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VERTEX (a, d)-ANTIMAGIC TOTAL LABELING ON CIRCULANT GRAPH $C_n(1, 2, 3)$

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Abstract. Let $G = (V, E)$ be a graph with order $|G|$ and size $|E|$. An (a, d)-vertex-antimagic total labeling is a bijection α from all vertices and edges to the set of consecutive integers $\{1, 2, ..., |V| + |E|\}$, such that the weights of the vertices form an arithmetic progression with the initial term a and the common difference d. If $\alpha(V(G)) = \{1, 2, \ldots, |V|\}$ then we call the labeling a super (a, d)-vertex antimagic total. In this paper we show how to construct such labelings for circulant graphs $C_n(1, 2, 3)$, for $d = 0, 1, 2, 3, 4, 8$.

Key words: Circulant graph, (a, d)-vertex antimagic total graph.

Abstrak. Misalkan $G = (V, E)$ adalah sebuah graf dengan orde $|G|$ dan ukuran $|E|$. Suatu pelabelan total antimagic (a, d)-titik adalah suatu bijeksi α dari semua titik-titik dan sisi-sisi ke himpunan dari bilangan bulat berurutan $\{1, 2, ..., |V| + |E|\}$, sedemikian sehingga bobot dari titik-titik membentuk sebuah barisan aritmatika dengan suku awal a dan beda d. Jika $\alpha(V(G)) = \{1, 2, \ldots, |V|\}$ maka kita menyebut pelabelan total antimagic (a, d)-titik super. Pada paper ini kami menunjukkan bagaimana mengkonstruksi pelabelan-pelabelan untuk graf-graf sirkulan $C_n(1, 2, 3)$, dengan $d = 0, 1, 2, 3, 4, 8$.

Kata kunci: Graf sirkulan, graf total antimagic (a, d)-titik.

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1. Introduction

All graphs which are discussed in this paper are simple and connected graphs. For a graph $G = G(V, E)$, we will denote the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. We use $n = |V(G)|$ and $e = |E(G)|$.

A *labeling* α of a graph G is a mapping that assigns elements of a graph to a set of positive integers. We will discuss a total labeling which means the domain of the mapping of α is $V \cup E$.

The *vertex-weight* $wt(x)$ of a vertex $x \in V$, under a labeling $\alpha : V \cup E \to \{1, 2, ..., n + e\}$, is the sum of values $\alpha(xy)$ assigned to all edges incident to a given vertex x together with the value assigned to x itself.

A bijection $\alpha : V \cup E \to \{1, 2, ..., n + e\}$ is called an (a, d)-*vertex-antimagic total* (in short, (a, d)-VAT) *labeling* of G if the set of vertex-weights of all vertices in G is $\{a, a + d, a + 2\,d, \ldots, a + (n - 1)\,d\}$, where $a > 0$ and $d \geq 0$ are two fixed nonnegative integers. If $d = 0$ then we call α a *vertex-magic total labeling.* The concept of the vertex-magic total labeling was introduced by MacDougall *et. al.* [**?**] in 2002.

An (a, d)-VAT labeling will be called *super* if it has the property that the vertex-labels are the integers $1, 2, \ldots, n$, the smallest possible labels. A graph which admits a (super) (a, d)-VAT labeling is said to be (super) (a, d)-VAT. These labelings were introduced in [2] as a natural extension of the vertex-magic total labeling (VAT labeling for $d = 0$) defined by MacDougall *et al.* [9] (see also [13]). Basic properties of (a, d)-VAT labelings are studied in [2]. In [11], it is shown how to construct super (a, d)-VAT labelings for certain families of graphs, including complete graphs, complete bipartite graphs, cycles, paths and generalized Petersen graphs.

In this paper, we specially focus on a special class of graphs which called circulant graphs. Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \left\lfloor \frac{n}{2} \right\rfloor$, where n and $a_i (i = 1, 2, \ldots, k)$ are positive integers. A *circulant graph* $C_n\,(a_1, a_2, \ldots, a_k)$ is a regular graph with $V = \{v_0, v_1, \ \ldots, \ v_{n-1}\}$ and $E = \{(v_i\, v_{i+a_j})\,(\bmod n) : i = 0, 1, 2, \ldots, n - 1, j = 1, 2, \ldots, k\}$.

Many known results on (a,d)-VAT labeling are already published. For more detail results the reader can see Gallian's dynamic survey on graph labeling [4]. Regarding of circulant graph $C_n(1, m)$, Balbuena *et. al* [3] have the following results.

Theorem 1.1. *For odd $n = 5$ and $m \in \{2, 3, \ldots, \frac{n-1}{2}\}$, circulant graphs $C_n(1, m)$ have a super vertex-magic total labeling with the magic constant $h = \frac{17\,n+5}{2}$.*

In the following, we will discuss on vertex (a, d)-antimagic total labeling of a class of circulant graphs $C_n(1, 2, 3)$ where n is an odd integer, for $d \in \{0, 1, 2, 3, 4, 8\}$

2. Vertex (a, d)-antimagic total labeling on circulant graph

The following lemma gives an upper bound for the value of d of vertex (a, d)-antimagic total labeling for $C_n(1, 2, 3)$.

Lemma 2.1. *Let $n \geq 5$ be odd integers. If $C_n(1,2,3)$ has vertex (a,d)-antimagic total labeling, then $d \leq 22$.*

The following theorems show that circulant graph $C_n(1,2,3)$ is a vertex (a,d)-antimagic graph for $n \geq 5$ and d=0, 1, 2, 3, 4, and 8.

Theorem 2.2. *For odd $n \geq 5$, circulant graphs $C_n(1,2,3)$ have a super vertex-magic total labeling with the magic constant $h = \frac{31n+7}{2}$.*

Proof. Let $C_n(1,2,3)$ be a subclass of circulant graphs with $n \geq 5$. Let $\{v_i : i = 0,1,\dots,n-1\}$ be the vertices of $C_n(1,2,3)$.

Label all the vertices and edges as follows:

$$\alpha_0(v_i) = \begin{cases} 3-i, & \text{for } i = 0,1,2, \\ n+3-i, & \text{for } i = 3,4,\dots,n-1, \end{cases}$$

$$\alpha_0(v_iv_{i+1}) = \begin{cases} 2n, & \text{for } i = 0, \\ \frac{3n+i}{2}, & \text{for } i = 1,3,\dots,n-2 \\ \frac{2n+i}{2}, & \text{for } i = 2,4,\dots,n-1, \end{cases}$$

$$\alpha_0(v_iv_{i+2}) = 3n-i, \text{for } i = 0,1,\dots,n-1,$$

$$\alpha_0(v_iv_{i+3}) = 3n+i+1, \text{for } i = 0,1,\dots,n-1.$$

The vertex and edge labels under the labeling α_0 are $\alpha_0(V) = \{1,2,\dots,n\}$ and $\alpha_0(E) = \{n+1,n+2,\dots,4n\}$. It means that the labeling α_0 is a bijection from the set $V(C_n(1,2,3)) \cup E(C_n(1,2,3))$ onto the set $\{1,2,\dots,4n\}$.

We consider the vertex-weights of $C_n(1,2,3)$ case by case.

Case 1. $i = 0,1,2$

a) For $i = 0$

$$\begin{aligned} wt_{\alpha_0}(v_0) = & \ (3)+(2n)+(3n)+(3n+1) \\ & +\tfrac{2n+(n-1)}{2}+(3n-(n-2))+(3n+(n-3)+1) \\ = & \ \tfrac{31n+7}{2}. \end{aligned}$$

b) For $i = 1$

$$\begin{aligned} wt_{\alpha_0}(v_0) = & \ (3-1)+\tfrac{3n+1}{2}+(3n-1)+(3n+1+1) \\ & +(2n)+(3n-(n-1))+(3n+(n-2)+1) \\ = & \ \tfrac{31n+7}{2}. \end{aligned}$$

c) For $i = 2$

$$\begin{aligned} wt_{\alpha_0}(v_0) = & \ (3-2)+\tfrac{2n+2}{2}+(3n-2)+(3n+2+1) \\ & +\tfrac{3n+1}{2}+(3n)+(3n+(n-1)+1) \\ = & \ \tfrac{31n+7}{2}. \end{aligned}$$

Case 2. i odd, $i \geq 3$

$$\begin{aligned} wt_{\alpha_0}(v_0) = & \ (n+3-i) + \tfrac{3n+i}{2} + (3n-i) + (3n+i+1) \\ & + \tfrac{2n+i-1}{2} + (3n-(i-2)) + (3n+(i-3)+1) \\ = & \ \tfrac{31n+7}{2}. \end{aligned}$$

Case 3. i even, $i \geq 4$

$$\begin{aligned} wt_{\alpha_0}(v_0) = & \ (n+3-i) + \tfrac{2n+i}{2} + (3n-i) + (3n+i+1) \\ & + \tfrac{3n+i}{2} + (3n-(i-2)) + (3n+(i-3)+1) \\ = & \ \tfrac{31n+7}{2}. \end{aligned}$$

Thus, we obtain $wt_{\alpha_0}(v_i) = \frac{31n+7}{2}$ for all cases. Consequently, it proves that α_0 is a vertex-magic total labeling for $C_n(1,2,3)$ with the magic constant $h = \frac{31n+7}{2}$. □

Theorem 2.3. *Let n be an odd integer, $n \geq 5$. The graph $C_n(1,2,3)$ admits a vertex $(\frac{29n+9}{2}, 1)$-antimagic total labeling.*

Proof. Let $C_n(1,2,3)$ be a subclass of circulant graphs with $n \geq 5$. Let $\{v_i : i = 0, 1, \ldots, n-1\}$ be the vertices of $C_n(1,2,3)$.

Label all the vertices and edges as follows:

$$\alpha_1(v_i) = \begin{cases} 5-2i, & \text{for } i = 0,1,2, \\ 2(n-i)+5, & \text{for } i = 3,4,\ldots,n-1, \end{cases}$$

$$\alpha_1(v_iv_{i+1}) = \begin{cases} 3n, & \text{for } i = 0, \\ \frac{5n+i}{2}, & \text{for } i = 1,3,\ldots,n-2, \\ \frac{4n+i}{2}, & \text{for } i = 2,4,\ldots,n-1, \end{cases}$$

$$\alpha_1(v_iv_{i+2}) = 4n-i, \text{for } i = 0,1,\ldots,n-1,$$
$$\alpha_1(v_iv_{i+3}) = 2(i+1), \text{for } i = 0,1,\ldots,n-1.$$

The vertex and edge labels under the labeling α_1 are $\alpha_1(V) = \{1,3,\ldots,2n-1\}$ and $\alpha_1(E) = \{2,4,\ldots,2n\} \cup \{2n+1, 2n+2, \ldots, 4n\}$. It means that the labeling α_1 is a bijection from the set $V(C_n(1,2,3)) \cup E(C_n(1,2,3))$ onto the set $\{1,2,\ldots,4n\}$.

We consider the vertex-weights of $C_n(1,2,3)$ case by case.

Case 1. $i = 0,1,2$

a) For $i = 0$

$$\begin{aligned} wt_{\alpha_1}(v_0) = & \ (5) + (3n) + (4n) + 2(0+1) \\ & + \tfrac{4n+(n-1)}{2} + (4n-(n-2)) + 2((n-3)+1) \\ = & \ \tfrac{29n+9}{2}. \end{aligned}$$

b) For $i = 1$

$$\begin{aligned} wt_{\alpha_1}(v_1) = & \ (5-2) + \tfrac{5n+1}{2} + (4n-1) + 2(1+1) \\ & + (3n) + (4n-(n-1)) + 2(n-2+1) \\ = & \ \tfrac{29n+11}{2}. \end{aligned}$$

c) For $i = 2$

$$\begin{aligned} wt_{\alpha_1}(v_2) = &\ (5-4) + \tfrac{4n+2}{2} + (4n-2) + 2(2+1) \\ &+ \tfrac{5n+1}{2} + (4n) + 2((n-1)+1) \\ = &\ \tfrac{29n+13}{2} + 3. \end{aligned}$$

Case 2. i odd, $i \geq 3$

$$\begin{aligned} wt_{\alpha_1}(v_i) = &\ 2(n-i) + 5 + \tfrac{5n+i}{2} + (4n-i) + 2(i+1) \\ &+ \tfrac{4n+(i-1)}{2} + (4n-(i-2)) + 2(i-3+1) \\ = &\ \tfrac{29n+9}{2} + i. \end{aligned}$$

Case 3. i even, $i \geq 4$

$$\begin{aligned} wt_{\alpha_1}(v_i) = &\ 2(n-i) + 5 + \tfrac{4n+i}{2} + (4n-i) + 2(i+1) \\ &+ \tfrac{5n+(i-1)}{2} + (4n-(i-2)) + 2((i-3)+1) \\ = &\ \tfrac{29n+9}{2} + i. \end{aligned}$$

Thus, we obtain that the vertex-weights form a sequence of consecutive integers: $\frac{29n+9}{2}, \frac{29n+9}{2}+1, \ldots, \frac{29n+9}{2}+n-1$. Consequently, circulant graph $C_n(1,2,3)$, $n \geq 5$, admits a $(\frac{29n+9}{2}, 1)$-VAT labeling.

□

Theorem 2.4. *Let n be an odd integer, $n \geq 5$. The graph $C_n(1,2,3)$ has a super $(\frac{29n+7}{2}, 2)$-VAT labeling.*

Proof. Let $C_n(1,2,3)$ be a subclass of circulant graphs with $n \geq 5$. Let $\{v_i : i = 0, 1, \ldots, n-1\}$ be the vertices of $C_n(1,2,3)$.

Label all the vertices and edges as follows:

$$\alpha_2(v_i) = \begin{cases} 3-i, & \text{for } i = 0,1,2, \\ n+3-i, & \text{for } i = 3,4,\ldots,n-1, \end{cases}$$

$$\alpha_2(v_iv_{i+1}) = \begin{cases} n+1, & \text{for } i = 0, \\ \frac{3n-i+2}{2}, & \text{for } i = 1,3,\ldots,n-2, \\ 2n+1-\frac{i}{2} & \text{for } i = 2,4,\ldots,n-1, \end{cases}$$

$$\alpha_2(v_iv_{i+2}) = 3n - i, \text{for } i = 0,1,\ldots,n-1,$$

$$\alpha_2(v_iv_{i+3}) = 3n + i + 1, \text{for } i = 0,1,\ldots,n-1.$$

The vertex and edge labels under the labeling α_2 are $\alpha_2(V) = \{1, 2, \ldots, n\}$ and $\alpha_2(E) = \{n+1, n+2, \ldots, 4n\}$. It means that the labeling α_2 is a bijection from the set $V(C_n(1,2,3)) \cup E(C_n(1,2,3))$ onto the set $\{1, 2, \ldots, 4n\}$.

We divide the vertex-weights of $C_n(1,2,3)$ in three cases.

Case 1. $i = 0, 1, 2$

a) For $i=0$

$$\begin{aligned} wt_{\alpha_2}(v_0) = &\ (3)+(n+1)+(3n)+(3n+1) \\ &+(2n+1-\tfrac{n-1}{2})+(3n-(n-2))+(3n+(n-3)+1) \\ = &\ \tfrac{29n+11}{2}. \end{aligned}$$

b) For $i=1$

$$\begin{aligned} wt_{\alpha_2}(v_1) = &\ (3-1)+\tfrac{3n-1+2}{2}+(3n-1)+(3n+1+1) \\ &+(n+1)+(3n-(n-1))+(3n+(n-2)+1) \\ = &\ \tfrac{29n+7}{2}. \end{aligned}$$

c) For $i=2$

$$\begin{aligned} wt_{\alpha_2}(v_2) = &\ (3-2)+2n+1-\tfrac{2}{2}+(3n-2)+(3n+2+1) \\ &+\tfrac{3n+1}{2}+(3n)+(3n+(n-1)+1) \\ = &\ \tfrac{33n+3}{2}. \end{aligned}$$

Case 2. i odd, $i \geq 3$

$$\begin{aligned} wt_{\alpha_2}(v_i) = &\ (n+3-i)+\tfrac{3n-i+2}{2}+(3n-i)+(3n+i+1) \\ &+(2n+1-\tfrac{i-1}{2})+(3n-(i-2)+(3n+(i-3)+1) \\ = &\ \tfrac{29n+13}{2}-2i. \end{aligned}$$

Case 3. i even, $i \geq 4$

$$\begin{aligned} wt_{\alpha_2}(v_i) = &\ (n+3-i)+2n+1-\tfrac{i}{2}+(3n-i)+(3n+i+1) \\ &+\tfrac{3n-(i-1)+2}{2}+(3n-(i-2))+(3n+(i-3)+1 \\ = &\ \tfrac{33n+13}{2}-2i. \end{aligned}$$

Then we obtain that the vertex weight form consecutive integers : $\frac{29n+7}{2}, \frac{29n+9}{2}+2, \ldots, \frac{29n+9}{2}+2n-1=\frac{33n+7}{2}$. Thus we obtain that $C_n(1,2,3)$, $n \geq 5$, has super $(\frac{29n+7}{2}, 2)$-VAT labeling. □

Theorem 2.5. *Let n be an odd integer,* $n \geq 5$*. The graph* $C_n(1,2,3)$ *admits a* $(\frac{27n+11}{2}, 3)$*-VAT labeling.*

Proof. Let $C_n(1,2,3)$ be a subclass of circulant graphs with $n \geq 5$. Let $\{v_i : i = 0, 1, \ldots, n-1\}$ be the vertices of $C_n(1,2,3)$.

Label all the vertices and edges as follows:

$$\alpha_3(v_i) = \begin{cases} 2n+2i-5, & \text{for } i = 0,1,2, \\ 2i-5, & \text{for } i = 3,4,\ldots,n-1, \end{cases}$$

$$\alpha_3(v_iv_{i+1}) = \begin{cases} 3n, & \text{for } i = 0, \\ \frac{5n+i}{2}, & \text{for } i = 1,3,\ldots,n-2, \\ \frac{4n+i}{2}, & \text{for } i = 2,4,\ldots,n-1, \end{cases}$$

$$\alpha_3(v_iv_{i+2}) = 4n-i, \text{for } i = 0,1,\ldots,n-1,$$

$$\alpha_3(v_iv_{i+3}) = 2(n-i), \text{for } i = 0, 1, \ldots, n-1.$$

The vertex and edge labels under the labeling α_3 are $\alpha_3(V) = \{1, 3, \ldots, 2n-1\}$ and $\alpha_3(E) = \{2, 4, \ldots, 2n\} \cup \{2n+1, 2n = 2, \ldots, 4n\}$. Then the labeling α_3 is a bijection from the set $V(C_n(1,2,3)) \cup E(C_n(1,2,3))$ onto the set $\{1, 2, \ldots, 4n\}$.

The vertex-weights of $C_n(1,2,3)$ will be calculated in three cases.

Case 1. $i = 0, 1, 2$

a) For $i = 0$

$$\begin{aligned} wt_{\alpha_3}(v_0) = &\ (2n-5) + (3n) + (4n) + 2(n) \\ &+ \tfrac{4n+(n-1)}{2} + (4n-(n-2)) + 2(n-(n-3)) \\ = &\ \tfrac{33n+5}{2}. \end{aligned}$$

b) For $i = 1$

$$\begin{aligned} wt_{\alpha_3}(v_1) = &\ (2n+2-5) + \tfrac{5n+1}{2} + (4n-1) + 2(n-1) \\ &+ (3n) + (4n-(n-1)) + 2(n-(n-2)) \\ = &\ \tfrac{33n-1}{2}. \end{aligned}$$

c) For $i = 2$

$$\begin{aligned} wt_{\alpha_3}(v_2) = &\ (2n+4-5) + \tfrac{4n+2}{2} + (4n-2) + 2(n-2) \\ &+ \tfrac{5n+1}{2} + (4n) + 2(n-(n-1)) \\ = &\ \tfrac{33n-7}{2}. \end{aligned}$$

Case 2. i odd, $i \geq 3$

$$\begin{aligned} wt_{\alpha_3}(v_1) = &\ (2i-5) + \tfrac{5n+i}{2} + (4n-i) + 2(n-i) \\ &+ \tfrac{4n+i-1}{2} + (4n-(i-2)) + 2(n-(i-3)) \\ = &\ \tfrac{33n+5}{2} - 3i. \end{aligned}$$

Case 3. i even, $i \geq 4$

$$\begin{aligned} wt_{\alpha_3}(v_2) = &\ (2i-5) + \tfrac{4n+i}{2} + (4n-i) + 2(n-i) \\ &+ \tfrac{5n+(i-1)}{2} + (4n-(i-2)) + 2(n-(i-3)) \\ = &\ \tfrac{33n+5}{2} - 3i. \end{aligned}$$

Thus, we conclude that $C_n(1,2,3)$, $n \geq 5$, has a vertex $(\frac{27n+11}{2}, 3)$-antimagic total labeling □

Theorem 2.6. *Let n be an odd integer, $n \geq 5$. The graph $C_n(1,2,3)$ admits a $(13n+6, 4)$-VAT labeling.*

Proof. Let $C_n(1,2,3)$ be a subclass of circulant graphs with $n \geq 5$. Let $\{v_i : i = 0, 1, \ldots, n-1\}$ be the vertices of $C_n(1,2,3)$.

Label all the vertices and edges as follows:

$$\alpha_4(v_i) = \begin{cases} 5 - 2i, & \text{for } i = 0, 1, 2, \\ 2(n-i) + 5, & \text{for } i = 3, 4, \ldots, n-1, \end{cases}$$

$$\alpha_4(v_iv_{i+1}) = \begin{cases} 2, & \text{for } i = 0, \\ n-i+2, & \text{for } i = 1, 3, \ldots, n-2, \\ 2(n+i)+1, & \text{for } i = 2, 4, \ldots, n-1, \end{cases}$$

$$\alpha_4(v_iv_{i+2}) = 4n - 2i, \text{for } i = 0, 1, \ldots, n-1,$$
$$\alpha_4(v_iv_{i+3}) = 2(n+i)+1, \text{for } i = 0, 1, \ldots, n-1.$$

The vertex and edge labels under the labeling α_4 are $\alpha_4(V) = \{1, 3, \ldots, 2n-1\}$ and $\alpha_4(E) = \{2, 4, \ldots, 2n\} \cup \{2n+1, 2n+2, \ldots, 4n\}$. It means that the labeling α_4 is a bijection from the set $V(C_n(1,2,3)) \cup E(C_n(1,2,3))$ onto the set $\{1, 2, \ldots, 4n\}$.

We consider the vertex-weights of $C_n(1,2,3)$ case by case.

Case 1. $i = 0, 1, 2$

a) For $i = 0$

$$\begin{aligned} wt_{\alpha_4}(v_0) = \;& (5) + (2) + (4n) + (2n+1) \\ & +2(n+1) - (n-1) + (4n - 2(n-2)) + (2(n + (n-3)) + 1) \\ =\;& 13n + 10. \end{aligned}$$

b) For $i = 1$

$$\begin{aligned} wt_{\alpha_4}(v_0) = \;& (5-2) + (n-1+2) + (4n-2) + (2(n+1)+1) \\ & +2 + (4n - 2(n-1)) + (2(n + (n-2)) + 1) \\ =\;& 13n + 6. \end{aligned}$$

c) For $i = 2$

$$\begin{aligned} wt_{\alpha_4}(v_0) = \;& (5-4) + 2(n+1) - 2 + (4n-4) + (2(n+2)+1) \\ & +(n-1+2) + (4n) + (2(n + (n-1)) + 1) \\ =\;& 17n + 2. \end{aligned}$$

Case 2. i odd, $i \geq 3$

$$\begin{aligned} wt_{\alpha_4}(v_0) = \;& (2(n-i)+5) + (n-i+2) + (4n-2i) + (2(n+i)+1) \\ & +(2(n+1) - (i-1)) + (4n - 2(i-2)) + (2(n + (i-3)) + 1) \\ =\;& 17n + 10 - 4i. \end{aligned}$$

Case 3. i even, $i \geq 4$

$$\begin{aligned} wt_{\alpha_4}(v_0) = \;& (2(n-i)+5) + (2(n+1)-i) + (4n-2i) + (2(n+i)+1) \\ & +(n-(i-1)+2) + (4n - 2(i-2)) + (2(n+i-3)+1) \\ =\;& 17n + 10 - 4i. \end{aligned}$$

The vertex weight set is $\{13n+6, 13n+10, \ldots, 13n+2+4(n-1) = 17n-2\}$. Thus, $C_n(1,2,3)$, $n \geq 5$, has vertex $(13n+6, 4)$-antimagic total labeling. □

Theorem 2.7. *Let n be an odd integer, $n \geq 5$. The graph $C_n(1,2,8)$ admits a $(10n+9, 8)$-VAT labeling.*

Proof. Let $C_n(1,2,3)$ be a subclass of circulant graphs with $n \geq 5$. Let $\{v_i : i = 0, 1, \ldots, n-1\}$ be the vertices of $C_n(1,2,3)$.

Label all the vertices and edges as follows:

$$\alpha_8(v_i) = \begin{cases} 9 - 4i, & \text{for } i = 0, 1, 2, \\ 4(n-i) + 9, & \text{for } i = 3, 4, \ldots, n-1, \end{cases}$$

$$\alpha_8(v_i v_{i+1}) = \begin{cases} 2, & \text{for } i = 0, \\ 2(n-i+1), & \text{for } i = 1, 3, \ldots, n-2, \\ 2(2n-i+1), & \text{for } i = 2, 4, \ldots, n-1, \end{cases}$$

$$\alpha_8(v_i v_{i+2}) = 4(n-i), \text{for } i = 0, 1, \ldots, n-1,$$

$$\alpha_8(v_i v_{i+3}) = 4i + 3, \text{for } i = 0, 1, \ldots, n-1.$$

The vertex and edge labels under the labeling α_8 are $\alpha_8(V) = \{1, 5, 9, \ldots, 4n-3\}$ and $\alpha_8(E) = \{2, 6, 10 \ldots, 4n-2\} \cup \{3, 7, 11 \ldots, 4n-1\} \cup \{4, 8, 12 \ldots, 4n\}$. It means that the labeling α_8 is a bijection from the set $V(C_n(1,2,3)) \cup E(C_n(1,2,3))$ onto the set $\{1, 2, \ldots, 4n\}$.

We consider the vertex-weights of $C_n(1,2,3)$ case by case.

Case 1. $i = 0, 1, 2$

a) For $i = 0$

$$\begin{aligned} wt_{\alpha_8}(v_0) = \; & (9) + (2) + (4n) + (3) \\ & +2(2n - (n-1) + 1) + 4(n - (n-2)) + (4(n-3) + 3) \\ = \; & 10n + 17. \end{aligned}$$

b) For $i = 1$

$$\begin{aligned} wt_{\alpha_8}(v_1) = \; & (9-4) + 2(n-1+1) + 4(n-1) + (4+3) \\ & +(2) + 4(n - (n-1)) + (4(n-2) + 3) \\ = \; & 10n + 9. \end{aligned}$$

c) For $i = 2$

$$\begin{aligned} wt_{\alpha_8}(v_2) = \; & (9-8) + 2(2n-2+1) + 4(n-2) + (8+3) \\ & +2(n-1+1) + 4(n) + (4(n-1) + 3) \\ = \; & 18n + 1. \end{aligned}$$

Case 2. i odd, $i \geq 3$

$$\begin{aligned} wt_{\alpha_8}(v_i) = \; & (4(n-i) + 9) + 2(n-i+1) + 4(n-i) + (4i+3) \\ & +2(2n - (i-1) + 1) + 4(n - (i-2)) + (4(i-3) + 3) \\ = \; & 18n + 17 - 8i. \end{aligned}$$

Case 3. i even, $i \geq 4$

$$\begin{aligned} wt_{\alpha_8}(v_i) = & \quad (4(n-i)+9)+2(2n-i+1)+4(n-i)+(4i+3) \\ & \quad +2(n-(i-1)+1)+4(n-(i-2))+(4(i-3)+3) \\ = & \quad 18n+17-8i. \end{aligned}$$

By calculating the vertex weights then $C_n(1,2,3)$, $n \geq 5$, has vertex $(10n+9,8)$-antimagic total labeling.

□

3. Concluding remark

As a final remark, we present some problems that are raised from this paper.

(1) Find the construction of vertex (a,d)-antimagic total labeling of $C_n(1,2,3)$ for $d = 5,6,7$ and for $9 \leq d \leq 22$.

(3) Find the construction of disjoint union of vertex (a,d)-antimagic total labeling of $C_{n_j}(1,2,3)$, for $j = 1,2,\ldots t$.

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