Numerical Study of Predator-Prey Model with Beddington-DeAngelis Functional Response and Prey Harvesting

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ABSTRACT

A modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and Michaelis-Menten type prey harvesting is studied. The equilibrium points of the system are investigated. To see the stability of each equilibrium point, we perform some numerical simulations. Our numerical simulations show that the extinction of prey or survival of both prey and predator are conditionally stable.

Keywords: *harvesting, Leslie-Gower, Michaelis-Menten, numerical simulations, predator-prey model*

INTRODUCTION

Every species interact with other species in their habitat. Interactions between species lead to a rise or reduction population densities. Predation is one form of interaction between the two species (predator and prey). In the case of predation, a predator population density is increasing. In contrast, the density of the prey population is decreasing. Species compete, evolve, and spread for the aim of seeking the source of food to sustain their existence [1].

Environmental resources constrain the growth of each species. If the population density increases, the growth rate of population is reduced and stopped at a certain population density that is known as the carrying capacity. In other words, carrying capacity is the maximum population density that can be supported by the environment [2].

In population dynamics, a functional response represents the predation rate of prey by predators per capita. Holling [3] has introduced a functional response that depends only on the prey species, namely Holling type I, II, and III. Beddington [4] and DeAngelis [5] have modified the Holling functional response. Beddington and DeAngelis suggested that predation rate is more realistic if it also depends on the predator species and environmental provides protection to the prey. Predator population densities are important to consider because the predators are competing

and sharing food.

Leslie [6] has introduced a predator-prey model with carrying capacity of the predator population is proportional to the number of prey population. The model is known as the Leslie-Gower predator-prey model. Aziz-Alaoui and Okiye [7] has modified the carrying capacity of the predator population in Leslie-Gower model. Aziz-Alaoui and Okiye involved the environment protection. They used Holling-type II functional responses. The existence of the economic value of the prey population encourages modification by adding prey harvesting. Gupta and Chandra [8] has modified Aziz-Alaoui and Okiye model by incorporating Michaelis-Menten type prey harvesting. Yu [9] has modified Aziz-Alaoui and Okiye model by the functional response into Beddington-DeAngelis-type.

In this article, we extend the predator-prey model proposed by Gupta and Chandra [8] and Yu [9] by incorporating a Bedington-DeAngelis functional response and the Michaelis-Menten type prey harvesting. The focus of this study is to give an overview of population densities in the long term based on numerical simulations. Fourth order Runge-Kutta method is used to solve the predator-prey system.

MATERIALS AND METHODS
Mathematical model

A mathematical model that represents the interaction between the prey species and the predator species will be constructed based on several assumptions. First, we assume that harvesting is performed only on prey species. Second, the coefficients of environmental protection for the prey and predator population are differ-

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ent.

Suppose x is the population density of prey and y is the population density of predator. It is assumed that prey grows with intrinsic growth rate r_1:

$$\frac{dx}{dt} = r_1 x. \tag{1}$$

The existence of competition among prey species to survive in their habitat with competition rate p causing the prey population growth rate was reduced as follow

$$\frac{dx}{dt} = -px^2. \tag{2}$$

Based on Yu [9], the rate of predation is given in the form of the Beddington-DeAngelis functional response:

$$p(x, y) = \frac{\alpha}{a + bx + cy},$$

where α and a are the maximum value of the per capita reduction rate of x due to y and the coefficient of environmental protection for the prey, respectively. b and c are suitable constants. Therefore, the prey population growth rate is reduced due to predation as follow

$$\frac{dx}{dt} = -\frac{\alpha xy}{a + bx + cy}. \tag{3}$$

Harvesting prey by humans given in a nonlinear form, known as the Michaelis-Menten type as in Gupta and Chandra [8]. The prey population growth rate is reduced due to harvesting as follow
where q is catchability coefficient, E is the effort

$$\frac{dx}{dt} = -\frac{qEx}{m_1 E + m_2 x}, \tag{4}$$

applied to harvest the prey species, and m_1 and m_2 are suitable constants. $m_1 E$ shows limited human effort to harvest and $m_2 x$ indicates the limited number of prey that harvested. Based on the equation (1), (2), (3), and (4), the prey population growth rate is formulated as follow
with all parameters are positive constant.

$$\frac{dx}{dt} = r_1 x - px^2 - \frac{\alpha xy}{a + bx + cy} - \frac{qEx}{m_1 E + m_2 x},$$

It is assumed that predator grows with intrinsic growth rate r_2:

$$\frac{dy}{dt} = r_2 y. \tag{5}$$

As Aziz-Alaoui and Okiye [7], the carrying capacity of the predator population is limited by prey population density and the extent to which the environmental provides protection to predator (k). If β is the competition rate among predator species to survive in their habitat, the predator population growth rate is reduced as follow

$$\frac{dy}{dt} = -\frac{\beta y^2}{x + k}. \tag{6}$$

Based on the equation (5) and (6), the predator population growth rate is formulated as follow

$$\frac{dy}{dt} = y\left(r_2 - \frac{\beta y}{x + k}\right).$$

Thus, predator-prey models involving Beddington-DeAngelis functional response and Michaelis-Menten type prey harvesting is

$$\frac{dx}{dt} = r_1 x - px^2 - \frac{\alpha xy}{a + bx + cy}$$
$$- \frac{qEx}{m_1 E + m_2 x}, \tag{7}$$
$$\frac{dy}{dt} = y\left(r_2 - \frac{\beta y}{x + k}\right).$$

Determination of the equilibrium point

The equilibrium point of the system (7) is the solution of that system when both population growth rates are zero. Equilibrium point illustrates a constant solution of the system.

Observing the stability of the equilibrium point

Stability of the equilibrium point is observed by the Jacobian matrix of the system. The stability property shows whether the equilibrium point is stable or not. If the equilibrium point is stable, then any solution of the system with different initial values will be convergent to it, and vice versa.

Numerical experiments

System (7) is a nonlinear system, so the analytical solution of system (7) is not easy to be determined. Therefore, numerical simulations will be performed to investigate the system behavior. Numerical solutions of the system can give an overview of population densities

in the long term. To determine the numerical solutions of the predator-prey model, we use the fourth order Runge-Kutta method.

RESULTS AND DISCUSSION
Equilibrium Point
In order to find the equilibrium point of the system (7), let

$$0 = \left(r_1 - px - \frac{\alpha y}{a + bx + cy} - \frac{qE}{m_1 E + m_2 x}\right)x,$$

$$0 = \left(r_2 - \frac{\beta y}{x + k}\right)y,$$

where x and y have same meaning as $x(t)$ and $y(t)$, respectively. Obviously, system (7) has five equilibrium points, as follow.

i. The extinction of both prey and predator point $E_0 = (0,0)$

ii. The predator extinction point $E_1 = (x^*_A, 0)$ and $E_2 = (x^*_B, 0)$ where

$$\Delta_1 = (m_2 r_1 - m_1 pE)^2 + 4 m_2 p(m_1 r_1 E - qE),$$

$$x^*_A = \frac{1}{2}\left(\frac{m_2 r_1 - m_1 pE + \sqrt{\Delta_1}}{m_2 p}\right),$$

$$x^*_B = \frac{1}{2}\left(\frac{m_2 r_1 - m_1 pE - \sqrt{\Delta_1}}{m_2 p}\right).$$

iii. The prey extinction point $E_3 = (0, r_2 k/\beta)$

iv. The positive equilibrium point E_4 $(x^*, r_2 (x^* + k)/\beta)$. x^* is a positive real root of

$$(x^*)^3 + \frac{B}{A}(x^*)^2 + \frac{C}{A}x^* + \frac{D}{A} = 0, \quad (8)$$

where

$$u = a\beta + cr_2 k,$$
$$v = b\beta + cr_2,$$
$$A = -m_2 pv,$$
$$B = m_2(r_1 v - pu) - m_1 Epv - m_2 \alpha r_2,$$
$$C = r_1 m_2 u + m_1 E(r_1 v - pu) - qEv$$
$$\quad\quad - \alpha r_2 (km_2 + m_1 E),$$
$$D = r_1 u m_1 E - \alpha r_2 km_1 E - qEu.$$

In contrast to all of the equilibrium point, E_4 $(x^*, r_2 (x^* + k)/\beta)$ shows that both the prey and the predator can survive in the long term.

Stability of Equilibrium Point
The Jacobian matrix of system (7) at equilibrium point (x,y) is

$$J = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

with

$$a_{11} = r_1 - 2p\hat{x} - \frac{\alpha \hat{y}(a + c\hat{y})}{(a + b\hat{x} + c\hat{y})^2} - \frac{qm_1 E^2}{(m_1 E + m_2 \hat{x})^2},$$

$$a_{12} = -\frac{\alpha \hat{x}(a + b\hat{x})}{(a + b\hat{x} + c\hat{y})^2},$$

$$a_{21} = \frac{\beta \hat{y}^2}{(\hat{x} + k)^2},$$

$$a_{22} = r_2 - \frac{2\beta \hat{y}}{\hat{x} + k}.$$

An equilibrium point (x,y) of system (7) is asymtotically stable if the real parts of all eigenvalues of $J(x,y)$ are negative. The Jacobian matrices for E_0, E_1, E_2 and E_3 are

$$J_0 = \begin{pmatrix} r_1 - \frac{q}{m_1} & 0 \\ 0 & r_2 \end{pmatrix},$$

$$J_1 = \begin{pmatrix} r_1 - 2px^*_A - \frac{qm_1 E^2}{(m_1 E + m_2 x^*_A)^2} & -\frac{\alpha x^*_A}{a + bx^*_A} \\ 0 & r_2 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} r_1 - 2px^*_B - \frac{qm_1 E^2}{(m_1 E + m_2 x^*_B)^2} & \frac{\alpha x^*_B}{a + bx^*_B} \\ 0 & r_2 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} r_1 - \frac{\alpha r_2 k}{a\beta + cr_2 k} - \frac{q}{m_1} & 0 \\ \frac{r_2^2}{\beta} & -r_2 \end{pmatrix},$$

respectively. r_2 is a positive eigenvalue of J_0, J_1 and J_2, so E_0, E_1 and E_2 are unstable equilibrium point. Therefore, the extinction of both prey and predator or predator extinction does not occur in the long term. We next investigate the stability of E_3 and E_4 by numerical simulations.

Numerical Simulations
Let $r_1 = 0.3$, a= 10, b = 1, c= 0.5, q= 0.3, E= 10, p= 0.05, m_1= 0.1, α= 0.02, m_2 = 0.6, r_2= 0.2, β = 0.5 and
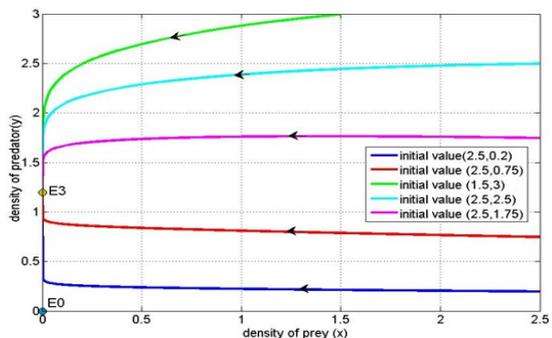

Figure 1. Numerical solution of system (7) with an asymptotically stable equilibrium point E_3.

k= 3. Based on those parameter values, the equilibrium point E_0 and E_3 are exist. The values of each point of equilibrium are E_0= (0,0) and E_3= (0,1.2). The eigenvalues of $J(E_0)$ are -2.7 and 0.2. Whereas, $J(E_3)$ the eigenvalues of are -0.2 and -2.70226. As a consequence of stability requirements, is unstable E_0 and E_3 is an asymptotically stable. These results are shown in Figure 1.

The initial values in Figure 1 show the initial density of both prey and predator. For various initial values, all numerical solutions of system (7) converge to E_3. In this numerical simulation, we choose small values of m_1 and m_2 which indicate the large prey harvesting. Clearly that relatively large harvesting leads to prey extinction. On the other hand, predators still survive in the habitat because there is a high environmental protection (k). The environmental protection includes the temperature, humidity, or other abiotic factors in the habitat. Without prey in the habitat, predators grow until it reaches carrying capacity (r_2k/β). Therefore, the prey extinction are predicted to occur in the long term when there is a large prey harvesting.

For the second simulation, we use the same parameter values as before except for parameter m_1 and m_2. In this simulation, the values of m_1 and m_2 are respectively 1.07 and 2. There are four equilibrium points i.e. E_0 = (0,0), E_1= (1.8101,0), E_3 = (0,1.2) and E_4 = (1.66299,1.86519). The eigenvalues of $J(E_0)$ are 0.0196 and 0.2, the eigenvalues of $J(E_1)$ are -0.03754 and 0.2, and the eigenvalues of $J(E_3)$ are -0.2 and -0.01736. On the other hand, the eigenvalues of $J(E_4)$ are -0.03321 and -0.198827.

Therefore, E_0, E_1 and E_3 are unstable and E_4 is asymptotically stable. In Figure 2, we plot some numerical solutions using several initial values. All numerical solutions of system (7) converge to E_4. Note that in this simulation, we use greater values of m_1 and m_2, meaning that the harvesting is decreased. Consequently, prey can survive although the intrinsic growth rate of prey is low. The presence of food (prey) causes predator to survive. Therefore, predator and prey can coexist in the long term if prey harvesting is reduced.

CONCLUSIONS

In this paper, we have studied a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and Michaelis-Menten type prey harvesting numerically. It is found that the model has five equilibrium points (E_0), namely the extinction of both prey and predator point the predator extinction point (E_1 and E_2), the prey extinction point (E_3), and
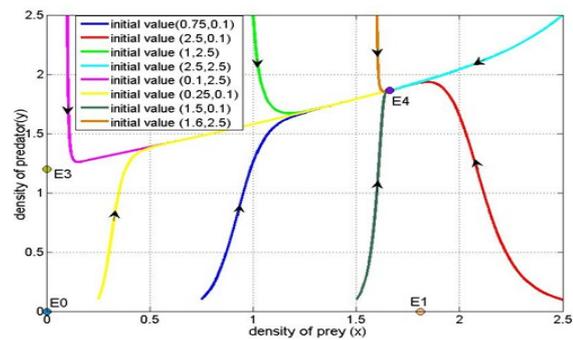

Figure 2. Numerical solution of system (7) with an asymptotically stable equilibrium point E_4.

the survival of both prey and predator point (E_4). Based on numerical simulations, there are two possible stable equilibrium points that are E_3 and E_4. E_3 and E_4 respectively indicated the prey extinction or the survival both prey and predator in the long term. In the future, we will explore the dynamics of the model analytically.

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