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ON SOME REFINEMENTS OF FEJÉR TYPE INEQUALITIES VIA SUPERQUADRATIC FUNCTIONS

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Abstract. In this paper some Fejér-type inequalities for superquadratic functions are established, we also get refinement of some known results when superquadratic function is positive and hence convex.

Key words: Hermite-Hadamard Inequality, convex function, Wright convex function, Fejér inequality, superquadratic function.

Abstrak. Pada paper ini dinyatakan beberapa ketaksamaan tipe Fejér untuk fungsi-fungsi superkuadratik, juga diperoleh perhalusan dari beberapa hasil yang telah diketahui untuk fungsi superkuadratik positif dan konveks.

Kata kunci: Ketaksamaan Hermite-Hadamard, fungsi konveks, fungsi konveks Wright, ketaksamaan Fejér, fungsi superkuadratik.

1. Introduction

For convex functions the following inequality has great significance in the field of inequalities: Let $f : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $a, b \in I$ with $a < b$, be a convex function then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

with the inequality reversed if f concave. The inequality (1.1) is known Hermite-Hadamard inequality.

The weighted generalization of (1.1) is the following inequality:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \leq \frac{1}{b-a}\int_a^b f(x)p(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)dx, \tag{1.2}$$

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where f as defined above and $p:[a,b]\to\mathbb{R}$ is non-negative integrable and symmetric about $x=\frac{a+b}{2}$. The inequality (1.2) is known in literature as Fejér's inequality. These inequalities have many extensions and generalizations, see [19]-[13] and [1]-[6].

Let us now define some mappings related to (1.2) and quote some Fejér-type inequalities from [3] and [11].

$$G(t)=\frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right],$$

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

$$H_p(t)=\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)p(x)dx,$$

$$L(t)=\frac{1}{2(b-a)}\int_a^b\left[f(ta+(1-t)x)+f(tb+(1-t)x)\right]dx$$

and

$$L_p(t)=\frac{1}{2}\int_a^b\left[f(ta+(1-t)x)+f(tb+(1-t)x)\right]p(x)dx,$$

where $f:[a,b]\to\mathbb{R}$ is a convex function and $p:[a,b]\to\mathbb{R}$ is non-negative integrable and symmetric about $x=\frac{a+b}{2}$.

Theorem 1.1. [3] *Let f , p, H_g be defined as above. Then H_p is convex, increasing on $[0,1]$ and for all $t\in[0,1]$, we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx=H_p(0)\le H_p(t)\le H_p(1)=\int_a^b f(x)\,p(x)\,dx. \tag{1.3}$$

Theorem 1.2. [11] *Let f, p, H_p be defined as above. Then:*

(1) *The following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx\le 2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,p\left(2x-\frac{a+b}{2}\right)dx\le\int_0^1 H_p(t)\,dt$$
$$\le\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx+\int_a^b f(x)\,p(x)\,dx\right]. \tag{1.4}$$

(2) *If f is differentiable on $[a,b]$ and p is bounded on $[a,b]$, then for all $t\in[0,1]$ we have the inequalities*

$$0 \le \int_a^b f(x)\,p(x)\,dx - H_p(t)$$
$$\le (1-t)\left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx\right]\|p\|_\infty, \quad (1.5)$$

where $\|p\|_\infty = \sup_{x\in[a,b]} |p(x)|$.

(3) *If f is differentiable on $[a,b]$, then for all $t\in[0,1]$ we have the inequalities*

$$0 \le \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx - H_p(t) \le \frac{\left(f'(a)-f'(b)\right)(b-a)}{4}\int_a^b p(x)\,dx. \quad (1.6)$$

Theorem 1.3. [11] *Let f, p, H_p, G be defined as above. Then:*

(1) *The following inequality holds for all $t\in[0,1]$:*

$$H_p(t) \le G(t)\int_a^b p(x)\,dx. \quad (1.7)$$

(2) *The following inequalities hold:*

$$2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,p\left(2x-\frac{a+b}{2}\right)dx \le \frac{1}{2}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right]\int_a^b p(x)\,dx$$
$$\le (b-a)\int_0^1 G(t)g((1-t)a+tb)dt \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b p(x)\,dx. \quad (1.8)$$

(3) *If f is differentiable on $[a,b]$ and p is bounded on $[a,b]$, then for all $t\in[0,1]$ we have the inequalities:*

$$0 \le H_p(t) - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \le (b-a)\,|H(t)-G(t)|\,\|p\|_\infty, \quad (1.9)$$

where $\|p\|_\infty = \sup_{x\in[a,b]} |p(x)|$.

Theorem 1.4. [11] *Let f, p, H_p, G, L_p be defined as above. Then:*

(1) *L_p is convex on $[0,1]$.*

(2) *We have the inequalities:*

$$G(t)\int_a^b p(x)\,dx \le L_p(t) \le (1-t)\int_a^b f(x)\,p(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx$$
$$\le \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx, \quad (1.10)$$

for all $t\in[0,1]$ *and*

$$\sup_{t\in[0,1]} L_p(t) = L_p(1) = \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx. \quad (1.11)$$

(3) *For all* $t\in[0,1]$, *we have the inequalities:*

$$H_p(1-t) \le L_p(t) \quad (1.12)$$

and

$$\frac{H_p(t)+H_p(1-t)}{2} \le L_p(t). \quad (1.13)$$

They used the following Lemma to prove the above results:

Lemma 1.5. [11, p.3] *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *be convex function and let* $a\le A\le C\le D\le B\le b$ *with* $A+B=C+D$. *Then*

$$f(A)+f(B)\le f(C)+f(D).$$

For the mappings

$$H_p(t) = \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)p(x)dx$$

and

$$Q(t) = \frac{1}{2}\int_a^b \left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)p\left(\frac{x+a}{2}\right)+f\left(\frac{1+t}{2}b+\frac{1-t}{2}b\right)p\left(\frac{x+b}{2}\right)\right]dx,$$

the following results hold for Wright convex functions see [13]. These result also hold for convex functions see [3, Remark 6] or [19].

Theorem 1.6. [13, Theorem 2.5] *Let* $f:[a,b)\longrightarrow\mathbb{R}$ *be a Wright-convex function and let* $p:[a,b)\to\mathbb{R}$ *be a non-negative integrable and symmetric about* $x=\frac{a+b}{2}$, *then* H *is Wright-convex, increasing on* $[0,1]$ *and*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx = H_p(0)\le H_p(t)\le H_p(1) = \int_a^b f(x)\,p(x)\,dx. \quad (1.14)$$

Theorem 1.7. [13, Theorem 2.7] *Let $f:[a,b)\longrightarrow\mathbb{R}$ be a Wright-convex function and let $p:[a,b)\to\mathbb{R}$ be a non-negative integrable and symmetric about $x=\frac{a+b}{2}$, then Q is Wright-convex, increasing on $[0,1]$ and*

$$\int_a^b f(x)\,p(x)\,dx=Q(0)\le Q(t)\le Q(1)=\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx. \tag{1.15}$$

In [13], the same Lemma 1 was used which also holds for Wright-convex functions to prove the above results.

Let us now state the definition, some of the properties and results related to superquadratic functions to be used in the sequel.

Definition 1.8. [16, Defintion 2.1] *Let $I=[0,a]$ or $[0,\infty)$ be an interval in $\mathbb{R}$. A function $f:I\longrightarrow\mathbb{R}$ is superquadratic if for each x in I there exists a real number $C(x)$ such that*

$$f(y)-f(x)\ge C(x)(y-x)+f(|y-x|) \tag{1.16}$$

for all $y\in I$. If $-f$ is superquadratic then f is called subquadratic.

For examples of superquadratic functions see [15, p. 1049].

Theorem 1.9. [16, Theorem 2.3] *The inequality*

$$f\left(\int g d\mu\right)\le\int\left(f(g(s)-f\left(\left|g(s)-\int g d\mu\right|\right)\right)d\mu(s), \tag{1.17}$$

holds for all probability measure μ and all non-negative μ-integrable function g, if and only if f is superquadratic.

The following is the discrete version of the above theorem which will be helpful in the sequel of the paper:

Lemma 1.10. [15, Lemma A, p.1049] *Suppose that f is superquadratic. Let $x_r\ge 0$, $1\le r\le n$, and let $\bar{x}=\sum_{r=1}^n\lambda_r x_r$ where $\lambda_r\ge 0$ and $\sum_{r=1}^n\lambda_r=1$. Then*

$$\sum_{r=1}^n\lambda_r f(x_r)\ge f(\bar{x})+\sum_{r=1}^n\lambda_r f(|x_r-\bar{x}|). \tag{1.18}$$

The following Lemma shows that positive superquadratic functions are also convex:

Lemma 1.11. [16, Lemma 2.2] *Let f be superquadratic function with $C(x)$ as in Definition 1. Then*

(1) $f(0)\le 0$.
(2) *If $f(0)=f^{'}(0)=0$ then $C(x)=f^{'}(x)$ whenever f is differentiable at $x>0$.*
(3) *If $f\ge 0$, then f convex and $f(0)=f^{'}(0)=0$.*

In [17] a converse of Jensen's inequality for superquadratic functions was proved:

Theorem 1.12. [17, Theorem 1] *Let* (Ω, A, μ) *be a measurable space with* $0 < \mu(\Omega) < \infty$ *and let* $f : [0, \infty) \to \mathbb{R}$ *be a superquadratic function. If* $g : \Omega \to [m, M] \subseteq [0, \infty)$ *is such that* $g,\ f \circ g \in L_1(\mu)$*, then we have for* $\bar{g} = \frac{1}{\mu(\Omega)} \int_\Omega g d\mu$*,*

$$\frac{1}{\mu(\Omega)} \int f(g) d\mu \leq \frac{M - \bar{g}}{M - m} f(m) + \frac{\bar{g} - m}{M - m} f(M) - \frac{1}{\mu(\Omega)} \frac{1}{M - m} \int_\Omega \left((M - g) f(g - m) + ((g - m) f(M - g)\right) d\mu. \tag{1.19}$$

The discrete version of this theorem is:

Theorem 1.13. [17, Theorem 2] *Let* $f : [0, \infty) \to \mathbb{R}$ *be a superquadratic function. Let* $(x_1, ..., x_n)$ *be an* n*-tuple in* $[m, M]^n$ $(0 \leq m \leq M < \infty)$*, and* $(p_1, ..., p_n)$ *be a non-negative* n*-tuple such that* $P_n = \sum_{i=1}^n p_i > 0$*. Denote* $\bar{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$*, then*

$$\frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) \leq \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M) - \frac{1}{P_n (M - m)} \sum_{i=1}^n p_i \left[(M - x_i) f(x_i - m) + (x_i - m) f(M - x_i)\right]. \tag{1.20}$$

Together with Theorems 7 and 8 and for $g(x) = x$ with measure μ defined on Ω by $\frac{1}{b-a} dt$, the following theorem was also proved in [17]:

Theorem 1.14. [17, Theorem 8] *Let* $f : [0, \infty) \to \mathbb{R}$ *be superquadratic function and let* $0 \leq a < b$*, then*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{b-a} \int_a^b f\left(\left|x - \frac{a+b}{2}\right|\right) dx \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} - \frac{1}{(b-a)^2} \int_a^b \left[(b-a) f(x-a) + (x-a) f(b-x)\right] dx. \tag{1.21}$$

The above theorem represents a refinement of (1.1) when superquadratic function f is positive and hence convex. The following inequality compares $S(t)$ of Theorem 10 with $S(0)$ and $S(1)$:

Theorem 1.15. [18, Theorem 4.2] *Let* $f : [0, \infty) \to \mathbb{R}$ *be superquadratic function and let* $g : [a, b] \to [0, \infty)$ *and* $p : [a, b] \to [0, \infty)$ *be integrable functions. Let*

$$S(t) = \frac{1}{P} \int_a^b p(x) f(t g(x) + (1 - t) \bar{g}) dx$$

where $P=\int_a^b p(x)\,dx$ *and* $\bar{g}=\frac{1}{P}\int_a^b p(x)\,g(x)dx$. *Then for* $0\le t\le 1$,

$$S(0)+\frac{1}{P}\int_a^b p(x)\,f(t\,|g(x)+\bar{g}|)\,dx\le S(t)$$
$$\le S(1)-(1-t)\frac{1}{P}\int_a^b p(x)\,f(|g(x)+\bar{g}|)\,dx$$
$$-\frac{1}{P}\int_a^b p(x)\,f((1-t)\,|g(x)+\bar{g}|)\,dx-\frac{1-t}{P}\int_a^b p(x)\,f(t\,|g(x)+\bar{g}|)\,dx. \quad (1.22)$$

By using Lemma 2 and Theorem 9 for $n=2$, S. Abramovich, J. Barić, J. Pečarić established the following results for superquadratic functions in [15, Theorem 1, p. 1051], and those results refine the results in Theorem 5 and Theorem 6 when superquadratic function is positive and hence convex.

Theorem 1.16. [15, Theorem 1, p. 1051] *Let* f *be superquadratic integrable function on* $[0,b]$ *and* $p(x)$ *be non-negative integrable and symmetric about* $x=\frac{a+b}{2}$, $0\le a<b$. *Let*

$$H_p(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\,\frac{a+b}{2}\right)p(x)dx.$$

Then for $0\le s\le t\le 1$, $t>0$,

$$H_p(s)\le H_p(t)-\int_a^b \frac{t+s}{2t}f\left((t-s)\left(\left|\frac{a+b}{2}-x\right|\right)\right)p(x)dx$$
$$-\int_a^b \frac{t-s}{2t}f\left((t+s)\left(\left|\frac{a+b}{2}-x\right|\right)\right)p(x)dx. \quad (1.23)$$

As a consequence it was shown in [15, p. 1052] that for superquadratic function f

$$\frac{f((1-t)A))+f((1+t)\,A)}{2}-f(A)-f(tA)\ge 0, A\ge 0, 0\le t\le 1. \quad (1.24)$$

gives sharp result than the inequality in Theorem 11 for $g(x)=x$.

Theorem 1.17. [15, Theorem 2, p. 1053] *Let* f *be defined as in Theorem 12. Let* $Q(t)$ *be*

$$Q(t)=\frac{1}{2}\int_a^b\left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)p\left(\frac{x+a}{2}\right)+f\left(\frac{1+t}{2}b+\frac{1-t}{2}b\right)p\left(\frac{x+b}{2}\right)\right]dx.$$

Then for $0 \le s \le t \le 1$, *we get that*

$$
\begin{aligned}
Q(s)-Q(t) \le & -\frac{1}{2}\int_a^b \left[\frac{(b-x)+\frac{t+s}{2}(x-a)}{b-x+t(x-a)} f\left(\frac{t-s}{2}(x-a)\right)\right. \\
& \left. + \frac{\frac{t-s}{2}(x-a)}{b-x+t(x-a)} f\left((b-x)+\frac{t+s}{2}(x-a)\right)\right] p\left(\frac{x+a}{2}\right) dx \\
& -\frac{1}{2}\int_a^b \left[\frac{(x-a)+\frac{t+s}{2}(b-x)}{x-a+t(b-x)} f\left(\frac{t-s}{2}(b-x)\right)\right. \\
& \left. + \frac{\frac{t-s}{2}(b-x)}{x-a+t(b-x)} f\left((x-a)+\frac{t+s}{2}(b-x)\right)\right] p\left(\frac{x+b}{2}\right) dx \\
= & -\int_a^b \frac{\left(1-\frac{t+s}{2}\right)|2x-a-b|+\frac{t+s}{2}(b-a)}{(1-t)|2x-a-b|+t(b-a)} f\left(\frac{t-s}{2}(b-a-|a+b-2x|)\right) p(x)\, dx \\
& -\int_a^b \frac{\frac{t-s}{2}(b-a-|a+b-2x|)}{(1-t)|2x-a-b|+t(b-a)} f\left(\left(1-\frac{t+s}{2}\right)|2x-a-b|\right. \\
& \left. + \frac{t+s}{2}(b-a)\right) p(x)\, dx. \quad (1.25)
\end{aligned}
$$

In this paper we deal with mappings $G(t)$, $H(t)$, $L_p(t)$ and $H_p(t)$ when f is superquadratic. In case when superquadratic function f is positive and therefore convex we get refinements of some parts of Theorem 2-Theorem 4.

2. **Main Results**

In this section we prove our main results by using the same techniques as used in [11] and [15]. Moreover, we assume that all the considered integrals in this section exist.

In order to prove our main results we go through some calculations as follows:

From Lemma 2 and Theorem 9 for $n = 2$, we get that

$$
f(z) \le \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \frac{M-z}{M-m} f(z-m) - \frac{z-m}{M-m} f(M-z) \quad (2.1)
$$

and

$$
f(M+m-z) \le \frac{z-m}{M-m} f(m) + \frac{M-z}{M-m} f(M) - \frac{z-m}{M-m} f(M-z) - \frac{M-z}{M-m} f(z-m) \quad (2.2)
$$

hold for superquadratic function f, $0 \le m \le z \le M$, $m < M$.

Therefore from (2.1) and (2.2), we have

$$f(z)+f(M+m-z) \le f(m)+f(M)-2\frac{z-m}{M-m}f(M-z)-2\frac{M-z}{M-m}f(z-m). \quad (2.3)$$

Now for $0 \le t \le \frac{1}{2}$ and $0 \le a \le x \le \frac{a+b}{2}$, we obtain from (2.3) the following inequalities:

By setting $z = \frac{a+b}{2}, M = \frac{3(a+b)}{4} - \frac{x}{2}, m = \frac{x}{2} + \frac{a+b}{4}$ in (2.3) we get that

$$2f\left(\frac{a+b}{2}\right) \le f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)-2f\left(\frac{1}{2}\left(\frac{a+b}{2}-x\right)\right) \quad (2.4)$$

holds.

Also, by replacing $z = \frac{x}{2} + \frac{a+b}{4}, M = tx + (1-t)\frac{a+b}{2}, m = t\left(\frac{a+b}{2}\right) + (1-t)x$ in (2.3), we observe that

$$\begin{aligned} &2f\left(\frac{x}{2}+\frac{a+b}{4}\right) \\ &\le f\left(t\left(\frac{a+b}{2}\right)+(1-t)x\right)+f\left(tx+(1-t)\frac{a+b}{2}\right)-2f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right) \end{aligned} \quad (2.5)$$

holds.

Further, for $z = \frac{3(a+b)}{4} - \frac{x}{2}, M = t\left(\frac{a+b}{2}\right) + (1-t)(a+b-x), m = t(a+b-x) + (1-t)\frac{a+b}{2}$, we get from (2.3) that

$$\begin{aligned} 2f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right) &\le f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right) \\ &+ f\left(t\left(\frac{a+b}{2}\right)+(1-t)(a+b-x)\right)-2f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right) \end{aligned} \quad (2.6)$$

holds.

Again, by replacing $z = t\left(\frac{a+b}{2}\right) + (1-t)x, M = \frac{a+b}{2}, m = x$ in (2.3), we have that

$$\begin{aligned} &f\left(t\left(\frac{a+b}{2}\right)+(1-t)x\right)+f\left(tx+(1-t)\frac{a+b}{2}\right) \\ &\le f(x)+f\left(\frac{a+b}{2}\right)-2tf\left((1-t)\left(\frac{a+b}{2}-x\right)\right)-2(1-t)f\left(t\left(\frac{a+b}{2}-x\right)\right) \end{aligned} \quad (2.7)$$

holds.

Finally, for $z = t(a+b-x) + (1-t)\frac{a+b}{2}, M = a+b-x, m = \frac{a+b}{2}$, we get from (2.3) that

$$f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)+f\left(t\left(\frac{a+b}{2}\right)+(1-t)(a+b-x)\right) \\ \leq f\left(\frac{a+b}{2}\right)+f(a+b-x)-2tf\left((1-t)\left(\frac{a+b}{2}-x\right)\right) \\ -2(1-t)f\left(t\left(\frac{a+b}{2}-x\right)\right) \quad (2.8)$$

holds.

Now we are ready to state and prove our first result based on the calculations given above.

Theorem 2.1. *Let f be superquadratic function on $[0,b]$ and $p(x)$ be non-negative and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Then we have the following inequalities:*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leq 2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,p\left(2x-\frac{a+b}{2}\right)dx - \int_a^b f\left(\frac{1}{2}\left|\frac{a+b}{2}-x\right|\right)p(x)\,dx, \quad (2.9)$$

$$2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,p\left(2x-\frac{a+b}{2}\right)dx \\ \leq \int_0^1 H_p(t)dt - \int_0^1\int_a^b f\left(\left|\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right|\right)p(x)\,dxdt \quad (2.10)$$

and

$$\int_0^1 H_p(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx + \int_a^b f(x)\,p(x)\,dx\right] \\ -\int_0^1\int_a^b t\left(f(1-t)\left|\frac{a+b}{2}-x\right|\right)p(x)\,dxdt - \int_0^1\int_a^b (1-t)\,f\left(t\left|\frac{a+b}{2}-x\right|\right)p(x)\,dxdt, \quad (2.11)$$

where

$$H_p(t) = \int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)p(x)dx,\ t\in[0,1].$$

PROOF. Since

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx = \int_a^{\frac{a+b}{2}} 2f\left(\frac{a+b}{2}\right)p(x)\,dx.$$

Therefore from (2.4), we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\int_a^b p(x)\,dx \\ &\leq \left[\int_a^{\frac{a+b}{2}} f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right]p(x)\,dx \\ &\qquad -2\int_a^{\frac{a+b}{2}} f\left(\frac{1}{2}\left(\frac{a+b}{2}-x\right)\right)p(x)\,dx. \end{aligned} \tag{2.12}$$

By the change of variable $x \to a+b-x$ together with the symmetry of $p(x)$, we get that

$$\int_a^{\frac{a+b}{2}} f\left(\frac{1}{2}\left(\frac{a+b}{2}-x\right)\right)p(x)\,dx = \int_{\frac{a+b}{2}}^b f\left(\frac{1}{2}\left(x-\frac{a+b}{2}\right)\right)p(x)\,dx. \tag{2.13}$$

By simple techniques of integration, we have that

$$\int_a^{\frac{a+b}{2}} \left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right]p(x)\,dx = 2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,p\left(2x-\frac{a+b}{2}\right)dx. \tag{2.14}$$

Therefore from (2.12), (2.13) and (2.14), we get (2.9). By simple techniques of integration, we have the following identity:

$$2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,p\left(2x-\frac{a+b}{2}\right)dx = \int_a^{\frac{a+b}{2}} \left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right]p(x)\,dx.$$

From (2.5), (2.6), integrating both sides over t on $\left[0,\frac{1}{2}\right]$, we get that

$$2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,p\left(2x-\frac{a+b}{2}\right)dx \le \int_{a}^{\frac{a+b}{2}}\int_{0}^{\frac{1}{2}}\left[f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,x\right)\right.$$
$$+f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)$$
$$+f\left(t\left(\frac{a+b}{2}\right)+(1-t)(a+b-x)\right)$$
$$\left.-4f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right)\right]p(x)\,dtdx.$$

By the change of variables $x \to a+b-x$ and $t \to 1-t$ and the symmetry of $p(x)$, we obtain (2.10). By simple techniques of integration, we have the following identity:

$$\int_0^1 H_p(t)dt = \int_0^{\frac{1}{2}}\int_a^{\frac{a+b}{2}}\left[f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,x\right)\right.$$
$$+f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)$$
$$\left.+\,f\left(t\left(\frac{a+b}{2}\right)+(1-t)(a+b-x)\right)\right]p(x)\,dxdt.$$

From (2.7) and (2.8) and by the change of variables $x \to a+b-x$ and $t \to 1-t$ and the symmetry of $p(x)$, we get that

$$\int_0^1 H_p(t)dt \le \int_0^{\frac{1}{2}}\int_a^{\frac{a+b}{2}}\left[f(x)+2f\left(\frac{a+b}{2}\right)+f(a+b-x)\right.$$
$$\left.-4tf\left((1-t)\left(\frac{a+b}{2}-x\right)\right)-4(1-t)\,f\left(t\left(\frac{a+b}{2}-x\right)\right)\right]p(x)\,dxdt$$
$$=\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx+\int_a^b f(x)\,p(x)\,dx\right]$$
$$-\int_0^1\int_a^b tf\left((1-t)\left|\frac{a+b}{2}-x\right|\right)p(x)\,dxdt-\int_0^1\int_a^b (1-t)\,f\left(t\left|\frac{a+b}{2}-x\right|\right)p(x)\,dxdt.$$

Hence the inequality (2.11) is also proved. This completes the proof of the theorem as well.

Remark 2.2. *If the superquadratic function f is positive and hence convex, then (2.9) represents a refinement of the first inequality in (1.4) of Theorem 2; (2.10)* *represents a refinement of the middle inequality in (1.4) of Theorem 2 and (2.11) represents a refinement of the last inequality in (1.4) of Theorem 2.*

Corollary 2.3. *Let f be superquadratic function on $[0,b]$. If $p(x)=\frac{1}{b-a}$, $x\in[a,b]$, $0\le a<b$, then*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx-\frac{1}{b-a}\int_a^b f\left(\frac{1}{2}\left|\frac{a+b}{2}-x\right|\right)dx\le\int_0^1 H(t)dt\\
&-\frac{1}{b-a}\int_0^1\int_a^b f\left(\left|\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right|\right)dxdt-\frac{1}{b-a}\int_a^b f\left(\frac{1}{2}\left|\frac{a+b}{2}-x\right|\right)dx\\
&\le\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)dx+\frac{1}{b-a}\int_a^b f(x)\,dx\right]-\frac{1}{b-a}\int_0^1\int_a^b tf\left((1-t)\left|\frac{a+b}{2}-x\right|\right)dxdt\\
&-\frac{1}{b-a}\int_0^1\int_a^b (1-t)\,f\left(t\left|\frac{a+b}{2}-x\right|\right)dxdt\\
&-\frac{1}{b-a}\int_0^1\int_a^b f\left(\left|\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right|\right)dxdt-\frac{1}{b-a}\int_a^b f\left(\frac{1}{2}\left|\frac{a+b}{2}-x\right|\right)dx,
\end{aligned}
\tag{2.15}
$$

where

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\,\frac{a+b}{2}\right)dx,\ t\in[0,1]\,.$$

.

Proof. It follows directly from the above theorem, since for $p(x)=\frac{1}{b-a}$, $x\in[a,b]$, $H_p(t)=H(t)$.

Remark 2.4. *If the superquadratic function f is positive and therefore convex, then form Corollary 1 represents a refinement of the inequality (1.3) in* [11, Theorem B, p. 2].

To proceed to our next results we again go through some similar calculations as given before Theorem 14.

For $0\le a\le x\le\frac{a+b}{2}$, we have that the following inequalities:

By setting $z = tx + (1-t)\frac{a+b}{2}$, $M = tb + (1-t)\frac{a+b}{2}$, $m = ta + (1-t)\frac{a+b}{2}$ in (2.3), we get that

$$f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \leq f\left(tb + (1-t)\frac{a+b}{2}\right) + f\left(ta + (1-t)\frac{a+b}{2}\right) - 2t\frac{x-a}{b-a}f(b-x) - 2t\frac{b-x}{b-a}f(x-a) \quad (2.16)$$

holds.

Similarly, by replacing $z = \frac{x}{2} + \frac{a+b}{4}$, $M = \frac{a+3b}{4}$, $m = \frac{3a+b}{4}$ in (2.3), we observe that

$$f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \leq f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) - 2\frac{x-a}{b-a}f(b-x) - 2\frac{b-x}{b-a}f(x-a) \quad (2.17)$$

holds.

Also, for $z = \frac{3a+b}{4}$, $M = \frac{2a+b-x}{2}$, $m = \frac{x+a}{2}$, we get from (2.3) that

$$2f\left(\frac{3a+b}{4}\right) \leq f\left(\frac{2a+b-x}{2}\right) + f\left(\frac{x+a}{2}\right) - 2f\left(\frac{a+b}{4} - \frac{x}{2}\right) \quad (2.18)$$

holds.

Again, for $z = \frac{a+3b}{2}$, $M = \frac{a+2b-x}{2}$, $m = \frac{x+b}{2}$, we get from (2.3) that

$$2f\left(\frac{a+3b}{4}\right) \leq f\left(\frac{a+2b-x}{2}\right) + f\left(\frac{x+b}{2}\right) - 2f\left(\frac{a+b}{4} - \frac{x}{2}\right) \quad (2.19)$$

holds.

Further, for $z = \frac{x+a}{2}$, $M = \frac{a+b}{2}$, $m = a$, we get from (2.3) that

$$f\left(\frac{x+a}{2}\right) + f\left(\frac{2a+b-x}{2}\right) \leq f(a) + f\left(\frac{a+b}{2}\right) - 2\frac{x-a}{b-a}f(b-x) - 2\frac{b-x}{b-a}f(x-a) \quad (2.20)$$

holds.

Finally, by replacing $z = \frac{x+b}{2}$, $m = \frac{a+b}{2}$, $M = b$, in (2.3) we observe that

$$f\left(\frac{x+b}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \leq f(b) + f\left(\frac{a+b}{2}\right) - 2\frac{x-a}{b-a}f(b-x) - 2\frac{b-x}{b-a}f(x-a) \quad (2.21)$$

holds.

Now we are ready to state and prove our next results based on the calculations done above.

Theorem 2.5. *Let f be superquadratic function on $[0,b]$ and p be non-negative symmetric about $x=\frac{a+b}{2}$, $0\le a<b$. Let H_p, G be defined as above. Then the following inequality holds for all $t\in[0,1]$:*

$$H_p(t)\le G(t)\int_a^b p(x)\,dx-\frac{t}{b-a}\int_a^b\left[(x-a)\,f(b-x)+(b-x)\,f(x-a)\right]p(x)dx. \tag{2.22}$$

The following inequalities hold:

$$\begin{aligned}2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} & f(x)\,p\left(2x-\frac{a+b}{2}\right)dx\\ &\le\frac{1}{2}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right]\int_a^b p(x)\,dx\\ &-\frac{1}{b-a}\int_a^b\left[(b-x)\,f(x-a)+(x-a)\,f(b-x)\right]p(x)dx,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\frac{1}{2}&\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right]\int_a^b p(x)\,dx\\ &\le(b-a)\int_0^1 G(t)p((1-t)a+tb)dt-\int_a^b f\left(\left|\frac{a+b}{4}-\frac{x}{2}\right|\right)p(x)dx\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}(b-a)&\int_0^1 G(t)p((1-t)a+tb)dt\\ &\le\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b p(x)\,dx\\ &-\frac{1}{b-a}\int_a^b\left[(x-a)\,f(b-x)+(b-x)\,f(x-a)\right]p(x)dx.\end{aligned} \tag{2.25}$$

Proof. By using (2.16), symmetry of p and the change of variable $x \to a+b-x$, we get that

$$\begin{aligned} H_p(t) &= \int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2} \right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2} \right) \right] p(x)dx \\ &\le \int_a^{\frac{a+b}{2}} \left[f\left(ta + (1-t)\frac{a+b}{2} \right) + f\left(tb + (1-t)\frac{a+b}{2} \right) \right] p(x)dx \\ &\quad - \int_a^{\frac{a+b}{2}} \left[2t\frac{x-a}{b-a} f(b-x) + 2t\frac{b-x}{b-a} f(x-a) \right] p(x)dx \\ &= G(t) \int_a^b p(x)\,dx - \frac{t}{b-a} \int_a^b \left[(x-a) f(b-x) + (b-x) f(x-a) \right] p(x)dx, \end{aligned}$$

for all $t \in [a,b]$. Thus (2.22) is established.

By the use of simple techniques of integration, we have that the following identity:

$$2 \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\, p\left(2x - \frac{a+b}{2}\right) dx = \int_a^{\frac{a+b}{2}} \left[f\left(\frac{x}{2} + \frac{a+b}{4} \right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2} \right) \right] p(x)\, dx.$$

Therefore from (2.17), by the use of techniques of integration, by the change of variable $x \to a+b-x$ and by the symmetry of p, we get that

$$\begin{aligned} 2 \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\, p\left(2x - \frac{a+b}{2}\right) dx &\le \int_a^{\frac{a+b}{2}} \left[f\left(\frac{3a+b}{4} \right) + f\left(\frac{a+3b}{4} \right) \right] p(x)dx \\ &\quad - \int_a^{\frac{a+b}{2}} \left[2\frac{x-a}{b-a} f(b-x) + 2\frac{b-x}{b-a} f(x-a) \right] p(x)dx \\ &= \frac{1}{2} \left[f\left(\frac{3a+b}{4} \right) + f\left(\frac{a+3b}{4} \right) \right] \int_a^b p(x)\, dx \\ &\quad - \frac{1}{b-a} \int_a^b \left[(x-a) f(b-x) + (b-x) f(x-a) \right] p(x)dx. \end{aligned}$$

Thus (2.23) is proved.

Now from the following identity, (2.18) and (2.19), we get that

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\int_a^b p(x)dx=\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\int_a^{\frac{a+b}{2}} p(x)dx$$

$$\leq \frac{1}{2}\int_a^{\frac{a+b}{2}}\left[f\left(\frac{a+2b-x}{2}\right)+f\left(\frac{x+b}{2}\right)\right.$$

$$\left.+f\left(\frac{a+2b-x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]p(x)dx-2\int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{4}-\frac{x}{2}\right)p(x)dx. \quad (2.26)$$

But

$$(b-a)\int_0^1 G(t)p((1-t)a+tb)dt$$

$$=\frac{b-a}{2}\left[\int_{\frac{1}{2}}^1 f\left(ta+(1-t)\frac{a+b}{2}\right)p(ta+(1-t)b)dt\right.$$

$$+\int_{\frac{1}{2}}^1 f\left(tb+(1-t)\frac{a+b}{2}\right)p(ta+(1-t)b)dt+\int_0^{\frac{1}{2}} f\left(ta+(1-t)\frac{a+b}{2}\right)p((1-t)a+tb)dt$$

$$\left.+\int_0^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}\right)p((1-t)a+tb)dt\right]$$

$$=\int_a^{\frac{a+b}{2}}\frac{1}{2}\left[f\left(\frac{2a+b-x}{2}\right)+f\left(\frac{x+a}{2}\right)+f\left(\frac{a+2b-x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]p(x)dx. \quad (2.27)$$

From (2.26) and (2.27) and by the change of variable $x\to a+b-x$ in the last integral and by the symmetry of p, we get (2.24).

From (2.20) and (2.21) and from (2.27), we get that

$$(b-a)\int_0^1 G(t)p((1-t)a+tb)dt\leq\frac{1}{2}\int_a^{\frac{a+b}{2}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]p((1-t)a+tb)dt$$

$$-2\int_a^{\frac{a+b}{2}}\left[\frac{x-a}{b-a}f(b-x)+\frac{b-x}{b-a}f(x-a)\right]p((1-t)a+tb)dt. \quad (2.28)$$

By the change of variable $x \to a+b-x$, we get from (2.28) that

$$(b-a)\int_0^1 G(t)p((1-t)a+tb)dt \le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\int_a^b p(x)dx - \frac{1}{b-a}\int_a^b \left[(x-a)f(b-x)+(b-x)f(x-a)\right]p(x)dx,$$

which is (2.25) and hence the theorem is proved.

Remark 2.6. *If the superquadratic function f is positive and hence convex, then from (2.22) we get refinement of the inequality (1.7) in Theorem 3; from (2.23) we get refinement of the first inequality in (1.8) of Theorem 3 and from (2.24) we get refinement of the middle inequality in (1.8) of Theorem 3 and from (2.25), we get refinement of the last inequality in (1.8) of Theorem 3.*

Corollary 2.7. *Let f be superquadratic function on $[0,b]$ and $p(x)=\frac{1}{b-a}$, $x\in[a,b]$, $0\le a<b$. Let G and H be defined as above. Then the following inequality holds for all $t\in[0,1]$.*

$$H(t) \le G(t) - \frac{t}{(b-a)^2}\int_a^b \left[(x-a)f(b-x)+(b-x)f(x-a)\right]dx. \tag{2.29}$$

The following inequalities hold:

$$\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx \le \frac{1}{2}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right] - \frac{1}{(b-a)^2}\int_a^b \left[(b-x)f(x-a)+(x-a)f(b-x)\right]p(x)dx, \tag{2.30}$$

$$\frac{1}{2}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right] \le \int_0^1 G(t)dt - \frac{1}{b-a}\int_a^b f\left(\left|\frac{a+b}{4}-\frac{x}{2}\right|\right)dx \tag{2.31}$$

and

$$\int_0^1 G(t)dt \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] - \frac{1}{(b-a)^2}\int_a^b \left[(x-a)f(b-x)+(b-x)f(x-a)\right]dx. \tag{2.32}$$

PROOF. It is a direct consequence of the above theorem.

Remark 2.8. *The results of the above corollary refine the results of inequalities (1.6) and (1.7) from* [11, Theorem C, p.2] *when superquadratic function f is positive and hence convex.*

Now we state and prove our last result of this section, before we proceed we go through again some calculations. For $x \in \left[a, \frac{a+b}{2}\right]$, $t \in [0,1]$ and by using (2.3), we have that the following inequalities hold for superquadratic function f:

$$2f\left(ta+(1-t)\frac{a+b}{2}\right) \le f\left(ta+(1-t)x\right)+f\left(ta+(1-t)(a+b-x)\right)-2f\left((1-t)\left(\frac{a+b}{2}-x\right)\right) \tag{2.33}$$

when $m = ta+(1-t)x$, $M = ta+(1-t)(a+b-x)$, $z = ta+(1-t)\frac{a+b}{2}$ and

$$2f\left(tb+(1-t)\frac{a+b}{2}\right) \le f\left(tb+(1-t)x\right)+f\left(tb+(1-t)(a+b-x)\right)-2f\left((1-t)\left(\frac{a+b}{2}-x\right)\right) \tag{2.34}$$

when $m = tb+(1-t)x$, $M = tb+(1-t)(a+b-x)$ and $z = tb+(1-t)\frac{a+b}{2}$.

Theorem 2.9. *Let f be superquadratic function on $[0,b]$ and $p(x)$ be non-negative and symmetric about $x = \frac{a+b}{2}$, $0 \le a < b$. Let G, L_p be defined as above, then we have the following inequality:*

$$G(t)\int_a^b p(x)\,dx \le L_p(t) - \int_a^b f\left((1-t)\left|x-\frac{a+b}{2}\right|\right)p(x)dx, \tag{2.35}$$

for all $t \in [0,1]$.

PROOF. By using the techniques of integration, under the assumptions on p, we have that the following identity:

$$G(t)\int_a^b p(x)\,dx = \int_a^{\frac{a+b}{2}}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right]p(x)\,dx,$$

holds for all $t \in [0,1]$. By using (2.33) and (2.34), we have that

$$G(t)\int_a^b p(x)\,dx \le \frac{1}{2}\int_a^{\frac{a+b}{2}} \left[f(ta+(1-t)x)+f(ta+(1-t)(a+b-x))\right]p(x)\,dx$$
$$+\frac{1}{2}\int_a^{\frac{a+b}{2}} \left[f(tb+(1-t)x)+f(tb+(1-t)(a+b-x))\right]p(x)\,dx$$
$$-2\int_a^{\frac{a+b}{2}} f\left((1-t)\left(\frac{a+b}{2}-x\right)\right)p(x)\,dx. \quad (2.36)$$

By the change of variable $x \to a+b-x$, under the assumptions on p, we get from (2.36) that

$$G(t)\int_a^b p(x)\,dx \le L_p(t) - \int_a^b f\left((1-t)\left|x-\frac{a+b}{2}\right|\right)p(x)dx.$$

Hence (2.35) is proved.

Remark 2.10. *If superquadratic function f is positive and therefore convex, then Theorem 17 refines the first inequality in (1.10) of Theorem 4.*

Corollary 2.11. *Let f be superquadratic function on $[0,b]$ and $p(x)=\frac{1}{b-a}$, $x \in [a,b]$, $0 \le a < b$. Let G and L be defined as above, then*

$$G(t) \le L(t) - \frac{1}{b-a}\int_a^b f\left((1-t)\left|x-\frac{a+b}{2}\right|\right)dx,$$

for all $t \in [0,1]$.

Proof. It follows directly from the above theorem, since for $p(x)=\frac{1}{b-a}$, $x \in [a,b]$, $L_p(t)=L(t)$.

Remark 2.12. *If superquadratic function f is convex, then the above corollary refines the first inequality in (1.9) from* [11, Theorem D, p.3].

3. Inequalities For Differentiable Superquadratic Functions

In this section we give results when f is a differentiable superquadratic function, those results refine the inequalities (1.5), (1.6) of Theorem 2 and refine the inequality (1.9) of Theorem 3 when superquadratic function f is positive and hence convex.

Theorem 3.1. *Let f be superquadratic function on $[0,b]$ and $p(x)$ be non-negative and symmetric about $x=\frac{a+b}{2}$, $0\le a<b$. If f is differentiable on $[a,b]$ such that $f(0)=f^{'}(0)=0$ and p is bounded on $[a,b]$, then we the following inequalities:*

$$\int_a^b f(x)\,p(x)\,dx - H_p(t) \le (1-t)\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right]\|p\|_\infty - \int_a^b f\left((1-t)\left|\frac{a+b}{2}-x\right|\right)p(x)\,dx, \quad (3.1)$$

where $\|p\|_\infty = \sup_{x\in[a,b]} |p(x)|$ and

$$\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx - H_p(t) \le \left[\frac{\left(f^{'}(a)-f^{'}(b)\right)(b-a)}{4} - f\left(\left|\frac{a-b}{2}\right|\right)\right]\int_a^b p(x)\,dx - \int_a^b f\left(t\left(\left|\frac{a+b}{2}-x\right|\right)\right)p(x)dx. \quad (3.2)$$

Proof. By integration by parts, we have

$$\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\left[f^{'}(a+b-x)-f^{'}(x)\right]dx = \int_a^b\left(\frac{a+b}{2}-x\right)f^{'}(x)\,dx = \frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)dx. \quad (3.3)$$

Using the substitution rules and by the assumptions on p, we have

$$\int_a^b f(x)\,p(x)\,dx = \int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]p(x)\,dx \quad (3.4)$$

and

$$H_p(t) = \int_a^{\frac{a+b}{2}}\left[f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right]p(x)\,dx. \quad (3.5)$$

By the assumptions on f, we have

$$\begin{aligned}
&\left[f\left(x\right)-f\left(tx+\left(1-t\right)\frac{a+b}{2}\right)\right]p(x)+[f\left(a+b-x\right)\\
&\qquad -f\left(t\left(a+b-x\right)+\left(1-t\right)\frac{a+b}{2}\right)\Big]\,p\left(x\right)\\
&\le\left(1-t\right)\left(\frac{a+b}{2}-x\right)f'\left(x\right)p\left(x\right)+\left(1-t\right)\left(\frac{a+b}{2}-x\right)f'\left(a+b-x\right)p\left(x\right)\\
&\quad -2f\left(\left(1-t\right)\left|\frac{a+b}{2}-x\right|\right)p\left(x\right)=\left(1-t\right)\left(\frac{a+b}{2}-x\right)\Big[f'\left(a+b-x\right)\\
&\quad -f'\left(x\right)\Big]\,p\left(x\right)-2f\left(\left(1-t\right)\left|\frac{a+b}{2}-x\right|\right)p\left(x\right)\le\left(1-t\right)\left(\frac{a+b}{2}-x\right)\\
&\qquad\left[f'\left(a+b-x\right)-f'\left(x\right)\right]\left\|p\right\|_{\infty}-2f\left(\left(1-t\right)\left|\frac{a+b}{2}-x\right|\right)p\left(x\right),
\end{aligned}\tag{3.6}$$

for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

Now from (3.3), (3.4), (3.5) and (3.6), under the assumptions on p and by the change of variables $x\to a+b-x$, in the last integral, we get (3.1).

By the assumptions on f and form Lemma 3, we get that

$$\frac{f(a)-f\left(\frac{a+b}{2}\right)}{2}\le\frac{a-b}{4}f'\left(a\right)-\frac{1}{2}f\left(\left|\frac{a-b}{2}\right|\right)$$

and

$$\frac{f(b)-f\left(\frac{a+b}{2}\right)}{2}\le\frac{b-a}{4}f'\left(b\right)-\frac{1}{2}f\left(\left|\frac{a-b}{2}\right|\right).$$

Adding these inequalities we get

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\le\frac{\left(f'(a)-f'(b)\right)(b-a)}{4}-f\left(\left|\frac{a-b}{2}\right|\right).$$

Thus

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}\int_a^b p\left(x\right)dx-f\left(\frac{a+b}{2}\right)\int_a^b p\left(x\right)dx\\
&\qquad\le\left[\frac{\left(f'(a)-f'(b)\right)(b-a)}{4}-f\left(\left|\frac{a-b}{2}\right|\right)\right]\int_a^b p\left(x\right)dx.
\end{aligned}\tag{3.7}$$

From (1.23) of Theorem 12, for $s=0$, we get

$$f\left(\frac{a+b}{2}\right)\int_a^b p\left(x\right)dx\le H_p(t)-\int_a^b f\left(t\left(\left|\frac{a+b}{2}-x\right|\right)\right)p(x)dx.\tag{3.8}$$

From (3.7) and (3.8), we have

$$\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx - H_p(t) \le \left[\frac{\left(f'(a)-f'(b)\right)(b-a)}{4} - f\left(\left|\frac{a-b}{2}\right|\right)\right]\int_a^b p(x)\,dx$$
$$-\int_a^b f\left(t\left(\left|\frac{a+b}{2}-x\right|\right)\right)p(x)dx.$$

Therefore (3.2) is also established. This completes the proof of the theorem.

Remark 3.2. *The Inequalities (3.1) and (3.2) represent refinements of the inequalities (1.5) and (1,6) of Theorem 2, when the superquadratic function f is positive and hence convex. Obviously when $p(x) = \frac{1}{b-a}$, $x \in [a,b]$, then from the above theorem, we get the following results:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx - H(t) \le \frac{1-t}{b-a}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]$$
$$-\frac{1}{b-a}\int_a^b f\left((1-t)\left|\frac{a+b}{2}-x\right|\right)dx$$

and

$$\frac{f(a)+f(b)}{2} - H(t) \le \frac{\left(f'(a)-f'(b)\right)(b-a)}{4}$$
$$-f\left(\left|\frac{a-b}{2}\right|\right) - \frac{1}{b-a}\int_a^b f\left(t\left(\left|\frac{a+b}{2}-x\right|\right)\right)dx,$$

which represent refinements of the inequalities (1.4) and (1.5) in [11, Theorem B, p. 2], *when superquadratic function f is positive and hence convex.*

Now we give our last result and summarize the results related to it in the remark followed by Theorem 19.

Theorem 3.3. *Let f be superquadratic function on $[0,b]$ and $p(x)$ be non-negative and symmetric about $x = \frac{a+b}{2}$, $0 \le a < b$. If f is differentiable on $[a,b]$ such that $f(0) = f'(0) = 0$ and p is bounded on $[a,b]$, then for all $t \in [0,1]$, we have the inequality:*

$$H_p(t) - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \le (b-a)\left[H(t)-G(t)\right]\|p\|_\infty - \int_a^b f\left(\left|\frac{a+b}{2}-x\right|\right)p(x)dx, \tag{3.9}$$

where $\|p\|_\infty = \sup_{x\in[a,b]} |p(x)|$.

space0.2mm Proof. By integration by parts we have

$$t\int_a^{\frac{a+b}{2}}\left[\left(x-\frac{a+b}{2}\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)\right.$$
$$\left.+\left(x-\frac{a+b}{2}\right)f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right]dx$$
$$=t\int_a^b\left(x-\frac{a+b}{2}\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)dx=(b-a)\left[G(t)-H(t)\right]. \quad (3.10)$$

By using the assumptions on f, we have that

$$\left[f\left(tx+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right]p(x)$$
$$+\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right]p(x)$$
$$\leq t\left(x-\frac{a+b}{2}\right)f'\left(tx+(1-t)\frac{a+b}{2}\right)p(x)$$
$$+t\left(\frac{a+b}{2}-x\right)f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)p(x)-2f\left(\left|\frac{a+b}{2}-x\right|\right)p(x)$$
$$=t\left(\frac{a+b}{2}-x\right)\left[f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right.$$
$$\left.-f'\left(tx+(1-t)\frac{a+b}{2}\right)\right]p(x)-2f\left(\left|\frac{a+b}{2}-x\right|\right)p(x)$$
$$\leq t\left(\frac{a+b}{2}-x\right)\left[f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right.$$
$$\left.-f'\left(tx+(1-t)\frac{a+b}{2}\right)\right]\|p\|_\infty-2f\left(\left|\frac{a+b}{2}-x\right|\right)p(x), \quad (3.11)$$

hold for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

Integrating (3.11) over x on $\left[a,\frac{a+b}{2}\right]$, using (3.10), by the change of variable $x\to a+b-x$ in the last integral, under the assumptions on p, we get

$$H_p(t)-f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx\leq(b-a)\left[G(t)-H(t)\right]\|p\|_\infty-\int_a^b f\left(\left|\frac{a+b}{2}-x\right|\right)p(x)dx.$$

This completes the proof of the theorem.

Remark 3.4. *The result of Theorem 18 refines the inequality (1.9) of Theorem 3, when superquadratic function f is positive and hence convex and if $p(x)=\frac{1}{b-a}$, $x\in[a,b]$ and superquadratic function f is positive and therefore convex, then we* *have the following inequality:*

$$H(t) - f\left(\frac{a+b}{2}\right) \leq G(t) - H(t) - \frac{1}{b-a}\int_a^b f\left(\left|\frac{a+b}{2} - x\right|\right) p(x)dx$$

the above inequality represents a refinement of the inequality (1.8) from [11, Theorem C, p.3].

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