J. Indones. Math. Soc.
Vol. 17, No. 1 (2011), pp. 29–38.

ON CLASSES OF ANALYTIC FUNCTIONS CONTAINING GENERALIZATION OF INTEGRAL OPERATOR

Maslina Darus[1,*] and Rabha W. Ibrahim[2]

[1]**School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia Bangi 43600, Selangor Darul Ehsan, Malaysia, maslina@ukm.my**

[2]**School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia Bangi 43600, Selangor Darul Ehsan, Malaysia, rabhaibrahim@yahoo.com**

Abstract. New classes containing generalization of integral operator are introduced. Characterization and other properties of these classes are investigated. Further, Fekete-Szegö functional for these classes are also given.

Key words and Phrases: Integral operator, analytic functions, differential operator, Fekete-Szegö functional, distortion theorem.

Abstrak. Pada paper ini diperkenalkan kelas baru yang memuat perumuman dari operator integral. Kemudian karakterisasi dan sifat lain juga dikaji dari kelas. Lebih lanjut, fungsional Fekete-Szegö untuk kelas ini juga diberikan.

Kata kunci: Perumuman operator integral, fungsi analitik, operator diferensial, fungsional Fekete-Szegö, teorema distorsi.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in U and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1}$$

2000 Mathematics Subject Classification: 30C 45.
Received: 17-12-2010, revised: 07-04-2011, accepted: 09-04-2011.
*Corresponding author

Now we introduce a differential operator defined as follows : $\mathbf{D}_{\lambda,\delta}^{k,\alpha,\beta} : \mathcal{A} \to \mathcal{A}$ by

$$\mathbf{D}_{\lambda,\delta}^{k,\alpha,\beta} f(z) = z + \sum_{n=2}^{\infty} [(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n) a_n z^n, \tag{2}$$

where $k \in \mathbb{N}_0, \alpha \geq 0,\ \lambda \geq 0,\ \delta \geq 0$,
and

$$C(\delta, n) = \left(\begin{array}{c} n+\delta-1 \\ \delta \end{array} \right) = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}.$$

Remark 1.1. When $\alpha = \beta = 1, \lambda = 0, \delta = 0$ or $\alpha = 0, \lambda = 1, \delta = 0$ we get Sălăgean differential operator [18], $k = 0$ gives Ruschewyh operator [17], $\alpha = 0, \delta = 0$ implies Al-Oboudi differential operator of order (k) [1], $\alpha = \beta = 1, \lambda = 0$ or $\alpha = 0, \lambda = 1$ operator (2) reduces to Al-shaqsi and Darus differential operator of order (k) [2] and $\alpha = 0$ poses the differential operator of order (k), which is given by the authors [3]. Note that the differential operator in [3] was initially introduced by Al-Shaqsi and Darus [20]. However, it was overlooked and restated again in [3]. Recently, the authors posed different kind of linear, differential and integral operators (see [4-13,19]).

Given two functions $f, g \in \mathcal{A},\ f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,\ \ z \in U.$$

And for several functions $f_1(z), ..., f_m(z) \in \mathcal{A}$

$$f_1(z) * ... * f_m(z) = z + \sum_{n=2}^{\infty} (a_{1n} ... a_{mn}) z^n,\ \ ,\ z \in U.$$

Analogous to $\mathbf{D}_{\lambda,\delta}^{k,\alpha,\beta} f(z),\ z \in U$ we define an integral operator $\mathbf{J}_{\lambda,\delta}^{k,\alpha} : \mathcal{A} \to \mathcal{A}$ as

follows.
Let

$$\phi(z) := \frac{z}{1-z} + \frac{\lambda z}{(1-z)^2} - \frac{\lambda z}{1-z},\ \ \lambda \geq 0.$$

$$F_k(z) = \underbrace{\phi(z) * ... * \phi(z)}_{k-times} * \underbrace{\frac{z\beta}{(1-z)^2} * ... * \frac{z\beta}{(1-z)^2}}_{\alpha k-times} * [\frac{z}{(1-z)^{\delta+1}}]$$

$$= z + \sum_{n=2}^{\infty} [(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n) z^n,$$

where $k,\ \alpha \in \mathbb{N}_0,\ \beta \geq 1,\ \delta \geq 0, \lambda \geq 0$.

And let $F_k^{(-1)}$ be defined such that

$$F_k(z) * F_k^{(-1)} = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n.$$

Note that $F_k^{(-1)}$ is needed to get the integral operator from differential operator.

Then

$$\begin{aligned}\mathbf{J}_{\lambda,\delta}^{k,\alpha,\beta} f(z) &= F_k^{(-1)} * f(z) \\ &= \Big[\underbrace{\phi(z) * ... * \phi(z)}_{k-times} * \underbrace{\frac{z\beta}{(1-z)^2} * ... * \frac{z\beta}{(1-z)^2}}_{\alpha k-times} * \frac{z}{(1-z)^{\delta+1}}\Big]^{(-1)} * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} z^n,\end{aligned} \tag{3}$$

where $k, \alpha \in \mathbb{N}_0$, $\beta \geq 1$, $\lambda \geq 0$, $\delta \geq 0$ $z \in U$.

Remark 1.2. When $\alpha = 0, \lambda = 1, \delta = 0$ we get an integral operator (see [18]), $k = 0$ gives Noor integral operator [15,16].

Some of relations for this integral operator are discussed in the next lemma.

Lemma 1.1. *Let $f \in \mathcal{A}$. Then*

(i) $\mathbf{J}_{\lambda,0}^{0,\alpha,\beta} f(z) = f(z)$,

(ii) $\mathbf{J}_{1,0}^{1,0,\beta} f(z) = \int_0^z \frac{f(t)}{t} dt = f(z) * (-log(1-z)) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$.

Note that (ii) is the type of Bernardi integral[21].

Definition 1.1. *Let $f(z) \in \mathcal{A}$. Then $f(z) \in \mathbf{S}_{\lambda,\delta}^{k,\alpha,\beta}(\mu)$ if and only if*

$$\Re\Big\{\frac{z[\mathbf{J}_{\lambda,\delta}^{k,\alpha,\beta} f(z)]'}{\mathbf{J}_{\lambda,\delta}^{k,\alpha,\beta} f(z)}\Big\} > \mu, \ \ 0 \leq \mu < 1, \ \ z \in U.$$

Definition 1.2. *Let $f(z) \in \mathcal{A}$. Then $f(z) \in \mathbf{C}_{\lambda,\delta}^{k,\alpha,\beta}(\mu)$ if and only if*

$$\Re\Big\{\frac{[z(\mathbf{J}_{\lambda,\delta}^{k,\alpha,\beta} f(z))']'}{(\mathbf{J}_{\lambda,\delta}^{k,\alpha,\beta} f(z))'}\Big\} > \mu, \ \ 0 \leq \mu < 1, \ \ z \in U.$$

The article is organized as follows: In section 2, we study the characterization and distortion theorems, and other properties of these classes. In section 3, we obtain sharp upper bound of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \nu a_2^2|$ for the classes $S^{k,\alpha,\beta}_{\lambda,\delta}(\mu)$ and $C^{k,\alpha,\beta}_{\lambda,\delta}(\mu)$. For this purpose we need the following result

Lemma 1.2.[14] *Let $p \in \mathcal{P}$, that is, p be analytic in U, be given by $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $\Re\{p(z)\} > 0$ for $z \in U$. Then*

$$|p_2 - \frac{p_1^2}{2}| \le 2 - \frac{|p_1|^2}{2}$$

and $|p_n| \le 2$ for all $n \in N$.

2. General Properties of $\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}$

In this section we study the characterization properties and distortion theorems for the function $f(z) \in \mathcal{A}$ to belong to the classes $\mathbf{S}^{k,\alpha,\beta}_{\lambda,\delta}(\mu)$ and $\mathbf{C}^{k,\alpha,\beta}_{\lambda,\delta}(\mu)$ by obtaining the coefficient bounds.

Theorem 2.1. *Let $f(z) \in \mathcal{A}$. If for $\alpha \ge 0, \beta \ge 1$, $\delta \ge 0$ and $\lambda \ge 0$*

$$\sum_{n=2}^{\infty} \frac{(n-\mu)|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \le 1 - \mu, \quad 0 \le \mu < 1, \tag{4}$$

then $f(z) \in \mathbf{S}^{k,\alpha,\beta}_{\lambda,\delta}(\mu)$. The result (4) is sharp.

Proof. Suppose that (4) holds. Since

$$\begin{aligned} 1 - \mu &\ge \sum_{n=2}^{\infty} (n-\mu) \frac{|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \\ &\ge \sum_{n=2}^{\infty} \mu \frac{|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \\ &\quad - \sum_{n=2}^{\infty} \frac{n|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \end{aligned}$$

then this implies that

$$\frac{1 + \sum_{n=2}^{\infty} \frac{n|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)}}{1 + \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)}} > \mu,$$

hence

$$\Re\Big\{ \frac{z[\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta} f(z)]'}{\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta} f(z)} \Big\} > \mu.$$

We also note that the assertion (4) is sharp and the extremal function is given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\mu)}{(n-\mu)} [(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n) z^n.$$

In the same way we can verify the following results

Theorem 2.2. *Let $f(z) \in \mathcal{A}$. If for $\alpha \geq 0$, $\beta \geq 1$, $\delta \geq 0$ and $\lambda \geq 0$*

$$\sum_{n=2}^{\infty} \frac{n(n-\mu)|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \leq 1-\mu, \quad 0 \leq \mu < 1, \tag{5}$$

then $f(z) \in \mathbf{C}^{k,\alpha,\beta}_{\lambda,\delta}(\mu)$. The result (8) is sharp.

Theorem 2.3. *Let the hypotheses of Theorem 2.1 be satisfy. Then for $z \in U$ and $0 \leq \mu < 1$*

$$|\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta} f(z)| \geq |z| - \frac{1-\mu}{2-\mu}$$

and

$$|\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta} f(z)| \leq |z| + \frac{1-\mu}{2-\mu}.$$

Proof. By using Theorem 2.1, one can verify that

$$\begin{aligned}(2-\mu)\sum_{n=2}^{\infty} &\frac{|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \\ &\leq \sum_{n=2}^{\infty} \frac{(n-\mu)|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \\ &\leq 1-\mu\end{aligned}$$

then

$$\sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \leq \frac{1-\mu}{2-\mu}.$$

Thus we obtain

$$\begin{aligned}|\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta} f(z)| &\leq |z| + \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)}|z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)}|z|^2 \\ &\leq |z| + [\frac{1-\mu}{2-\mu}]|z|^2\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}
|\mathbf{J}_{\lambda,\delta}^{k,\alpha,\beta} f(z)| &= |z + \sum_{n=2}^{\infty} \frac{a_n}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} z^n| \\
&\geq |z - \sum_{n=2}^{\infty} \frac{a_n}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} z^n| \\
&\geq |z| - \sum_{n=2}^{\infty} \frac{|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} |z|^n \\
&\geq |z| - \sum_{n=2}^{\infty} (n-\mu)[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)|a_n||z|^2 \\
&\geq |z| - [\frac{1-\mu}{2-\mu}]|z|^2
\end{aligned}$$

This complete the proof.

Theorem 2.4. *Let the hypotheses of Theorem 2.21 be satisfy. Then for $z \in U$ and $0 \leq \mu < 1$*

$$|\mathbf{J}_{\lambda,\delta}^{k,\alpha,\beta} f(z)| \geq |z| - \frac{(1-\mu)}{2(2-\mu)}|z|^2$$

and

$$|\mathbf{J}_{\lambda,\delta}^{k,\alpha,\beta} f(z)| \leq |z| + \frac{(1-\mu)}{2(2-\mu)}|z|^2.$$

Theorem 2.5. *Let the hypotheses of Theorem 2.1 be satisfy. Then*

$$\frac{(n-\mu)}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \geq 1, \ \forall n \geq 2 \text{ and } 0 \leq \mu < 1$$

implies

$$|f(z)| \geq |z| - (1-\mu)|z|^2$$

and

$$|f(z)| \leq |z| + (1-\mu)|z|^2.$$

Proof. In virtue of Theorem 2.1, we have

$$\sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \frac{(n-\mu)|a_n|}{[(\beta n)^{\alpha} + (n-1)(\beta n)^{\alpha}\lambda]^k C(\delta, n)} \leq (1-\mu)$$

then

$$\sum_{n=2}^{\infty} |a_n| \leq (1-\mu).$$

Thus we obtain

$$\begin{aligned}|f(z)| &= |z+\sum_{n=2}^{\infty}a_nz^n| \\ &\le |z|+\sum_{n=2}^{\infty}|a_n||z|^2 \\ &\le |z|+(1-\mu)|z|^2\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}|f(z)| &\ge |z-\sum_{n=2}^{\infty}a_nz^n| \\ &\ge |z|-\sum_{n=2}^{\infty}|a_n||z|^2 \\ &\ge |z|-(1-\mu)|z|^2.\end{aligned}$$

This complete the proof.

Theorem 2.6. *Let the hypotheses of Theorem 2.2 be satisfy. Then*

$$\frac{(n-\mu)}{[(\beta n)^\alpha+(n-1)(\beta n)^\alpha\lambda]^kC(\delta,n)}\ge 1,\ \forall n\ge 2 \ \textit{and}\ 0\le\mu<1$$

poses

$$|f(z)|\ge|z|-\frac{(1-\mu)}{2}|z|^2$$

and

$$|f(z)|\le|z|+\frac{(1-\mu)}{2}|z|^2.$$

3. Fekete-Szegö for the Classes $\mathbf{S}_{\lambda,\delta}^{k,\alpha,\beta}(\mu)$ *and* $\mathbf{C}_{\lambda,\delta}^{k,\alpha,\beta}(\mu)$

In this section we determine the sharp upper bound for $|a_2|$ for the classes $\mathbf{S}_{\lambda,\delta}^{k,\alpha,\beta}(\mu)$ and $\mathbf{C}_{\lambda,\delta}^{k,\alpha,\beta}(\mu)$. Moreover, we calculate the Fekete-Szegö $|a_3-\nu a_2^2|$ functional for them.

Theorem 3.1. *Let the hypotheses of Theorem 2.1 be satisfy. Then*

$$|a_2|\le\frac{2(1-\mu)[(\beta 2)^\alpha(1+\lambda)]^kC(\delta,2)}{1+\mu}$$

and for all $\nu\in\mathbb{C}$ *the following bound is sharp*

$$\begin{aligned}|a_3-\nu a_2^2| &\le 2(1-\mu)[(3\beta)^\alpha(1+2\lambda)]^kC(\delta,3) \\ & max\Big\{1,|1+\frac{2(1-\mu)}{(1+\mu)[(3\beta)^\alpha(1+2\lambda)]^kC(\delta,3)} \\ & -2\nu\frac{1-\mu}{(1+\mu)^2}\frac{\{[(2\beta)^\alpha(1+\lambda)]^kC(\delta,2)\}^2}{[(3\beta)^\alpha(1+2\lambda)]^kC(\delta,3)}|\Big\}.\end{aligned}$$

Proof. Since $f \in \mathbf{S}^{k,\alpha,\beta}_{\lambda,\delta}(\mu)$ then the condition

$$\Re\Big\{\frac{z[\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}f(z)]'}{\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}f(z)}\Big\} > \mu, \quad 0 \le \mu < 1, \ \ z \in U$$

is equivalent to

$$z[\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}f(z)]' = (1-\mu)p(z)\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}f(z), \ \ z \in U,$$

for some $p \in \mathcal{P}$. Equating coefficients we obtain $a_2 = \frac{(1-\mu)Ap_1}{1+\mu}$, $a_3 = (1-\mu)B(p_2 + \frac{(1-\mu)}{2+\mu}p_1^2)$ where $A := [(2\beta)^\alpha(1+\lambda)]^k C(\delta, 2)$, $B := [(3\beta)^\alpha(1+2\lambda)]^k C(\delta, 3)$ and further, for $C := \frac{(1-\mu)^2}{1+\mu} + \frac{(1-\mu)B}{2} - \nu(\frac{(1-\mu)A}{1+\mu})^2$ and by using Lemma 1.2 we have $|a_3 - \nu a_2^2| \le H(x) = 2(1-\mu)B + (C - \frac{(1-\mu)B}{2})x^2$, $x := |p_1|$. Consequently, we receive

$$|a_3 - \nu a_2^2| = \begin{cases} H(0) = 2(1-\mu)B, & C \le \frac{(1-\mu)B}{2} \\ \\ H(2) = 4C, & C > \frac{(1-\mu)B}{2}. \end{cases}$$

Equality is attained for functions given by

$$\frac{z[\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}f(z)]'}{\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}f(z)} = \frac{1 + z^2(1-2\mu)}{1-z^2}$$

and

$$\frac{z[\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}f(z)]'}{\mathbf{J}^{k,\alpha,\beta}_{\lambda,\delta}f(z)} = \frac{1 + z(1-2\mu)}{1-z}$$

respectively.
For $\mu = 0$ we receive the following corollary.

Corollary 3.1. *Let the assumptions of Theorem 3.1 hold. Then for* $\mu = 0$

$$|a_2| \le 2[(2\beta)^\alpha(1+\lambda)]^k C(\delta, 2)$$

and

$$|a_3 - \nu a_2^2| \le 2[(3\beta)^\alpha(1+2\lambda)]^k C(\delta, 3)$$
$$max\Big\{1, |1 + \frac{2}{[(3\beta)^\alpha(1+2\lambda)]^k C(\delta, 3)}$$
$$- 2\nu\frac{\{[(2\beta)^\alpha(1+\lambda)]^k C(\delta, 2)\}^2}{[(3\beta)^\alpha(1+2\lambda)]^k C(\delta, 3)}|\Big\}.$$

In the similar manner we can prove the following result.

Theorem 3.2. *Let the hypotheses of Theorem 2.2 be satisfy. then*

$$|a_2| \le \frac{2(1-\mu)[(2\beta)^\alpha(1+\lambda)]^k C(\delta, 2)}{3+\mu}$$

and for all $\nu \in \mathbb{C}$ the following bound is sharp

$$|a_3 - \nu a_2^2| \leq \frac{2(1-\mu)}{3(2+\mu)}[(3\beta)^\alpha(1+2\lambda)]^k C(\delta,3)$$
$$max\Big\{1, |\frac{1}{2} + \frac{2(1-\mu)}{(3+\mu)}$$
$$- \frac{3\nu(1-\mu)}{(3+\mu)^2}\{[(2\beta)^\alpha(1+\lambda)]^k C(\delta,2)\}^2|\Big\}.$$

For $\mu = 0$ we receive the following corollary.

Corollary 3.2. *Let the assumptions of Theorem 3.2 hold. Then for $\mu = 0$*

$$|a_2| \leq \frac{2}{3}[(2\beta)^\alpha(1+\lambda)]^k C(\delta,2)$$

and

$$|a_3 - \nu a_2^2| \leq \frac{1}{3}[(3\beta)^\alpha(1+2\lambda)]^k C(\delta,3) max\Big\{1, |\frac{1}{2} + \frac{2}{3} - \frac{\nu}{3}\{[(2\beta)^\alpha(1+\lambda)]^k C(\delta,2)\}^2|\Big\}.$$

Acknowledgement: The work here is fully supported by MOHE: UKM-ST-06-FRGS0244-2010.

References

[1] Al-Oboudi, F., "On univalent functions defined by a generalized Salagean operator", *I.J.M.M.S*, **27** (2004), 1429–1436.

[2] Al-Shaqsi, K. and Darus, M., "An operator defined by convolution involving polylogarithms functions", *Journal of Mathematics and Statistics*, **4** (2008), 46–50.

[3] Darus, M. and Ibrahim, R.W., "Generalization of differential operator", *Journal of Mathematics and Statistics*, **4** (2008), 138–144.

[4] Darus, M. and Ibrahim, R.W., "On subclasses of uniformly Bazilevic type functions involving generalised differential and integral operators", *Far East Journal of Mathematical Sciences (FJMS)*, **33** (2009), 401–411.

[5] Darus, M. and Ibrahim, R.W., "On subclasses for generalized operators of complex order", *Far East Journal of Mathematical Sciences (FJMS)*, **33** (2009), 299–308.

[6] Darus, M. and Ibrahim, R.W., "On inclusion properties of generalized integral operator involving Noor integral", *Far East Journal of Mathematical Sciences (FJMS)*, **33** (2009), 309–321.

[7] Darus, M. and Ibrahim, R.W., "Ibrahim and Iqbal H. Jebril, Bounded turning for generalized integral operator", *Int. J. Open Problems Complex Analysis*, **1** (2009), 1–10.

[8] Darus, M. and Ibrahim, R.W., "New Classes Containing Generalization of Differential Operator", *Applied Mathematical Sciences*, **3:51** (2009), 2507–2515.

[9] Darus, M. and Ibrahim, R.W., "Integral operator defined by $k-$th Hadamard product", *ITB J. Sci.*, **42:2** (2010), 135–152.

[10] Darus, M. and Ibrahim, R.W. , "On differential operator of multivalent functions", *Inequality Theory and Applications, NOVA Publishers*, **6** (2010).

[11] Darus, M. and Ibrahim, R.W., "Multivalent functions based on linear integral operator", *Miskolc N.N*, **11:1** (2010), 43–52.

[12] Darus, M. and Ibrahim, R.W., "Operator defined by convolution with Zeta functions", *Far East Journal of Mathematical Sciences*, **40:1** (2010), 93–105.

[13] Darus, M. and Ibrahim, R.W., "On univalent functions defined by a generalized differential operator", *J. Anal. Appl.*, **16:2** (2010), 305–313.

[14] Duren, P.L., *Univalent functions*, Springer-Verlag, New York, 1983.

[15] Noor, K.I., "On new classes of integral operators", *J. Nat. Geomet.*, **16** (1999), 71–80.

[16] Noor, K.I. and Noor, M.A., "On integral operators", *J. of Math. Anal. Appl.*, **238** (1999), 341–352.

[17] Ruscheweyh, S., "New criteria for univalent functions", *Proc. Amer. Math. Soc.*, **49** (1975), 109–115.

[18] Sălăgean, G. S., "Subclasses of univalent functions", *Lecture Notes in Math., 1013*, Springer-Verlag, Berlin, 1983, 362–372.

[19] Srivastava, H. M., Darus, M., and Ibrahim, R.W., "Classes of analytic functions with fractional powers defined by means of a certain linear operator", *Integ. Tranc. Special Funct.*, **22** (2011), 17–28.

[20] Darus M. and Al-Shaqsi K., "Differential sandwich theorem with generalised derivative operator", *Int. J. Comput. Math. Sci.*, **2:2** (2008), 75–78.

[21] Bernardi, S.D., "Convex and starlike univalent functions", *Trans. Amer. Math. Soc.*, **135** (1969), 429–446.