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DECOMPOSITION OF COMPLETE GRAPHS AND COMPLETE BIPARTITE GRAPHS INTO COPIES OF P_n^3 OR $S_2(P_n^3)$ AND HARMONIOUS LABELING OF $K_2 + P_n$

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Abstract. In this paper, the graphs P_n^3 and $S_2(P_n^3)$ are shown to admit an α-valuation, where P_n^3 is the graph obtained from the path P_n by joining all the pairs of vertices u, v of P_n with $d(u, v) = 3$ and $S_2(P_n^3)$ is the graph obtained from P_n^3 by merging the centre of the star S_{n_1} and that of the star S_{n_2} respectively at the two unique 2-degree vertex of P_{n_3} (the origin and terminus of the path P_n contained in P_n^3). It follows from the significant theorems due to Rosa [1967] and EI-Zanati and Vanden Eynden [1996] that the complete graphs K_{2cq+1} or the complete bipartite graphs $K_{mq,nq}$ can be cyclically decomposed into the copies of P_n^3 or copies of $S_2(P_n^3)$, where c, m, n are arbitrary positive integer and q denotes either $|E(P_n^3)|$ or $|E(S_2(P_n^3))|$. Further, it is shown that join of complete graph K_2 and path P_n, denoted $K_2 + P_n$, for $n \geq 1$ is harmonious graph.

Key words: α-labeling, harmonious labeling, P_n^3 graphs, join, path.

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Abstrak. Pada paper ini, graf-graf P_n^3 dan $S_2(P_n^3)$ ditunjukkan mempunyai nilai-α, dengan P_n^3 adalah graf yang diperoleh dari lintasan P_n dengan menghubungkan semua pasangan titik u, v dari P_n dengan $d(u, v) = 3$ dan $S_2(P_n^3)$ adalah graf yang diperoleh dari P_n^3 dengan menggabungkan secara berurutan pusat dari bintang S_{n_1} dan dari bintang S_{n_2} pada dua titik berderajat-2 tunggal dari P_{n_3} (awal dan akhir dari lintasan P_n termuat di P_n^3). Dengan mengikuti teorema-teorema yang terkenal dari Rosa [1967] dan EI-Zanati dan Vanden Eynden [1996] bahwa graf-graf lengkap K_{2cq+1} atau graf-graf bipartit lengkap $K_{mq,nq}$ dapat didekomposisikan secara siklis menjadi kopi-kopi dari P_n^3 atau kopi-kopi $S_2(P_n^3)$, dengan c, m, n adalah bilangan bulat positif tertentu dan q menyatakan $|E(P_n^3)|$ atau $|E(S_2(P_n^3))|$. Lebih jauh, ditunjukkan juga bahwa join dari graf lengkap K_2 dan lintasan P_n, dinotasikan dengan $K_2 + P_n$, untuk $n \geq 1$ adalah graf harmonis.

Kata kunci: Pelabelan-α, pelabelan harmonis, graf-graf P_n^3, join, lintasan.

1. **Introduction**

In [1964], Ringel [9] conjectured that the complete graph K_{2m+1} can be decomposed into $2m + 1$copies of any Tree with m edges. In an attempt to solve the Ringel conjecture, Rosa [1967] introduced hierarchy of labeling called ρ, σ, β and α-labeling. Later in [1972], Golomb [6] called β-labeling as Graceful and this term is widely used. A function f is called a *graceful labeling* of a graph G with q edges if f is an injection from the set of vertices of G to the set $\{0, 1, 2, \cdots, q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct.

A stronger version of the graceful labeling is the α-labeling. A graceful labeling f of a graph $G = (V, E)$ is said to be an α-valuation (interlaced or balanced) if there exists a λ such that $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$ for every edge $uv \in E(G)$.

A graph which admits an α-labeling is necessarily a bipartite graph. In his classical paper Rosa [10] proved the significant theorem **Theorem A:** If a graph G with q edges admits α-labeling, then the complete graphs K_{2cq+1} can be cyclically decomposed into $2cq + 1$ copies of G, where c is an arbitrary positive number.

Later in 1996, EI-Zanati and Vanden Eynden [3] extended the cyclic decomposition for the complete bipartite graphs and proved the following significant theorem. **Theorem B:** If a graph G with q edges admits an α-valuation, then the complete bipartite graphs $K_{mq,nq}$ can be cyclically decomposed into copies of G where $q = |E(G)|$. These two results motivate to construct graphs which would admit an α-labeling. Many interesting families of graphs where proved to admit an α-labeling [5]. In this paper we show that P_n^3 and $S_2(P_n^3)$ admit an α-valuation. Here P_n^3 is the graph obtained from the path P_n by joining all the pairs of vertices u, v of P_n with $d(u, v) = 3$ and $S_2(P_n^3)$ is the graph obtained from P_n^3 by merging the center of the S_{n_1} and that of star S_{n_2} respectively at the two unique 2-degree vertex of P_n^3 (the origin and terminus of the path P_n contained in P_n^3).

In [1980] Graham and Sloane [4] introduced harmonious labeling in connection with their study in error correcting codes. Recently, it is established that recognizing a graph is harmonious is a NP-complete problem [7]. Thus it motivates to construct graphs admitting harmonious labeling. Number of interesting results where proved in this direction [1,2,4,5,6,8,11]. Here we show that join of K_2 and P_n, denoted $K_2 + P_n$ is harmonious graph for all $n \geq 1$.

A function f is called a *harmonious* if f is an injection from the set of vertices of graph G to the group of integer modulo $q, \{0, 1, 2, \cdots, q-1\}$, such that when each edge uv is assigned the label $(f(u) + f(v)) \pmod{q}$ the resulting edges labels are distinct.

2. α-Valuation of the Graph P_n^3 and the Graph $S_2(P_n^3)$

Here, in this section we show that P_n^3 and $S_2(P_n^3)$ admit an α-valuation. Let $v_1, v_2, \cdots v_n$ be vertices of P_n^3. Observe that in P_n^3, each $v_i$is adjacent to v_{i+1} for $1 \leq i \leq n-1$ and it is also adjacent to v_{i+3} for $1 \leq i \leq n-3$. It is clear that P_n^3 has n vertices and $2n-4$ edges. The graph P_n^3 is given in Figure 1.

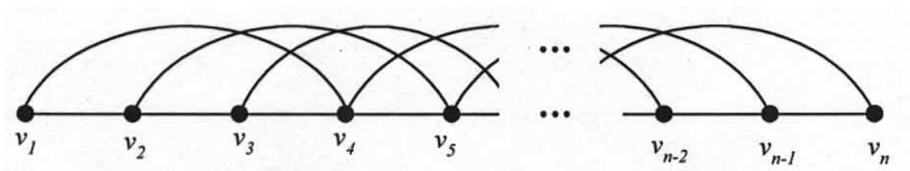

FIGURE 1. The Graph P_n^3.

Theorem 2.1 *For $n \geq 4$, the graph P_n^3 admits an $\alpha-$ valuation.*
Proof. $f : V(G) \to \{0, 1, 2, \cdots, M\}$ by

$$f(v_i) = \begin{cases} \dfrac{i-1}{2}, \text{ for } 1 \leq i \leq n \text{ and } i \text{ odd} \\ M - 3\left(\dfrac{i-2}{2}\right), \text{ for } 1 \leq i \leq n-1 \text{ and } i \text{ even} \\ M - 3\left(\dfrac{n-2}{2}\right) + 1, \text{ for } i = n \text{ and even} \end{cases} \tag{2.1}$$

Observe that the sequence $f(v_i)$, $1 \leq i \leq n$ and i even, form a monotonically decreasing sequence.

Further, when n is odd,

$$\begin{aligned}
\max\{f(v_i) \mid 1 \le i \le n \text{ with } i \text{ odd }\} &= \frac{n-1}{2} \quad \text{and} && (2)\\
\min\{f(v_i) \mid 1 \le i \le n \text{ with } i \text{ even }\} &= M - 3\left(\frac{n-1-2}{2}\right) \\
&= 2n-4-\left(\frac{3n-9}{2}\right) \\
&= \frac{4n-8-3n+9}{2} \\
&= \frac{n+1}{2}. && (3)
\end{aligned}$$

Therefore, from equation (2.2) and (2.3), we have

$$\min\{f(v_i) \mid 1 \le i \le n \text{ with } i \text{ even }\} > \max\{f(v_i) \mid 1 \le i \le n \text{ with } i \text{ odd }\}. \quad (2.4)$$

Also, when n is even,

$$\begin{aligned}
\max\{f(v_i) \mid 1 \le i \le n \text{ with } i \text{ odd}\} &= \frac{n-2}{2} \\
&= \frac{n}{2} - 1 && (5)
\end{aligned}$$

and

$$\begin{aligned}
\min\{f(v_i) \mid 1 \le i \le n \text{ with } i \text{ even }\} &= M - 3\left(\frac{n-2}{2}\right) + 1 \\
&= 2n-4-\frac{(3n-6)}{2}+1 \\
&= \frac{4n-8-3n+6+2}{2} \\
&= \frac{n}{2} && (6)
\end{aligned}$$

Therefore, from equations (2.5) and (2.6), it follows

$$\min\{f(v_i) \mid 1 \le i \le n \text{ with } i \text{ even }\} = \max\{f(v_i) \mid 1 \le i \le n \text{ with } i \text{ odd }\} + 1 \quad (2.7)$$

Since $f(v_i), 1 \le i \le n$, with i odd, is a monotonically increasing sequence and $f(v_i), 1 \le i \le n$ with i even, is a monotonically decreasing sequence and from equations (2.4) and (2.7), it follows $f(v_i), 1 \le i \le n$ are all distinct.

Let A be the set of edges $v_iv_{i+1}, 1 \le i \le n-1$ along the path and B be the set of edges $v_iv_{i+3}, 1 \le i \le n-3$ of G.

Observe from the definition of $f$that when n is even, the member of A get the values $\{M, M-1, M-4, M-5, M-8, M-9, \cdots, 4, 3, 1\}$ and when n is odd, the members of A get the values $\{M, M-1, M-4, M-5, M-8, M-9, \cdots, 6, 5, 2, 1\}$.

Similarly, when n is even, the members of B get the values $\{M-3, M-2, M-7, M-6, \cdots, 5, 6, 2\}$ and when n is odd, the number of B get the values $\{M-3, M-2, M-7, M-6, \cdots, 7, 8, 3, 4\}$.

Thus, it is clear that the edge values of all the edges of P_n^3 are distinct and range from 1 and M. Hence P_n^3 is graceful.

From the definition of f, observe that in the above labeling, when n is even, if we consider $\lambda = \frac{n}{2} - 1$ then $f(u) \leq \lambda < f(v)$ for every edge uv of P_n^3 and when n is odd, if we consider $\lambda = \frac{n-1}{2}$ then $f(u) \leq \lambda < f(v)$ for every edge uv of P_n^3.

Thus P_n^3 admits an α-valuation.

Hence the theorem.

The following two corollaries are immediate consequence of Rosa's theorem (1967) and the theorem of El-Zanati and Vanden Eynden (1996) respectively.

Corollary 1. *If a graph P_n^3 with q edges has an α-valuation, then there exists a cyclic decomposition of the edges of the complete graphs K_{2cq+1} into sub-graphs isomorphic to P_n^3, where c is an arbitrary positive integer.*

Corollary 2. *If a graph P_n^3 with q edges has an α-valuation, then there exists a decomposition of the edges of the complete bipartite graphs $K_{mq,nq}$ into subgraphs isomorphic to P_n^3, where m and n are arbitrary positive integers.*

Illustrative example of labeling given in the proof of Theorem 1 are given in Figures 2,3.

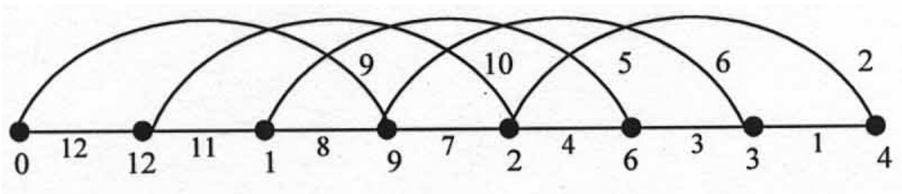

FIGURE 2. α-valuation of P_8^3.

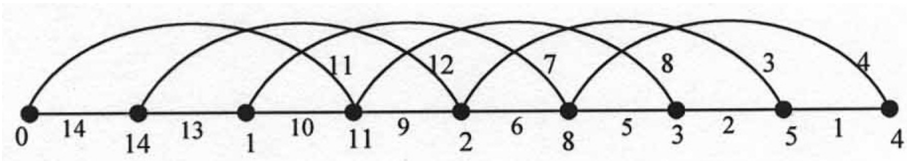

FIGURE 3. α-valuation of P_9^3.

Let $S_2(P_n^3)$ denote the graph obtained from P_n^3 by attaching the centre of the stars S_{n_1} and S_{n_2} at end the vertices v_1 and v_n of P_n^3.

As in the last theorem we assume that $v_1, v_2, \cdots, v_n$ be the vertices of P_n^3. Let $v_{1,1}, v_{1,2}, \cdots, v_{1,n_1}$ be the n_1 pendant vertices of the star S_{n_1} attached at v_1 of (P_n^3) and let $v_{n,1}, v_{n,2}, \cdots, v_{n,n_2}$ be the n_2 pendent vertices of star S_{n_2} attached at

v_n of (P_n^3). It is clear that $S_2(P_n^3)$ has $n+n_1+n_2$ vertices and $2n+n_1+n_2-4$ edges.

Theorem 2.2. *For $n \geq 4$, the graph $S_2(P_n^3)$, admits an $\alpha-$ valuation.*

Proof. For $n \geq 4$, let G be the graph $S_2(P_n^3)$. Let $M = |E(G)| = 2n+n_1+n_2-4$. Define $f : V(G) \to \{0, 1, 2, \cdots, M\}$ by

$$f(v_{1,j}) = M - (j-1), \text{ for } 1 \leq j \leq n_1 \tag{2.8}$$

$$f(v_i) = \begin{cases} \dfrac{i-1}{2}, \text{ if } 1 \leq i \leq n \text{ with } i \text{ odd} \\ (M-n_1) - 3\left(\dfrac{i-2}{2}\right), \text{ If } 1 \leq i \leq n-1 \text{ with } i \text{ even} \\ (M-n_1) - 3\left(\dfrac{n-2}{2}\right) + 1, \text{ if } i = n \text{ and even} \end{cases} \tag{2.9}$$

$$f(v_{n,j}) = \begin{cases} \dfrac{n-1}{2} + j, \text{ for } 1 \leq j \leq n_2 \text{ when } n \text{ is odd} \\ \dfrac{n-2}{2} + j, \text{ for } 1 \leq j \leq n_2 \text{ when } n \text{ is even.} \end{cases} \tag{2.10}$$

From the above definition of f, observe that the sequence $f(v_{1,j})$, $1 \leq j \leq n_1$ and $f(v_i)$, $1 \leq i \leq n$ when i is even, form monotonically decreasing sequence and the sequence $f(v_i)$, $1 \leq i \leq n$ when i is odd and $f(v_{n,j})$, $1 \leq j \leq n_2$, form monotonically increasing sequence.

Further, when n is odd

$$\max(\{f(v_i) \mid 1 \leq i \leq n \text{ and } i \text{ odd }\} \cup \{f(v_{n,j}) \mid 1 \leq j \leq n_2\}) = n_2 + \frac{n-1}{2} \tag{2.11}$$

$$\min(\{f(v_{1,j}) \mid 1 \leq j \leq n_1\} \cup \{f(v_i) \mid 1 \leq i \leq n \text{ and } i \text{ even }\}) = n_2 + \frac{n+1}{2}. \tag{2.12}$$

Therefore, from equations (2.11) and (2.12), it follows

$$\begin{aligned} &\min(\{f(v_{1,j}) \mid 1 \leq j \leq n_1\} \cup \{f(v_i) \mid 1 \leq i \leq n \text{ and } i \text{ even }\}) \\ &\quad = \max(\{f(v_i) \mid 1 \leq i \leq n \text{ and } i \text{ odd }\} \cup \{f(v_{n,j}) \mid 1 \leq j \leq n_2\}) + 1. \end{aligned} \tag{13}$$

When n is even,

$$\max(\{f(v_i) \mid 1 \leq j \leq n \text{ and } i \text{ odd }\} \cup \{f(v_{n,j}) \mid 1 \leq j \leq n_2\}) = n_2 + \frac{n}{2} - 1 \tag{2.14}$$

$$\min(\{f(v_{1,j}) \mid 1 \leq j \leq n_1\} \cup \{f(v_i) \mid 1 \leq i \leq n \text{ and } i \text{ even }\}) = n_2 + \frac{n}{2}. \tag{2.15}$$

Therefore, from the equations (2.14) and (2.15), it follows:

$$\begin{aligned} &\min(\{f(v_{1,j}) \mid 1 \leq j \leq n_1\} \cup \{f(v_i) \mid 1 \leq i \leq n \text{ and } i \text{ even }\}) \\ &\quad = \max(\{f(v_i) \mid 1 \leq i \leq n \text{ and } i \text{ odd and } f(v_{n,j}) \mid 1 \leq j \leq n_2\}) + 1. \end{aligned} \tag{16}$$

Since the sequences $f(v_{1,j}), 1 \leq j \leq n_1$ and $f(v_i), 1 \leq i \leq n$ with i even, form a monotonically decreasing sequence and the sequences $f(v_i), 1 \leq i \leq n$ with i odd and $f(v_{n,j}), 1 \leq j \leq n_2$, form a monotonically increasing sequence and from the equations (2.13) and (2.16), it follows $f(v_{1,j}), 1 \leq j \leq n_1, f(v_i)$, $1 \leq i \leq n, f(v_{n,j})$, $1 \leq j \leq n_2$, are all distinct.

Let A be the set of edges in S_{n_1} and B be the set of edges v_iv_{i+1}, $1 \leq i \leq n-1$ along the path C be the set of edges $v_iv_{i+3}, 1 \leq i \leq n-3$ and D be the set of edges in S_{n_2}.

Observe from the definition of f that the members of A get the value $\{M, M-1, M-2, \cdots, M-(n_1-1)\}$. The members of B get the value $\{M-n_1, M-n_1-1, M-n_1-4, M-n_1-5, M-n_1-8, M-n_1-9, \cdots, n_2+4, n_2+3, n_2+1\}$ when n is even and when n is odd, the members of B get the value $\{M-n_1, M-n_1-1, M-n_1-4, M-n_1-5, \cdots, n_2+2, n_2+1\}$.

The members of C get the value $\{M-n_1-3, M-n_1-2, M-n_1-7, M-n_1-6, \cdots, n_2+5, n_2+6, n_2+2\}$ when n is even and when n is odd, the members of C get the value $\{M-n_3, M-n_1-2, M-n_1-7, M-n_1-6, \cdots, n_2+3, n_2+4\}$. The members of D get the value $\left\{\frac{n+1}{2}, \frac{n+3}{2}, \cdots, \frac{n+n_2-1}{2}\right\}$ when n is odd and when n is even, the members of D get the value $\left\{\frac{n+2}{2}, \frac{n+4}{2}, \cdots, \frac{n+2n_2}{2}\right\}$. Thus it is clear that the edge values of all the edges of P_n^3 are distinct and range from 1 to M.

Hence $S_2(P_n^3)$ is graceful.

We consider $\lambda = n$ or $\frac{n-1}{2}$ according as n is even or odd. Then by the definition of f, it is clear that $f(u) \leq \lambda < f(v)$ for every edge uv of $S_2(P_n^3)$.

Thus, the graph $S_2(P_n^3)$ is graceful and admits an α-valuation. Hence, the theorem.

The following corollary is an immediate consequence of Rosa's theorem.

Corollary 3. *There exists a cyclic decomposition of the complete graphs K_{2cq+1} into subgraphs isomorphic to $S_2(P_n^3)$, where c is an arbitrary positive integer.*

Due to the theorem if El-Zanati and Vanden Eynden (1996) we have the following corollary.

Corollary 4. *There exists a partition of the complete bipartite graphs $K_{mq,nq}$ into subgraphs isomorphic to $S_2(P_n^3)$, where m and n are arbitrary positive integers.*

Illustrative example of labeling given in the Proof of Theorem 2 are shown in Figures 4,5,6.

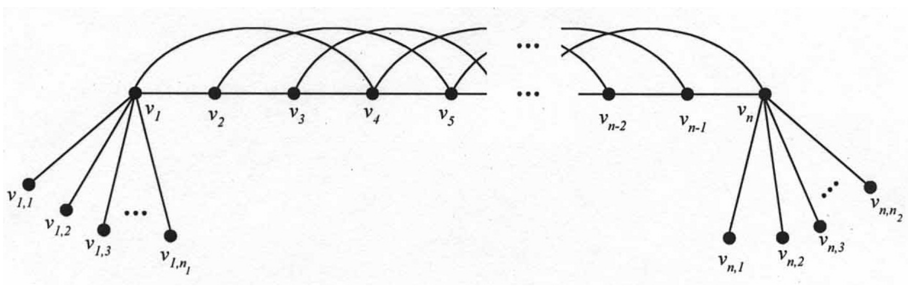

FIGURE 4. The Graph $S_2\left(P_n^3\right)$.

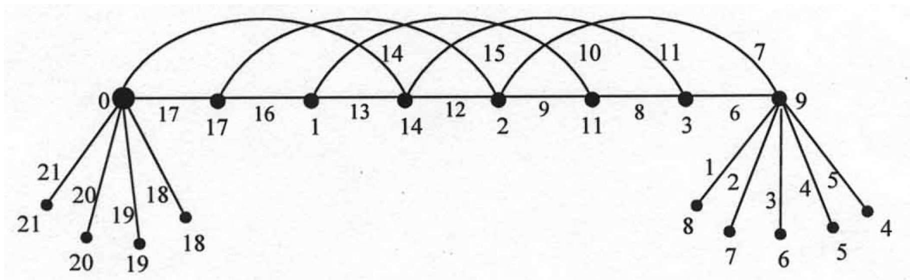

FIGURE 5. α-valuation of $S_2\left(P_8^3\right)$.

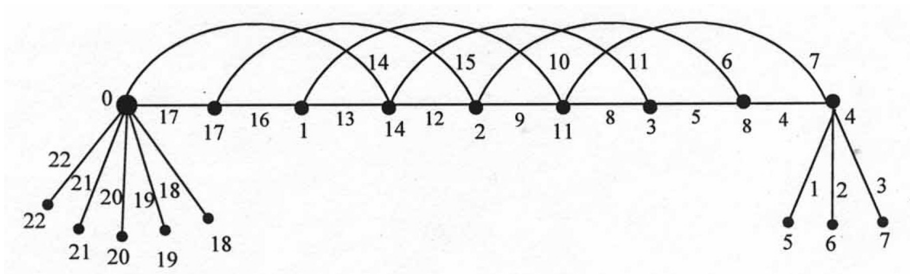

FIGURE 6. α-valuation of $S_2\left(P_9^3\right)$.

3. Harmonious Labeling of $K_2 + P_n$ for $n \geq 1$

In this section it is shown that join of complete graph K_2 and path P_n, denoted $K_2 + P_n$ is harmonious for all n.

Theorem 3.1 *Join of $K_2 + P_n$ is harmonious, for $n \geq 1$.*

Proof: For $n \geq 1$, let G be a graph $K_2 + P_n$. Let u_1 and u_2 be the vertices of K_2 and $v_1, v_2, \cdots, v_n$ be the vertices of P_n. Then G has $|E(G)| = M = 3n$ edges. We define vertex labeling f in two cases depends on n is odd or even.

Case (i) n is odd

$$\begin{aligned} \text{Define } f(u_1) &= 0 \\ f(u_2) &= M-1 \\ f(v_i) &= 3i-2, \; 1 \leq i \leq n. \end{aligned}$$

Then, it is clear that the vertex labeling $f(u_i), i = 1, 2$ and $f(v_i), 1 \leq i \leq n$ are distinct.

Further,

$$\begin{aligned} f(u_1u_2) &= 3n-1, \\ f(u_1v_i) &= 3i-2, \quad 1 \le i \le n, \\ f(u_2u_i) &= 3i-3, \quad 1 \le i \le n \\ f(v_iv_{i+1}) &= 6i-1 (\bmod M),\ 1 \le i \le n-1. \end{aligned}$$

That is

$$\begin{aligned} f(v_iv_{i+1}) &= 6i-1,\ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor, \\ f(v_iv_{i+1}) &= (6i-1)(\bmod M),\ \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n-1, \\ &= (6i-1), i = \left\lfloor \frac{(n-1)}{2} \right\rfloor + j,\ 1 \le j \le \left\lfloor \frac{(n-1)}{2} \right\rfloor \\ &= 6\left(\left\lfloor \frac{(n-1)}{2} \right\rfloor + j \right) - 1,\ 1 \le j \le \left\lfloor \frac{(n-1)}{2} \right\rfloor, \\ &= 3n-6j-4(\bmod M), \quad 1 \le j \le \left\lfloor \frac{(n-1)}{2} \right\rfloor, \\ &= 6j-4,\ 1 \le j \le \left\lfloor \frac{(n-1)}{2} \right\rfloor. \end{aligned}$$

Observe that,

$$\begin{aligned} f(u_1u_2) &= \{3n-1\} \\ \{f(u_1v_i) \mid 1 \le i \le n\} &= \{1, 4, 7, 10, \cdots, 3n-2\}, \\ \{f(u_2v_i) \mid 1 \le i \le n\} &= \{0, 3, 6, 9, \cdots, 3n-3\}, \end{aligned}$$

and

$$\begin{aligned} \{f(v_iv_{i+1}) \mid 1 \le i \le n-1\} &= \{(6i-1) \mid 1 \le i \le n-1\}, \\ &= \{(6i-1) \mid 1 \le i \le (n-1)/2\} \cup \{(6i-1) \mid 1 \le i \le (n-1)/2\} \\ &= \{(6i-1) \mid 1 \le i \le (n-1)/2\} \cup \{6(((n-1)/2)+j)-1 \mid 1 \le j \le (n-1)/2\} \\ &= \{(6i-1) \mid 1 \le i \le (n-1)/2\} \cup \{3n+6j-4 \mid 1+j \le (n-1)/2\} \\ &= \{(6i-1) \mid 1 \le i \le (n-1)/2\} \cup \{(6j-4) \mid 1 \le j \le (n-1)/2\} \\ &= \{5, 11, 17, \cdots, 3n-4\} \cup \{2, 8, 14, \cdots, 3n-7\} \\ &= \{2, 5, 8, 11, 14, \cdots, 3n-7, 3n-4\}. \end{aligned}$$

From the above sets of the edge values, it follows that edge labeling of each edge is distinct and edge values ranges from 0 to $M-1$.

Case (ii), n is even

Case(ii)(a) n is even and $n = 4k, n \ge 1$

Define

$$\begin{aligned}
f(u_1) &= 0, \\
f(u_2) &= M-1, \\
f(v_i) &= 3(i-1)+1, \qquad 1 \le i \le 2k-1, \\
f(v_i) &= 6(i-k)+1, \qquad 2k \le i \le 3k-1, \\
f(v_{3k}) &= 3(n-1)+1, \\
f(v_i) &= 3(n-2(i-3k)-1)+1, \qquad 3k+1 \le i \le 4k.
\end{aligned}$$

It is clear that the vertex labeling $f(v_i)$ are distinct, for$1 \le i \le n$.
Further,

$$\begin{aligned}
f(u_1u_2) &= M-1, \\
f(u_1v_i) &= 3(i-1)+1,\ 1 \le i \le 2k-1 \\
f(u_1v_i) &= 3(2(i-k))+1,\ 2k \le i \le 3k+1 \\
f(u_1v_{3k}) &= 3(n-1)+1, \\
f(u_1v_{3k+i}) &= 3(n-2i-1)+1,\ 1 \le i \le k \\
f(u_2v_i) &= 3(i-1),\ 1 \le i \le 2k-1, \\
f(u_2v_i) &= 3(2(i-k)),\ 2k \le i \le 3k-1, \\
f(u_2v_{3k}) &= 3(n-1) \\
f(u_2v_i) &= 3(n-2(i-3k)-1),\ 3k+1 \le i \le 4k
\end{aligned}$$

and

$$\begin{aligned}
f(v_iv_{i+1}) &= (3(i-1)+1)+(3i+1),\ 1 \le i \le 2k-2, \\
&= 3(2i-1)+2,\ 1 \le i \le 2k-2 \\
f(v_{2k-1}v_{2k}) &= 3(4k-2)+2 \\
f(v_iv_{i+1}) &= (3(2i-k))+1+3(2(i+1)-k))+1)(\text{mod } M), 2k \le i \le 3k-2, \\
&= (3(2(2i-2k+1)+2))(\text{mod } M), 2k \le i \le 3k-2, \\
f(v_{3k-1}, v_{3k}) &= (3(2(3k+1-k))+1+3n-2)(\text{mod } M), \\
&= (12k-6+1-2+3n)(\text{mod } M), \\
&= (3(n-3)+2)(\text{mod } M), \\
f(v_{3k}, v_{3k+1}) &= (3n-2+3n-6-2)(\text{mod } M), \\
&= (3n-2+3n-8)(\text{mod } M) \\
&= (6n-10)(\text{mod } M), \\
&= (3n-10) \\
&= 3(n-4)+2 \\
f(v_iv_{i+1}) &= (3(n-4(i-3k)-4)+2)(\text{mod } M), 3k+1 \le i \le 4k-1.
\end{aligned}$$

Similarly it follows that edge labeling are distinct and edge value ranges from 0 to $M-1$.

Case (ii) (b) n is even and $n = 4k+2, k \ge 1$.

Define

$$\begin{aligned}
f(u_1) &= 0,\\
f(u_2) &= M-1\\
f(v_i) &= 3(i-1)+1,\ 1 \le i \le 2k,\\
f(v_i) &= 3(2(i-k)-1)+1,\ 2k+1 \le i \le 3k+1,\\
f(v_{3k+2} &= 3(n-2)+1,\\
f(v_i) &= (3(2(k-i))+1)(\text{mod } M),\ 3k+3 \le i \le 4k+2.
\end{aligned}$$

It is clear that the vertex labeling $f(v_i)$ are distinct for $1 \le i \le n$.
Further, observe that

$$\begin{aligned}
f(u_1u_2) &= M-1\\
f(u_1v_i) &= 3(i-1)+1,\ 1 \le i \le 2k,\\
f(u_1v_i) &= 3(2(i-k)-1)+1,\ 2k+1 \le i \le 3k+1,\\
f(u_1v_{3k+2}) &= 3(n-2)+1,\\
f(u_1v_i) &= (3(2k-i)+1)(\text{mod } M),\ 3k+3 \le i \le 4k+2\\
f(u_2v_i) &= M-1+3(i-1)+1,\\
&= 3(i-1),\ 1 \le i \le 2k,\\
f(u_2v_i) &= 3(2(i-k)-1),\ 2k+1 \le i \le 3k+1\\
f(u_2v_{3k+2}) &= 3(n-2),\\
f(u_2v_i) &= 3(2(k-i))(\text{mod } M),\ 3k+3 \le i \le 4k+2
\end{aligned}$$

and

$$\begin{aligned}
f(v_iv_{i+1}) &= (3(i-1)+(3i+1)),\\
&= 6i-1,\\
&= 3(2i-1)+2,\ 1 \le i \le 2k-1,\\
f(v_{2k}v_{2k+1}) &= 3(2k-1)+1+3(2(k+1)-1)+1 = 3(4k)+2\\
f(v_iv_{i+1}) &= 6(i-k)-2+6(i+1-k)-2,\\
&= (3(4(i-k)+2)(\text{mod } M),\ 2k+1 \le i \le 3k,\\
f(v_{3k+1}v_{3k+2}) &= (3(2(3k+1-k)+1)+3n-5)(\text{mod } M),\\
&= 12k-1,\\
f(v_{3k+2}v_{3k+3}) &= (3n-5+6(k-i)+1)(\text{mod } M)\\
&= (3n-16)(\text{mod } M),\\
f(v_iv_{i+1}) &= 6k-6i+1+6k-6i-6+1(\text{mod } M)\\
&= 12k-12i-4(\text{mod } M)\\
&= 12k+6-12i-4-6(\text{mod } M),\\
&= -12i-10(\text{mod } M),\\
&= -(12i+10)(\text{mod } M),\ 3k+3 \le i \le 4k+1.
\end{aligned}$$

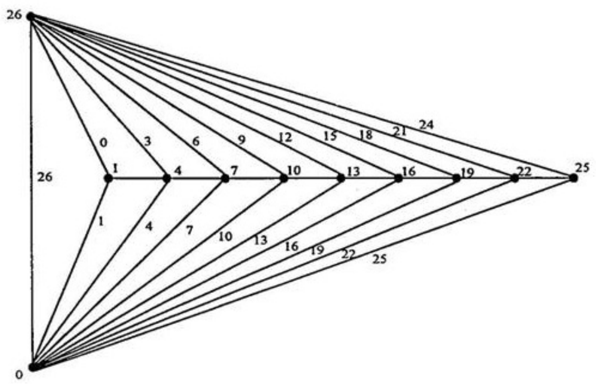

FIGURE 7. Harmonious labeling of $K_2 + P_9$.

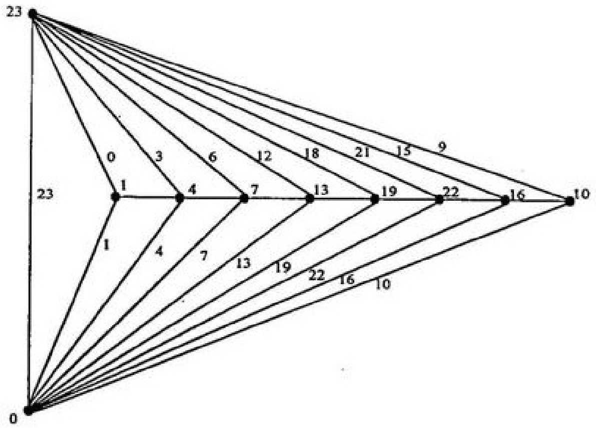

FIGURE 8. Harmonious labeling of $K_2 + P_8$.

It follows that edge labelings are distinct and edge values ranges from 0 to $M-1$. When $n=2, G$ is K_4, which is harmonious.

Hence G is harmonious.

Illustrative example of labeling given in the proof of Theorem 3 are given in Figures 7,8,9.

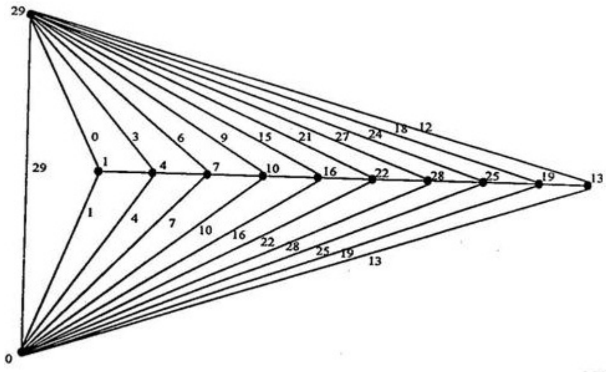

FIGURE 9. Harmonious labeling of $K_2 + P_{10}$.

4. **Discussion**

In our paper we have shown that P_n^3 and $S_2(P_n^3)$ admit an α-valuation. We believe that it is possible to prove that P_n^t and $S_2(P_n^t)$, for t, $2 \leq t \leq n-2$ admit an α-valuation. Thus, we end this paper with the following conjecture.
Conjecture: P_n^t and $S_2(P_n^t)$ admit an α-valuation for t, $2 \leq t \leq n-2$.

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