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LIVING WITH THE LABELING DISEASE FOR 25 YEARS

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Abstract. In this article I trace my involvement with graph labeling for the past 25 years. I provide some statistical information about the growth in interest in graph labeling and some open problems that I believe are accessible.

Key words: Graph labeling, graceful graphs, harmonious graphs.

Abstrak. Pada paper ini saya menapaktilas keterlibatan saya dengan pelabelan graf sejak 25 tahun yang lalu. Saya memberikan beberapa informasi secara statistik tentang pertumbuhan minat pada pelabelan graf dan beberapa masalah terbuka yang saya yakini dapat dikerjakan.

Kata kunci: Pelabelan graf, graf-graf graceful, graf-graf harmonis.

1. My Background

I obtained a PhD in finite groups in 1971 from the University of Notre Dame. While I was a student the notion of a graph never arose so when I came to the University of Minnesota Duluth (UMD) in 1972 I did not even know definition of a graph. My first encounter with graphs occurred in the mid 1970s when I was working on a problem that concerned generating the elements of a 2-generated group of order n by listing a sequence of length n of the two generators in such a way that the n partial products are distinct and the last product is the inverse of one of the two generators [15]. A referee's report on the paper I wrote on this problem mentioned that what I was doing was finding a Hamiltonian circuit in the Cayley digraph of the group with the given 2-element generating set. Upon looking into this topic I came to the conclusion that problems about Hamiltonian circuits in Cayley digraphs of groups was something interesting to me and that such problems were more accessible to undergraduate students than the typical research problems

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in finite group theory. After immersing myself in this area I thought it would useful to me and my research students if I wrote a survey paper on the subject to help them assimilate the concepts and the known results quickly. This became my first of two survey papers on Hamiltonian circuits in Cayley graphs and digraphs [29] and [5]–see below.

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A SURVEY: HAMILTONIAN CYCLES IN CAYLEY GRAPHS

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It has been conjectured there is a hamiltonian cycle in every Cayley graph. Interest in this and other closely related questions has grown in the past few years. We survey the results, techniques and open problems in the field.

1. What is a Cayley graph?

Let S generate a finite group G. We define the *Cayley digraph* $\mathrm{Cay}(S:G)$ of the generators S on G as follows. The vertices of $\mathrm{Cay}(S:G)$ are the elements of G, and there is an arc from g to gs whenever $g \in G$ and $s \in S$. The *Cayley graph* $\mathrm{Cay}_g(S:G)$ of S on G is obtained by replacing each arc in $\mathrm{Cay}(S:G)$ with an (undirected) edge. One can identify $\mathrm{Cay}_g(S:G)$ with $\mathrm{Cay}(S \cup S^{-1}:G)$, where $S^{-1}=\{s^{-1}:s \in S\}$.

We often write $\mathrm{Cay}(G)$ instead of $\mathrm{Cay}(S:G)$, especially when S is the 'standard' generating set for G. We use Z_n to denote the cyclic group of order n. The (undirected) *cycle* $\mathrm{Cay}_g(Z_n)$ is denoted C_n and the (directed) *circuit* $\mathrm{Cay}(Z_n)$ is denoted Z_n (the standard generating set for a cyclic group has just one element).

2. Techniques

In this section we discuss a few of the ideas which have been developed to establish the existence of hamiltonian cycles and circuits in Cayley graphs and digraphs. These techniques will be demonstrated in later sections. The notation introduced here will be used in proofs throughout the paper.

2.1. *Notation*

A path in a digraph can be specified either by the sequence $(v_i: 0 \leq i \leq n)$ of vertices encountered, or by a list $[a_i: 1 \leq i \leq n]$ of the arcs traversed. In $\mathrm{Cay}(S:G)$, each arc of the form (g, gs) is labelled s; and it is more convenient to list the labels of the arcs, rather than the arcs themselves. Then, to determine a unique path, it is also necessary to specify the initial vertex v_0. When $[a_i: 1 \leq i \leq$

My experience has been that survey papers are coveted by journal editors and researchers and they are read and cited far more often than standard research papers. As of November 2010 my first on Cayley digraphs was cited on Google scholar 80 times and my second survey on Cayley digraphs was cited 74 times.

I first heard of graph labeling from Ron Graham at a talk he gave on harmonious labeling of graphs in the early 1980s. I immediately thought that labeling problems would be interesting and accessible to undergraduate research students. One of the first papers on graph labeling I read was by Roberto Frucht on graceful labelings of wheels [10]. In that paper Frucht mentioned that he had also found graceful labelings for prisms $C_n \times P_2$. When I wrote to him asking for a copy of his paper on prisms he replied that he had not formally written up his results because he desired to find simpler labelings and he asked if I wanted to join him in starting over from scratch. That lead to my first paper on graph labeling (see [11]).

In 1986 I had two extremely strong undergraduate summer research students named Doug Jungreis and Mike Reid whom I wanted to challenge. Both of them were medal winners in the International Mathematical Olympiad in 1983 and 1984. Of course, I knew that the graceful tree conjecture was considered to be notoriously difficult but I thought that maybe these two exceptional problem solvers might be able to come up with a fresh approach to this problem. They worked hard on the problem but were not able to make any substantial progress. This convinced me that the graceful tree conjecture deserved its notoriety for being difficult. Nevertheless, Jungreis and Reid [19] were able to produce some significant results on graceful, α, and harmonious labelings of grids, prisms, and tori. (An *α-labeling* is a graceful labeling with the additional property that there exists an integer k so that for each edge xy either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$.)

2. **Graph Labeling Surveys**

My first labeling survey paper was published in the *Journal of Graph Theory* in 1989 [12]. It was 14 pages in length and covered only graceful and harmonious labelings. In the introduction I mention that about 150 papers had been written on graph labeling. It contained 78 references. As of November 2010 this survey was cited 61 times on Google Scholar.

A Survey: Recent Results, Conjectures, and Open Problems in Labeling Graphs

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ABSTRACT

In this paper we organize and summarize much of the work done on graceful and harmonious labelings of graphs. Many open problems and conjectures are included.

1. INTRODUCTION

Interest in graph labeling problems began in the mid-1960s with a conjecture of G. Ringel [64] and a paper by A. Rosa [66]. In the intervening two decades, well over 150 papers on this topic have appeared. The so-called Ringel–Kotzig conjecture that all trees are graceful has been the focus of many of these (see [6,11,46,78]). Despite the large number of papers, there are relatively few general results or methods on graph labelings. Indeed, most of the results focus on particular classes of graphs and utilize ad hoc methods. Frequently, the same classes have been done by several authors. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, and communication network addressing (see [12] and [13] for details). In this paper we organize and summarize much of the work done to date, and offer a plethora of open problems and conjectures. Earlier surveys include [6,11,46,78].

My second labeling survey paper appeared in the *Discrete Applied Mathematics* in 1994 [13]. It updated my first survey and added new topics such as cordial, sequential, etc. It was 17 pages long with 74 references. I mention in the introduction that over 200 papers had been published on graph labeling. Through November 2010 it has been cited 21 times on Google scholar.

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A guide to the graph labeling zoo

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Abstract

In this paper we survey many of the variations of graceful and harmonious labeling methods that have been introduced and summarize much of what is known about each kind.

1. Introduction

A vertex *labeling, valuation* or *numbering* of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels $f(x)$ and $f(y)$. Most graph labeling methods trace their origin to one introduced by Rosa [Ga66][1] in 1967, or one given by Graham and Sloane [Ga36] in 1980. Rosa [Ga66] called a function f a *β-valuation* of a graph G with q edges if f is an injection from the vertices of G to the set $\{0,1,\ldots,q\}$ such that, when each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct. (Golomb [Ga32] subsequently called such labelings *graceful* and this term is now the popular one.) Rosa introduced β-valuations as well as a number of other valuations as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [Ga64] that K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges. Harmonious graphs naturally arose in the study by Graham and Sloane [Ga36] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be *harmonious* if there is an injection f from the vertices of G to the group of integers modulo q such that, when each edge xy is assigned the label $f(x)+f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices.

In the intervening years, close to 200 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few

3. **Dynamic Survey on Graph Labeling**

When I first heard of the idea of a dynamic survey (that is, an electronic paper that is occasionally updated without undergoing further review by referees and available free on the web) I thought graph labeling was a perfect candidate– the field was growing rapidly with papers appearing in journals from around the world. In my initial dynamic article I updated my 1994 survey and expanded the range of labelings included. Although I submitted it to the Electronic Journal of Combinatorics (EJC) in September 1996 it was not accepted until November 1997 [14]. In fact, about a year after submitting it I received a letter from the editor of EJC saying they were rejecting it because they could not find anyone to agree to referee it. I wrote the editor back suggesting several people who I thought might be willing to referee it. One of them was Alex Rosa, who had been helpful to me in some earlier occasions about where to submit labeling papers. Of course, I was not told who finally agreed to serve as referee but it was accepted a few months later.

One obvious problem with a dynamic survey is that the journal does not retain a permanent copy of earlier versions. It did not occur to me to save copies of previous editions so I do not know how many pages the first edition had nor the number of references it included. The same is true for several later editions. Following is the only information I have about the various editions.

First edition 1997 (? pages)
Second edition 1998 (43 pages); about 200 references
Third edition 1999 (52 pages)
Four edition 2000 (? pages)
Fifth edition 2000 (58 pages)
Sixth edition 2001 (74 pages)
Seventh 2002 (106 pages)
Eighth edition 2003 (147 pages)
Ninth edition 2005 (155 pages)
Tenth edition 2006 (196 pages); 794 references
Eleventh edition 2008 (? pages); 865 references
Twelfth edition 2009 (219 pages); 1013 references
IWOGL edition 2010 (233 pages); 1197 references.

Through November 2010 the dynamic survey has been cited 404 times on Google Scholar.

4. **Data Mining Google Scholar**

For the 2010 graph labeling workshop in Duluth I thought it might be of interest to prepare a list of the most cited graph labeling papers I could find using Google Scholar. Following are the results. The number of citations for each is given in parentheses.

- A. Rosa [25], On certain valuations of the vertices of a graph, 1966 (285)
- R. Graham and N. Sloane [18], Additive bases and harmonious graphs, 1980 (124)
- R. P. Stanley [26], Linear homogeneous Diophantine equations and magic labelings of graphs, 1973 (122)
- A. Kotzig and A. Rosa [20], Magic valuations of finite graphs, 1970 (120)
- W. D. Wallis [27], Magic Graphs, 2001 (99)
- H. Enomoto, A. S. Llado, T. Nakamigawa [8], and G. Ringel, Super edge-magic graphs, 1998 (66)
- R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle [9], The place of super edge-magic labelings among other classes of labelings, 2001 (53)
- T. Grace [17], On sequential labelings of graphs, 1983 (53)
- M. Doob [7], Characterizations of regular magic graphs, 1978 (48)
- S. P. Lo [22], On edge-graceful labelings 1985, (44)
- J. A. MacDougall, M. Miller, and W. D. Wallis [23], Vertex-magic total labelings of graphs, 2004 (44)
- G. S. Bloom [3], A chronology of the Ringel-Kotzig conjecture and the continuing quest to call all trees graceful, 1979 (44).

The only surprise to me on this list is the paper by Stanley.

5. **Data Mining Dynamic Survey**

Following is a list of the authors in the IWOGL edition of my dynamic survey that have 10 or more papers cited in the survey along with the number of papers cited.

S. M. Lee	135	Vilfred	15
Bača	82	Barrientos	15
Miller	67	Kotzig	14
Sethuraman	30	Cahit	14
Slamin	27	El-Zanati	12
Hegde	32	Koh	12
Ryan	25	Bu	11
Acharya	22	Rogers	11
Shiu	21	Figueroa-Centeno	11
Youssef	21	Ichishima	11
Wallis	19	Muntaner-Batle	11
Shetty	19	Gray	10
Baskoro	19	Kathiresan	10
MacDougall	18	Kwong	10
Seoud	18	Seah	10
Singh	16	Selvaraju	10
Ng	16		

The above lists raises the question of which countries are the most active in graph labeling research. My guess is that is the top five (in alphabetical order) are Australia, China, India, Indonesia, and the United States.

6. Graph Labeling Time Line

Here is a time line for the introduction of various concepts in graph labeling.

1966	Magic: Sedláček
1967	Graceful: Rosa
1967	Super magic: Steward
1970	Edge-magic total: Kotzig and Rosa
1980	Prime labeling: Entringer
1980	Harmonious: Graham and Sloane
1981	Sequential (strongly c-harmonious): Grace; Chang, Hsu, Rogers
1981	Elegant: Chang, Hsu, and Rogers
1982	k-graceful: Slater; Maheo and Thuillier
1983	Magic labelings of Types (a, b, c): Lih
1985	Edge-graceful: Lo
1987	Cordial: Cahit
1990	(k, d)-arithmetic, (k, d)-indexable: Acharya, Hegde
1990	Antimagic: Hartsfield and Ringel
1990	k-equitable: Cahit
1990	Sum graphs: Harary
1990	Mod sum graphs: Boland, Laskar, Turner, and Domke
1991	Skolem-graceful: Lee and Shee
1991	Odd-graceful: Gnanajothi
1991	Felicitous: Choo
1993	(a, d)-antimagic: Bodendiek and Walther
1994	Integral sum graphs: Harary
1998	Super edge-magic: Enomoto, Llado, Nakamigana and Ringel
1999	Vertex-magic total: MacDougall, Miller, Slamin, and Wallis
2000	(a, d)-vertex-antimagic total labeling: Bača, Bertault, MacDougall, Miller, Simanjuntak, and Slamin
2001	Radio labeling: Chartrand, Erwin, Zhang, and Harary

7. Some Partially Done Problems I Would Like to See Finished

Following is a list of labeling problems for basic families of graphs that have only partially been done that I would like to see finished.

- Finish the remaining cases for graceful and harmonious labelings of $C_m \times C_n$.
- Finish the remaining cases for graceful and harmonious labelings of $C_m \times P_n$.
- Finish the remaining cases for harmonious cycles with a P_k-cord ($k > 2$) (the graceful analog is done).
- Finish the remaining cases of super edge-magic labeling of $C_m \times C_n$.

- Finish the remaining cases of prime labelings of $P_m \times P_n$. (A graph with vertex set V is said to have a *prime labeling* if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge xy the labels assigned to x and y are relatively prime.)
- Finish the remaining cases of graceful and harmonious labelings of lobsters. This problem is probably very hard. All caterpillars have been shown to be graceful and harmonious
- Finish the remaining cases of graceful labelings of windmill graphs $K_4^{(m)}$ $m > 3$ (the one-point union of m copies of K_4). This has been done up to $m = 1000$. The harmonious case is done.
- Finish the remaining cases of graceful and harmonious labelings of P_n^k. The graceful problem is open for $k > 2$; the harmonious problem is open for some even k; the harmonious odd case is done.

8. Untouched Open Area

Here is an area of graph labeling that has been untouched except for cycles. Let G be a graph with q edges and H a finite Abelian group (under addition) of order q. Define G to be *H-harmonious* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x) + f(y)$ the resulting edge labels are distinct. When G is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley, and Jungreis [2] have shown that if H is a finite Abelian group of order $n > 2$ then C_n is H-harmonious if and only if H has a non-cyclic or trivial Sylow 2-subgroup and H is not of the form $Z_2 \times Z_2 \times \cdots \times Z_2$. Thus, for example, C_{12} is not Z_{12}-harmonious but is $(Z_2 \times Z_2 \times Z_3)$-harmonious.

9. Universal Conjecture on Graph Labeling

When any new graph labeling concept is introduced one of the first problems to be considered is which trees have the desired labeling. If some examples of trees that do not have the labeling are not quickly found it is typically conjectured that every tree has the labeling. Specifically, it is conjectured that every tree has the following kinds of labelings: graceful; harmonious; k-graceful for some k; odd-graceful; odd-harmonious; triangular graceful; edge-magic total labelings; antimagic (except P_2); and $(a, 1)$-edge-antimagic total labeling prime. (See [14] for details.)

On the other hand, it has been proved that every tree is cordial; all trees are indexable; not all trees have an α-labeling; and not all trees are elegant.

10. New Topic Included in IWOGL Survey

The IWOGL edition includes a new topic that has been around for quite a while but largely has been ignored by people who have worked on graceful/harmonious/magic/antimagic kinds of labelings. The notion of the ranking number of a graph arose in late 1980s in connection with very large scale integration (VLST) layout designs and parallel processing. A *ranking of a graph* is a labeling of the vertices with positive integers such that every path between two vertices with the same label contains a vertex with a greater label. (See Figure 1.) The *rank number* of a graph is the smallest possible number of labels in a ranking.

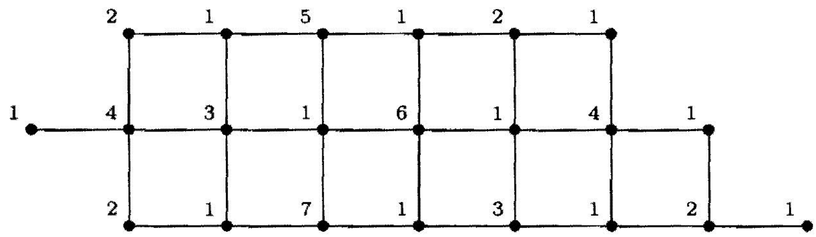

Figure 1: Ranking of a partial grid.

Here is a partial list of the known results on rank numbers of graphs.

- (1996) Laskar and Pillone [21] proved that determining rankings numbers is NP-complete. It is NP-hard even for bipartite graphs.
- (1997) Wang [28] found the rank number for paths and joins.
- (2004) Dereniowski [6] found the rank numbers of stars, cycles, wheels, and complete k-partite graphs.
- (2009) Novotny, Ortiz, and Narayan [24] determined the rank number of P_n^2 and asked about P_n^k.
- (2009) Alpert [1] and independently, C.- W. Chang, Kuo, and H.-C. Lin [4] determined the rank numbers of P_n^k, C_n^k, $P_2 \times P_n$, and $P_2 \times C_n$. Chang et al. also determined the rank numbers of caterpillars.
- (2009) Alpert [1] used recursive methods to find rank numbers of Möbius ladders, $K_s \times P_n$, $P_3 \times P_n$, and found bounds for rank numbers of general grid graphs $P_m \times P_n$.

Here are some open ranking number problems.

- Finish open cases for grid graphs $P_m \times P_n$ (probably hard)
- $P_m \times C_n$ (probably hard)
- $C_m \times C_n$ (probably very hard)

11. How to Help me with the Dynamic Survey

As the number of topics in the survey expands with each edition and the number of people working on graph labeling grows, the survey is becoming increasingly difficult for me to keep up to date. You can assist me by sending me preprints, corrections, typos, and updates of citation information. To be most helpful to me please send me an expanded abstract that includes the definitions and summarizes the main results. Send me what you would like to see in my survey. I might make some edits but such a summary would make it easy for me to include your results.

Thank you for your contributions to graph labeling and continue to spread the "disease."

12. **Acknowledgment**

I wish to thank the organizing committee of the IWOGL: Dalibor Froncek, Mirka Miller, Sergei Bezrukov, Suresh Hegde, Yuqing Lin, and Steve Rosenberg for the kind invitation to be a featured speaker.

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