2.1 must be invoked to rule out  $P_2$ .)

In particular,

**Corollary 2.10.** *The only totally magic cycle is  $K_3$ .*

**Theorem 2.11.** [6] Suppose  $G$  contains two vertices,  $x_1$  and  $x_2$ , that are each adjacent to precisely the same set  $\{y_1, y_2, \dots, y_d\}$  of other vertices. (It is not specified whether  $x_1$  and  $x_2$  are adjacent.) If  $d > 1$  then  $G$  is not totally magic.

**Corollary 2.12.** The only totally magic complete graphs are  $K_1$  and  $K_3$ . The only totally magic complete bipartite graph is  $K_{1,2}$ .

**Theorem 2.13.** [6] Suppose  $G$  contains two vertices,  $x$  and  $y$ , with a common neighbor. If  $x$  and  $y$  are nonadjacent and each have degree 2, or are adjacent and each have degree 3, then  $G$  is not totally magic.

(The case where  $x$  and  $y$  are adjacent could be rephrased, “a totally magic graph  $G$  cannot contain a triangle with two vertices of degree 3”.)

**Theorem 2.14.** [6] Suppose the totally magic graph  $G$  contains a triangle. Then the sum of the labels of all edges outside the triangle and incident with any one vertex of the triangle is the same, whichever vertex is chosen.

**Corollary 2.15.** If the totally magic graph  $G$  contains a triangle with one vertex of degree 2, then the triangle is a component of  $G$ .

Observe that Theorems 2.8, 2.11, 2.13 and 2.14 are essentially *forbidden configuration* theorems. If a graph  $G$  is in violation of one of them, then not only is  $G$  not totally magic, but  $G$  cannot be a component or union of components in any totally magic graph.

**2.3. Unions of triangles.** In [6] we constructed two infinite families of totally magic graphs, both based on triangles. We showed:

**Theorem 2.16.** The union of an odd number of triangles is always totally magic, and so is the graph constructed by deleting one edge from such a union.

However, the corresponding graphs constructed from an *even* number of triangles are never magic.

**2.4. Small graphs.** A complete search for small totally magic graphs is described in [6]. The result is that no examples with ten or fewer vertices exist, other than the four graphs given in Section 2.1 and the two nine-vertex graphs constructed in Theorem 2.16.

The complete search was carried out in two stages. First, using *nauty* [15], lists were prepared of all connected graphs (up to ten vertices) not ruled out by Theorems 2.8, 2.11 and 2.13 and Corollary 2.15. Second, the survivors were tested exhaustively. There were four survivors with six or fewer vertices ( $K_1, K_3$  and  $P_3$  and the graph shown in Figure 2), 42 with seven, 1,070 with eight, 61,575 with nine and 4,579,637 with 10. The graph of Figure 2 was eliminated using Theorem 2.14; probably many of the other survivors could also be eliminated by *ad hoc* applications of that Theorem.

Exhaustive testing is very time-consuming. However, a shortcut is available. If a totally magic labelling exists, it must retain the magic properties after reduction

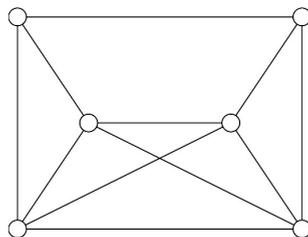


FIGURE 2. This graph can be eliminated using Theorem 2.14

modulo 2. So we tested all mod 2 possibilities. Label 1 or 0 is assigned to each vertex and to the constant  $k$ . Then every edge-label can be calculated. Next, one can check whether all the vertex weights are congruent (mod 2). Moreover, the total number of vertices and edges labeled 1 must either equal the number labeled 0 or exceed that number by 1. This process is quite fast (for example, only  $2^9$  cases need to be examined in the eight-vertex case), and eliminated over 254, and so on. For example, sieving the 1070 eight-vertex graphs mod 2 eliminated 307 graphs, sieving mod 3 eliminated 351 more, and so on: one graph survived after sieving modulo 7, and it was eliminated mod 8.

There were very few disconnected graphs to consider. Except for the graph  $K_1 \cup P_3$ , the only possibilities are made up of at most one star, copies of  $K_3$ , and survivors with more than three vertices. The only cases not already discussed are  $K_{1,n} \cup K_3$  for  $n = 3, 4, 5, 6$ ,  $K_{1,4} \cup 2K_3$ , and the 42 unions of a triangle and a 7-vertex survivor. None of these is totally magic, so there are no further totally magic graphs with ten or fewer vertices.

A further investigation using a variant of simulated annealing has been carried out. This procedure quickly finds the graphs we have described, but has so far found no other examples. This might suggest that we have found all totally magic graphs. However, for larger numbers of vertices (more than 20, say), it appears that the search gets “nearer” to satisfaction (no, I do not wish to clarify this vague description!), so perhaps there are large totally magic graphs yet to be discovered.

### 3. Totally magic injections

Are there any other totally magic graphs? If so, could we possibly construct one from smaller components?

Toward this end, we define a *magic injection* to be a one-to-one map from the vertices and edges of a graph to the positive integers — like a magic labelling in which some numbers are missing. We shall qualify these labelings as “edge”, “vertex”, “totally”, in the usual way.

A graph with a totally magic injection will be denoted a *TMI graph*.

**Theorem 3.1.** *If  $G$  is a TMI graph with no isolate then  $G \cup K_1$  is TMI.*

**Proof.** The new vertex takes label  $h$ . Clearly the largest label is smaller than  $h$ . (Say the largest label is  $m$ . This label is not on an isolate, so there is some vertex whose weight is  $m+$  (something positive). So  $m < h$ .)

**Theorem 3.2.** *Every star with 3 or more edges is TMI.*

**Proof.** To label  $K_{1,n}$ , label the centre 1, and label the edges  $2, 3, \dots, n + 1$ . Then, considering the centre vertex, the only possibility is  $h = (n + 1)(n + 2)/2$ . Therefore the outer vertices are labeled  $h - 2, h - 3, \dots, h - n - 1$ . To see that this is an injection, it is only necessary to check that

$$(n + 1)(n + 2)/2 - n - 1 > n + 1.$$

But this is obvious when  $n > 2$ . So we have a totally magic injection, and  $k = h + 1$ .

Two other easy results are:

**Theorem 3.3.** *The graph  $nK_3$ , the disjoint union of  $n$  triangles, has a totally magic injection for every  $n$ .*

**Theorem 3.4.** *The graph  $P_3 \cup nK_3$  has a totally magic injection for every  $n$ .*

So here are the known TMI graphs, so far:

	TMGs	Others
1 vertex:	$K_1$	none
2 vertices:	none	none
3 vertices:	$K_3, P_3$	none
4 vertices:	$P_3 \cup K_1$	$K_3 \cup K_1, K_{1,3}$
5 vertices:	none	$K_{1,3} \cup K_1, K_{1,4}$
6 vertices:	none	$K_3 \cup K_3, K_3 \cup P_3, K_{1,4} \cup K_1, K_{1,5}$
7 vertices:	none	$K_3 \cup K_3 \cup K_1, K_3 \cup P_3 \cup K_1, K_{1,5} \cup K_1, K_{1,6}$

A complete search shows that there are no other TMI graphs of any other order up to 6. At order 7, the size of the search becomes quite large.

### 3.1. Order seven.

**Theorem 3.5.** [21]

Given a graph  $G$  with vertices  $V_1, V_2, \dots, V_v$  and edges  $E_1, E_2, \dots, E_e$ , the mapping  $\lambda(V_i) = x_i, \lambda(E_j) = y_j$ . is a totally magic injection if and only if the system of equations

$$\begin{aligned} wt(V_i) - h &= 0 & \text{for all } i \in \{1, 2, \dots, v\} \\ wt(E_j) - k &= 0 & \text{for all } j \in \{1, 2, \dots, e\} \end{aligned}$$

has a solution with all labels distinct positive integers.

This can be considered as a system of  $v + e$  equations in the  $v + e + 2$  variables  $h, k, \dots, x_i, \dots, y_j, \dots$

Write  $M_G$  for the matrix of these equations for graph  $G$ , and  $\overline{M}_G$  for the matrix obtained from it by deleting the  $h$  and  $k$  columns.

**Theorem 3.6.** *If  $\overline{M_G}$  is invertible, and if  $G$  has a totally magic injection, then each edge label is an integer multiple of*

$$\frac{2h - k}{\det \overline{M_G}}.$$

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**Theorem 3.7.** *If  $\overline{M_G}$  is invertible, and if  $G$  has a totally magic injection, then each edge label is an integer multiple of*

$$\frac{2h - k}{\det \overline{M_G}}.$$

This Theorem allows us to eliminate most 7-vertex graphs. The exception is shown in Figure 3. The graph has

$$\det \overline{M_G} = 789,$$

so  $2h - k$  is a multiple of 789. Then some tedious arithmetic occurs, leading to the an injection with smallest possible constants are  $h = 577, k = 365$ .

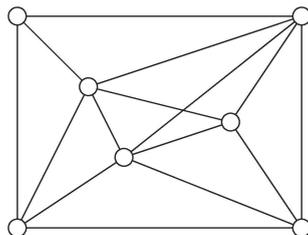


FIGURE 3. A candidate for a totally magic injection

We define the deficiency of a TMI by

$$\text{deficiency} = (\text{largest label}) - (v + e).$$

That is, the deficiency is the number of integers “missed” in the labelling.

The smallest labeling for the above graph gives deficiency 181. the labeling is shown in Figure 4.

The question of order 8, and all larger orders, remains open.

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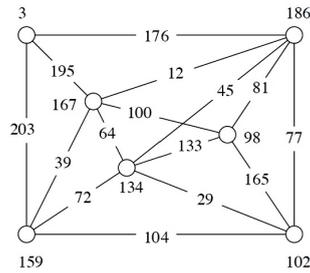


FIGURE 4. A candidate for a totally magic injection

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