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INSTABILITY FOR A CERTAIN FUNCTIONAL DIFFERENTIAL EQUATION OF SIXTH ORDER

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Abstract. Sufficient conditions are obtained for the instability of the zero solution of a certain sixth order nonlinear functional differential equation by the Lyapunov-Krasovskii functional approach.

Key words: Instability, Lyapunov-Krasovskii functional, delay differential equation, sixth order.

Abstrak. Dalam paper ini diperoleh syarat cukup untuk ketakstabilan solusi nol dari persamaan diferensial fungsional nonlinear orde keenam tertentu dengan pendekatan fungsional Lyapunov-Krasovskii.

Kata kunci: Ketakstabilan, fungsional Lyapunov-Krasovskii, persamaan diferensial tunda, orde keenam.

1. Introduction

First, in 1982, Ezeilo [2] discussed instability of the zero solution of the sixth order nonlinear differential equation without delay,

$$x^{(6)}(t) + a_1 x^{(5)}(t) + a_2 x^{(4)}(t) + e(x(t), x'(t), x''(t), x'''(t), x^{(4)}(t), x^{(5)}(t))x'''(t)$$

$$+f(x'(t))x''(t) + g(x(t), x'(t), x''(t), x'''(t), x^{(4)}(t), x^{(5)}(t))x'(t) + h(x(t)) = 0.$$

Later, in 1990, Tiryaki [4] proved an instability theorem for the sixth order nonlinear differential equation without delay,

$$x^{(6)}(t) + a_1 x^{(5)}(t) + f_1(x(t), x'(t), x''(t), x'''(t), x^{(4)}(t), x^{(5)}(t))x^{(4)}(t) + f_2(x''(t))x'''(t)$$

$$+f_3(x(t), x'(t), x''(t), x'''(t), x^{(4)}(t), x^{(5)}(t))x''(t) + f_4(x'(t)) + f_5(x(t)) = 0. \tag{1}$$

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On the other hand, recently, Tunç [10] established some sufficient conditions to ensure the instability of the zero solution of the sixth order nonlinear delay differential equation,

$$x^{(6)}(t) + a_1 x^{(5)}(t) + a_2 x^{(4)}(t)$$
$$+e(x(t-r), x'(t-r), x''(t-r), x'''(t-r), x^{(4)}(t-r), x^{(5)}(t-r))x'''(t) + f(x'(t))x''(t)$$
$$+g(x(t-r), x'(t-r), x''(t-r), x'''(t-r), x^{(4)}(t-r), x^{(5)}(t-r))x'(t) + h(x(t-r)) = 0.$$

In this paper, instead of Eq. (1), we consider the sixth order nonlinear delay differential equation

$$x^{(6)}(t) + a_1 x^{(5)}(t) + f_2(x(t-r), x'(t-r), x''(t-r), x'''(t-r), x^{(4)}(t-r), x^{(5)}(t-r))x^{(4)}(t)$$
$$+f_3(x''(t))x'''(t) + f_4(x(t-r), x'(t-r), x''(t-r), x'''(t-r), x^{(4)}(t-r), x^{(5)}(t-r))x''(t)$$
$$+f_5(x'(t-r)) + f_6(x(t-r)) = 0. \tag{2}$$

We write Eq. (2) in system form as follows

$$x_1' = x_2, x_2' = x_3, x_3' = x_4, x_4' = x_5, x_5' = x_6,$$
$$x_6' = -a_1 x_6 - f_2(x_1(t-r), x_2(t-r), x_3(t-r), x_4(t-r), x_5(t-r), x_6(t-r))x_5$$
$$-f_3(x_3)x_4 - f_4(x_1(t-r), x_2(t-r), x_3(t-r), x_4(t-r), x_5(t-r), x_6(t-r))x_3$$
$$-f_5(x_2) + \int_{t-r}^{t} f_5'(x_2(s))x_3(s)ds - f_6(x_1) + \int_{t-r}^{t} f_6'(x_1(s))x_2(s)ds, \tag{3}$$

which was obtained as usual by setting $x = x_1$, $x' = x_2$, $x'' = x_3$, $x''' = x_4$, $x^{(4)} = x_5$ and $x^{(5)} = x_6$ from (2), where r is a positive constant, a_1 is a constant, the primes in Eq. (2) denote differentiation with respect to t, $t \in \Re_+$, $\Re_+ = [0, \infty)$; f_2, f_3, f_4, f_5 and f_6 are continuous functions on $\Re^6$, $\Re$, $\Re^6$, $\Re$ and $\Re$, respectively, with $f_5(0) = f_6(0) = 0$, and satisfy a Lipschitz condition with all their arguments. Hence, the existence and uniqueness of the solutions of Eq. (2) are guaranteed (see Èl'sgol'ts [1, pp.14, 15]). We assume in what follows that the functions f_5 and f_6 are differentiable, and $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, $x_5(t)$ and $x_6(t)$ are abbreviated as x_1, x_2, x_3, x_4, x_5 and x_6, respectively.

Besides, for some works achieved on the instability of solutions of various sixth order nonlinear differential equations without delay, the reader can refer to the papers of Tunç [5-9] and E. Tunç and C. Tunç [11]. It should be noted that the basic reason to investigate this topics here is that functional differential equations play a key role in applied sciences. However, we only study the theoretical aspects of the topic here. Our purpose is to get through the result established in[2] to nonlinear delay differential equation (2) for the instability of its zero solution. Finally, we did not find any work on the instability of the solutions of sixth order linear and nonlinear delay differential equations in the literature except that of Tunç [10]. This paper is the second attempt on the same topic in the literature.

Let $r \geq 0$ be given, and let $C = C([-r, 0],\ \Re^n)$ with

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|, \quad \phi \in C.$$

For $H > 0$, define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r,\ A] \to \Re^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t+s), -r \leq s \leq 0.$$

Let G be an open subset of C and consider the general autonomous differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F : G \to \Re^n$ is a continuous function that maps closed and bounded sets into bounded sets. It follows from the conditions on F that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 1.1. *The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.*

2. Main Results

We prove here the following theorem.

Theorem 2.1. *In addition to the all the basic assumptions imposed on the functions f_2, f_3, f_4, f_5 and f_6 that appearing in Eq. (2), we assume that there exist positive constants a_5, $\bar{a}_5$, a_6, $\bar{a}_6$ and δ such that the following conditions hold:*

$$f_6(0) = 0, f_6(x_1) \neq 0,\ (x_1 \neq 0),$$

$$f_5(0) = 0, f_5(x_2) \neq 0,\ (x_2 \neq 0),$$

$$-a_6 \leq f_6'(x_1) \leq -\bar{a}_6 < 0 \text{ for all } x_1,$$

$$-a_5 \leq f_5'(x_2) \leq -\bar{a}_5 < 0 \text{ for all } x_2,$$

$$f_4(x_1(t-r), ..., x_6(t-r)) - \frac{1}{4} f_2^2(x_1(t-r), ..., x_6(t-r)) \geq \delta > 0$$

for all $x_1(t-r), ..., x_6(t-r)$.
Then, the zero solution of Eq. (2) is unstable for all arbitrary a_1 and f_3 .

Remark 2.2. *Note that the proof of the above theorem is based on the instability criteria of Krasovskii [3]. According to these criteria, it is necessary to show here that there exists a Lyapunov functional* $V \equiv V(x_{1t}, x_{2t}, ..., x_{6t})$ *which has Krasovskii properties, say* (P_1), (P_2) *and* (P_3) *:*

(P_1) In every neighborhood of $(0,0,0,0,0,0)$, *there exists a point* $(\xi_1, \xi_2, ..., \xi_6)$ *such that* $V(\xi_1, \xi_2, ..., \xi_6) > 0$,

(P_2) the time derivative $\dot{V} \equiv \frac{d}{dt}V(x_{1t}, x_{2t}, ..., x_{6t})$ *along solution paths of (3) is positive semi-definite,*

(P_3) the only solution $(x_1, x_2, ..., x_6) = (x_1(t), x_2(t), ..., x_6(t))$ *of (3) which satisfies* $\frac{d}{dt}V(x_{1t}, x_{2t}, ..., x_{6t}) = 0$, $(t \geq 0)$, *is the trivial solution* $(0,0,0,0,0,0)$.

PROOF. We define a Lyapunov functional $V = V(x_{1t}, x_{2t}, x_{3t}, x_{4t}, x_{5t}, x_{6t})$ given by

$$\begin{aligned} V \;=\; & -x_3x_6 - a_1x_3x_5 + x_4x_5 + \tfrac{1}{2}a_1x_4^2 - \int_0^{x_3} f_3(s)s\,ds - \int_0^{x_2} f_5(u)du - f_6(x_1)x_2 \\ & -\lambda_1 \int_{-r}^{0}\int_{t+s}^{t} x_2^2(\theta)d\theta ds - \lambda_2 \int_{-r}^{0}\int_{t+s}^{t} x_3^2(\theta)d\theta ds, \end{aligned} \tag{4}$$

where s is a real variable such that the integral $\int_{-r}^{0}\int_{t+s}^{t} x_2^2(\theta)d\theta ds$ and $\int_{-r}^{0}\int_{t+s}^{t} x_3^2(\theta)d\theta ds$ are non-negative, and λ_1 and λ_2 are some positive constants which will be determined later in the proof.

Hence, it is clear from the definition of V that

$$V(0,0,0,0,0,0) = 0$$

and

$$V(0,0,0,\varepsilon^2,\varepsilon,0) = \varepsilon^3 + \frac{1}{2}a_1\varepsilon^4 > 0$$

for all sufficiently arbitrary small ε so that every neighborhood of the origin in the $(x_1, x_2, x_3, x_4, x_5, x_6)-$ space contains points $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ such that $V(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) > 0$.

Let

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$$

be an arbitrary solution of (3).

By an elementary differentiation, time derivative of the Lyapunov functional V in (4) along the solutions of (3)yields

$$\begin{aligned}
\dot{V} &\equiv \tfrac{d}{dt}V(x_{1t},x_{2t},x_{3t},x_{4t},x_{5t},x_{6t}) \\
&= x_5^2 + f_2(x_1(t-r),...,x_6(t-r))x_3x_5 + f_4(x_1(t-r),...,x_6(t-r))x_3^2 - f_6'(x_1)x_2^2 \\
&-x_3\int\limits_{t-r}^{t} f_5'(x_2(s))x_3(s)ds - x_3\int\limits_{t-r}^{t} f_6'(x_1(s))x_2(s)ds \\
&-\lambda_1 r x_2^2 + \lambda_1\int\limits_{t-r}^{t} x_2^2(s)ds - \lambda_2 r x_3^2 + \lambda_2\int\limits_{t-r}^{t} x_3^2(s)ds \\
&= \left(x_5 + \tfrac{1}{2}f_2(x_1(t-r),...,x_6(t-r))x_3\right)^2 \\
&+\{f_4(x_1(t-r),...,x_6(t-r)) - \tfrac{1}{4}f_2^2(x_1(t-r),...,x_6(t-r))\}x_3^2 \\
&-f_6'(x_1)x_2^2 - x_3\int\limits_{t-r}^{t} f_5'(x_2(s))x_3(s)ds - x_3\int\limits_{t-r}^{t} f_6'(x_1(s))x_2(s)ds \\
&-\lambda_1 r x_2^2 + \lambda_1\int\limits_{t-r}^{t} x_2^2(s)ds - \lambda_2 r x_3^2 + \lambda_2\int\limits_{t-r}^{t} x_3^2(s)ds.
\end{aligned}$$

Making use of the assumptions $-a_5 \le f_5'(x_2) \le -\bar{a}_5 < 0$, $-a_6 \le f_6'(x_1) \le -\bar{a}_6 < 0$ and the estimate $2\,|mn| \le m^2 + n^2$, we get

$$\begin{aligned}
-x_3\int\limits_{t-r}^{t} f_6'(x_1(s))x_2(s)ds &\ge |x_3| \int\limits_{t-r}^{t} f_6'(x_1(s))\ |x_2(s)|\,ds \\
&\ge -\tfrac{1}{2}a_6 r x_3^2 - \tfrac{1}{2}a_6\int\limits_{t-r}^{t} x_2^2(s)ds
\end{aligned}$$

and

$$\begin{aligned}
-x_3\int\limits_{t-r}^{t} f_5'(x_2(s))x_3(s)ds &\ge |x_3| \int\limits_{t-r}^{t} f_5'(x_2(s))\ |x_3(s)|\,ds \\
&\ge -\tfrac{1}{2}a_5 r x_3^2 - \tfrac{1}{2}a_5\int\limits_{t-r}^{t} x_3^2(s)ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{V} &\ge (\bar{a}_6 - \lambda_1 r)x_2^2 + \{\delta - (\lambda_2 + \tfrac{1}{2}a_5 + \tfrac{1}{2}a_6)r\}x_3^2 \\
&+ \left(\lambda_1 - \tfrac{1}{2}a_6\right)\int\limits_{t-r}^{t} x_2^2(s)ds + \left(\lambda_2 - \tfrac{1}{2}a_5\right)\int\limits_{t-r}^{t} x_3^2(s)ds.
\end{aligned}$$

Let $\lambda_1 = \frac{1}{2}a_6$, $\lambda_2 = \frac{1}{2}a_5$ and $r < 2\min\{\frac{\bar{a}_6}{a_6}, \frac{\delta}{2a_5+a_6}\}$. Then, it follows that

$$\dot{V} \ge (\bar{a}_6 - \frac{1}{2}a_6 r)x_2^2 + \{\delta - (a_5 + \frac{1}{2}a_6)r\}x_3^2 > 0$$

On the other hand, it follows that $\frac{d}{dt}V(x_{1t}, x_{2t}, x_{3t}, x_{4t}, x_{5t}, x_{6t}) = 0$ if and only if $x_2 = 0$, which implies that

$$x_1 = \xi_1 \text{ (constant)}, x_2 = x_3 = x_4 = x_5 = x_6 = 0. \tag{5}$$

The substitution of the estimate (5) in system (3) leads $f_6(\xi_1) = 0$. In view of the assumption of the theorem, $f_6(0) = 0$, $f_6(x_1) \neq 0$, $(x_1 \neq 0)$, it is seen that $f_6(\xi_1) = 0$ if and only if $\xi_1 = 0$. Hence, we can easily conclude that the only solution $(x_1, ..., x_6) = (x_1(t), ..., x_6(t))$ of system (3) which satisfies $\frac{d}{dt}V(x_{1t}, x_{2t}, ..., x_{6t}) = 0, \quad (t \geq 0)$, is the trivial solution $(0, 0, 0, 0, 0, 0)$. Thus, the estimate $\frac{d}{dt}V(x_{1t}, x_{2t}, x_{3t}, x_{4t}, x_{5t}, x_{6t}) = 0$ implies $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$. Hence, we see that the functional V satisfies the property (P_3).

In view of the whole discussion made above, it follows that the functional V has all Krasovskii properties, (P_1), (P_2) and (P_3). Thus, one can conclude that the zero solution of Eq. (2) is unstable. The proof of the theorem is completed.

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