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OPTIMAL GENERALIZED LOGARITHMIC MEAN BOUNDS FOR THE GEOMETRIC COMBINATION OF ARITHMETIC AND HARMONIC MEANS

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Abstract. In this paper, we answer the question: for $\alpha \in (0,1)$, what are the greatest value $p = p(\alpha)$ and least value $q = q(\alpha)$, such that the double inequality $L_p(a,b) \le A^{\alpha}(a,b)H^{1-\alpha}(a,b) \le L_q(a,b)$ holds for all a, $b > 0$? where $L_p(a,b)$, $A(a,b)$, and $H(a,b)$ are the p-th generalized logarithmic, arithmetic, and harmonic means of a and b, respectively.

Key words: Generalized logarithmic mean, arithmetic mean, harmonic mean.

Abstrak. Dalam paper ini, kami menjawab pertanyaan: untuk $\alpha \in (0,1)$, berapa nilai terbesar $p = p(\alpha)$ dan nilai terkecil $q = q(\alpha)$, sehingga ketidaksamaan ganda $L_p(a,b) \le A^{\alpha}(a,b)H^{1-\alpha}(a,b) \le L_q(a,b)$ dipenuhi untuk semua a, $b > 0$? dengan $L_p(a,b)$, $A(a,b)$, dan $H(a,b)$ secara berturut-turut adalah rata-rata logaritmik yang diperumum, aritmatik, dan harmonik ke-p dari a and b.

Kata kunci: Rata-rata logaritmik yang diperumum, rata-rata aritmatik, rata-rata harmonik.

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1. Introduction

For $p \in \mathbb{R}$ the generalized logarithmic mean $L_p(a,b)$ of two positive numbers a and b is defined by

$$L_p(a,b) = \begin{cases} a, & a=b, \\ \left[\frac{a^{p+1}-b^{p+1}}{(p+1)(a-b)}\right]^{1/p}, & p \neq 0, p \neq -1, a \neq b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & p=0, a \neq b, \\ \frac{b-a}{\ln b - \ln a}, & p=-1, a \neq b. \end{cases}$$

It is well-known that $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. In the recent past, the generalized logarithmic mean has been the subject of intensive research. In particular, many remarkable inequalities for L_p can be found in the literature [1, 8, 9, 13, 14, 19, 20, 22, 23]. It might be surprising that the generalized logarithmic mean has applications in economics, physics and even in meteorology [10, 17, 18]. In [10] the authors study a variant of Jensen's functional equation involving L_p, which appear in a heat conduction problem.

Let $A(a,b) = (a+b)/2$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $L(a,b) = (b-a)(\ln b - \ln a)$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the arithmetic, identric, logarithmic, geometric, and harmonic means of two positive numbers a and b with $a \neq b$, respectively. Then

$$\begin{aligned} \min\{a,b\} < H(a,b) < G(a,b) = L_{-2}(a,b) < L(a,b) = L_{-1}(a,b) \\ < I(a,b) = L_0(a,b) < A(a,b) = L_1(a,b) < \max\{a,b\}. \end{aligned}$$

In [7, 11, 21] the authors present bounds for $L(a,b)$ and $I(a,b)$ in terms of $G(a,b)$ and $A(a,b)$.

Theorem 1.1. *For all positive real numbers a and b with $a \neq b$ we have*

$$A^{\frac{1}{3}}(a,b)G^{\frac{2}{3}}(a,b) < L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b)$$

and

$$\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) < I(a,b).$$

The proof of the following Theorem 1.2 can be found in [5].

Theorem 1.2. *For all positive real numbers a and b with $a \neq b$ we have*

$$\sqrt{G(a,b)A(a,b)} < \sqrt{L(a,b)I(a,b)} < \frac{1}{2}(L(a,b) + I(a,b)) < \frac{1}{2}(G(a,b) + A(a,b)).$$

For $p \in \mathbb{R}$, the p-th power mean $M_p(a,b)$ of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

The main properties of these means are given in [5]. Several authors discussed the relationship of certain means to $M_p(a,b)$. The following sharp bounds for L, I, $(IL)^{1/2}$ and $(I+L)/2$ in terms of power means are proved in [2, 3, 6, 12, 15, 16].

Theorem 1.3. *For all positive real numbers a and b with $a \neq b$ we have*

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b), \quad M_{2/3}(a,b) < I(a,b) < M_{\ln 2}(a,b),$$

$$M_0(a,b) < I^{1/2}(a,b)L^{1/2}(a,b) < M_{1/2}(a,b)$$

and

$$\frac{1}{2}[I(a,b) + L(a,b)] < M_{1/2}(a,b).$$

The following Theorems 1.4-1.6 were established by Alzer and Qiu in [4].

Theorem 1.4. *The inequalities*

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b)$$

hold for all positive real numbers a and b with $a \neq b$ if and only if

$$\alpha \leq 2/3 \quad \textit{and} \quad \beta \geq 2/e = 0.73575\cdots.$$

Theorem 1.5. *Let a and b be real numbers with $a \neq b$. If $0 < a,\ b \leq e$, then*

$$[G(a,b)]^{A(a,b)} < [L(a,b)]^{I(a,b)} < [A(a,b)]^{G(a,b)}.$$

And, if $a, b \geq e$, then

$$[A(a,b)]^{G(a,b)} < [I(a,b)]^{L(a,b)} < [G(a,b)]^{A(a,b)}.$$

Theorem 1.6. *For all positive real numbers a and b with $a \neq b$ we have*

$$M_c(a,b) < \frac{1}{2}(L(a,b) + I(a,b))$$

with the best possible parameter $c = \ln 2/(1+\ln 2) = 0.40938\cdots$.

It is the aim of this paper to answer the question: for $\alpha \in (0,1)$, what are the greatest value $p = p(\alpha)$ and least value $q = q(\alpha)$, such that the double inequality

$$L_p(a,b) \leq A^{\alpha}(a,b)H^{1-\alpha}(a,b) \leq L_q(a,b)$$

holds for all $a,\ b > 0$?

2. Main Results

Theorem 2.1. *For $\alpha \in (0,1)$ and all $a, b > 0$ we have*

(1) $L_{6\alpha-5}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{-1/\alpha}(a,b)$ *for* $\alpha = 1/3$ *or* $\alpha = 1/2$*;*

(2) $L_{-1/\alpha}(a,b) \le A^{\alpha}(a,b)H^{1-\alpha}(a,b) \le L_{6\alpha-5}(a,b)$ *for* $\alpha \in (0,1/3)\bigcup(1/2,1)$ *and* $L_{-1/\alpha}(a,b) \ge A^{\alpha}(a,b)H^{1-\alpha}(a,b) \ge L_{6\alpha-5}(a,b)$ *for* $\alpha \in (1/3,1/2)$*, with equality if and only if* $a = b$*, and the parameters* $-1/\alpha$ *and* $6\alpha-5$ *in either case are best possible.*

Proof. (1) If $\alpha = 1/3$ or $\alpha = 1/2$, and $a = b$, then we clearly see that $L_{6\alpha-5}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{-1/\alpha}(a,b) = a$.

If $\alpha = 1/3$ and $a \ne b$, then we have

$$L_{6\alpha-5}(a,b) = L_{-1/\alpha}(a,b) = L_{-3}(a,b)$$

$$= \frac{2^{1/3}(ab)^{2/3}}{(a+b)^{1/3}} = A^{1/3}(a,b)H^{2/3}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b).$$

If $\alpha = 1/2$ and $a \ne b$, then we get

$$L_{6\alpha-5}(a,b) = L_{-1/\alpha}(a,b) = L_{-2}(a,b)$$

$$= (ab)^{1/2} = A^{1/2}(a,b)H^{1/2}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b).$$

(2) If $\alpha \in (0,1)$ and $a = b$, then we clearly see that $L_{-1/\alpha}(a,b) = A^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{6\alpha-5}(a,b)$. Without loss of generality, we assume that $t = a/b > 1$ in the following discussion.

If $\alpha = 2/3$, then one has

$$\begin{aligned} & \ln L_{6\alpha-5}(a,b) - \ln[A^{\alpha}(a,b)H^{1-\alpha}(a,b)] \\ = & \ln L_{-1}(a,b) - \ln[A^{2/3}(a,b)H^{1/3}(a,b)] \\ = & \ln(\frac{t-1}{\ln t}) - \ln[2^{-1/3}(1+t)^{1/3}t^{1/3}]. \qquad (1) \end{aligned}$$

Let $f_1(t) = \ln(\frac{t-1}{\ln t}) - \ln[2^{-1/3}(1+t)^{1/3}t^{1/3}]$, then simple computations lead to

$$\lim_{t\to 1^+} f_1(t) = 0, \qquad (2)$$

$$f_1'(t) = \frac{g_1(t)}{3t(t-1)(t+1)\ln t}, \qquad (3)$$

where $g_1(t) = (t^2+4t+1)\ln t - 3(t^2-1)$.

$$g_1(1) = 0, \qquad (4)$$

$$g_1'(t) = \frac{h_1(t)}{t}, \qquad (5)$$

where $h_1(t) = 2t(t+2)\ln t - 5t^2 + 4t + 1$.

$$\begin{aligned} g_1'(1) &= h_1(1) = 0, && (6)\\ h_1'(t) &= 4(t+1)\ln t - 8(t-1), \\ h_1'(1) &= 0, && (7)\\ h_1''(t) &= \frac{4}{t} v_1(t), && (8) \end{aligned}$$

where $v_1(t) = t\ln t - t + 1$.

$$h_1''(1) = v_1(1) = 0 \quad (9)$$

and

$$v_1'(t) = \ln t > 0 \quad (10)$$

for $t > 1$.

From (1)-(10) we know that $L_{6\alpha-5}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha = 2/3$ and $a \neq b$.

If $\alpha = 5/6$, then

$$\begin{aligned} & \ln L_{6\alpha-5}(a,b) - \ln[A^{\alpha}(a,b)H^{1-\alpha}(a,b)] \\ = \ & \ln L_0(a,b) - \ln[A^{5/6}(a,b)H^{1/6}(a,b)] \\ = \ & \ln[\frac{1}{e}t^{\frac{t}{t-1}}] - \ln[2^{-2/3}t^{1/6}(1+t)^{2/3}]. \quad (11) \end{aligned}$$

Let $f_2(t) = \ln[\frac{1}{e}t^{\frac{t}{t-1}}] - \ln[2^{-2/3}t^{1/6}(1+t)^{2/3}]$, then elementary calculations yield

$$\begin{aligned} \lim_{t\to 1^+} f_2(t) &= 0, && (12)\\ f_2'(t) &= \frac{g_2(t)}{6t(t-1)^2(t+1)}, && (13)\\ g_2(t) &= (t^3 + 9t^2 - 9t - 1) - 6t(t+1)\ln t, \\ g_2(1) &= 0, && (14)\\ g_2'(t) &= 3t^2 + 12t - 6(2t+1)\ln t - 15, \\ g_2'(1) &= 0, && (15)\\ g_2''(t) &= \frac{6}{t}h_2(t), && (16)\\ h_2(t) &= t^2 - 2t\ln t - 1, \\ g_2''(1) &= h_2(1) = 0, && (17)\\ h_2'(t) &= 2(t - \ln t - 1), \\ h_2'(1) &= 0 && (18) \end{aligned}$$

and

$$h_2''(t) = 2(1 - \frac{1}{t}) > 0 \quad (19)$$

for $t > 1$.

From (11)-(19) we know that $L_{6\alpha-5}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha = 5/6$ and $a \neq b$.

If $\alpha \in (0,1)\backslash\{2/3, 5/6\}$, then

$$\begin{aligned}
&\ln L_{6\alpha-5}(a,b) - \ln[A^{\alpha}(a,b)H^{1-\alpha}(a,b)] \\
= \ & \frac{1}{6\alpha-5}\ln\frac{t^{6\alpha-4}-1}{(6\alpha-4)(t-1)} - \ln[2^{1-2\alpha}t^{1-\alpha}(t+1)^{2\alpha-1}]. \qquad (20)
\end{aligned}$$

Let $f_3(t) = \frac{1}{6\alpha-5}\ln\frac{t^{6\alpha-4}-1}{(6\alpha-4)(t-1)} - \ln[2^{1-2\alpha}t^{1-\alpha}(t+1)^{2\alpha-1}]$, then simple computations lead to

$$\lim_{t\to 1^+} f_3(t) = 0, \qquad (21)$$

$$f_3'(t) = \frac{g_3(t)}{t(t^2-1)(t^{6\alpha-4}-1)}, \qquad (22)$$

where $g_3(t) = (1-\alpha)t^{6\alpha-2} + \frac{4(1-\alpha)(1-3\alpha)}{6\alpha-5}t^{6\alpha-3} - \frac{(1-2\alpha)(1-3\alpha)}{6\alpha-5}t^{6\alpha-4} + \frac{(1-2\alpha)(1-3\alpha)}{6\alpha-5}t^2 - \frac{4(1-\alpha)(1-3\alpha)}{6\alpha-5}t - (1-\alpha)$.

$$g_3(1) = 0, \qquad (23)$$

$$\begin{aligned}
g_3'(t) = \ & 2(1-\alpha)(3\alpha-1)t^{6\alpha-3} + \frac{12(1-\alpha)(1-3\alpha)(2\alpha-1)}{6\alpha-5}t^{6\alpha-4} \\
& -\frac{2(1-2\alpha)(1-3\alpha)(3\alpha-2)}{6\alpha-5}t^{6\alpha-5} + \frac{2(1-2\alpha)(1-3\alpha)}{6\alpha-5}t \\
& -\frac{4(1-\alpha)(1-3\alpha)}{6\alpha-5},
\end{aligned}$$

$$g_3'(1) = 0, \qquad (24)$$

$$\begin{aligned}
g_3''(t) = \ & 6(1-\alpha)(1-2\alpha)(1-3\alpha)t^{6\alpha-4} \\
& +\frac{24(1-\alpha)(1-2\alpha)(1-3\alpha)(2-3\alpha)}{6\alpha-5}t^{6\alpha-5} \\
& -2(1-2\alpha)(1-3\alpha)(3\alpha-2)t^{6\alpha-6} + \frac{2(1-2\alpha)(1-3\alpha)}{6\alpha-5},
\end{aligned}$$

$$g_3''(1) = 0 \qquad (25)$$

and

$$g_3'''(t) = -12(1-\alpha)(1-2\alpha)(1-3\alpha)(2-3\alpha)t^{6\alpha-7}(t-1)^2. \qquad (26)$$

If $\alpha \in (0,1/3)\bigcup(1/2,2/3)$, then

$$t^{6\alpha-4} - 1 < 0 \qquad (27)$$

and (26) implies that

$$g_3'''(t) < 0 \qquad (28)$$

for $t > 1$.

From (20)-(21) and (27)-(28) we know that $L_{6\alpha-5}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (0,1/3)\bigcup(1/2,2/3)$ and $a \neq b$.

If $\alpha \in (2/3,5/6)\bigcup(5/6,1)$, then

$$t^{6\alpha-4} - 1 > 0 \tag{29}$$

and (26) implies that

$$g_3'''(t) > 0 \tag{30}$$

for $t > 1$.

From (20)-(25) and (29)-(30) we conclude that $L_{6\alpha-5}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (2/3,5/6)\bigcup(5/6,1)$ and $a \neq b$.

If $\alpha \in (1/3,1/2)$, then (27) and (30) again hold. From (20)-(25) and (27) together with (30) we know that $L_{6\alpha-5}(a,b) < A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (1/3,1/2)$ and $a \neq b$.

Next we compare the values of $L_{-1/\alpha}(a,b)$ with $A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (0,1)$ and $a \neq b$. It is not difficult to verify that

$$\begin{aligned} & \ln L_{-1/\alpha}(a,b) - \ln[A^{\alpha}(a,b)H^{1-\alpha}(a,b)] \\ = & \ \alpha \ln \frac{(1-1/\alpha)(t-1)}{t^{1-1/\alpha}-1} - \ln[2^{1-2\alpha}t^{1-\alpha}(1+t)^{2\alpha-1}]. \end{aligned} \tag{31}$$

Let $f_4(t) = \alpha \ln \frac{(1-1/\alpha)(t-1)}{t^{1-1/\alpha}-1} - \ln[2^{1-2\alpha}t^{1-\alpha}(1+t)^{2\alpha-1}]$, then elementary calculations yield

$$\lim_{t\to 1^+} f_4(t) = 0, \tag{32}$$

$$f_4'(t) = \frac{g_4(t)}{(t^2-1)(t^{1/\alpha-1}-1)}, \tag{33}$$

where $g_4(t) = (3\alpha-1)t^{1/\alpha-1} + (1-\alpha)t^{1/\alpha-2} + (\alpha-1)t + (1-3\alpha)$.

$$g_4(1) = 0, \tag{34}$$

$$g_4'(t) = \frac{(1-\alpha)(3\alpha-1)}{\alpha}t^{1/\alpha-2} + \frac{(1-\alpha)(1-2\alpha)}{\alpha}t^{1/\alpha-3} + (\alpha-1),$$

$$g_4'(1) = 0, \tag{35}$$

$$g_4''(t) = \frac{(1-\alpha)(1-2\alpha)(3\alpha-1)}{\alpha^2}t^{1/\alpha-4}(t-1). \tag{36}$$

If $\alpha \in (0,1/3)\bigcup(1/2,1)$, then(36) implies

$$g_4''(t) < 0 \tag{37}$$

for $t > 1$.

From (31)-(35) and (37) we know that $L_{-1/\alpha}(a,b) < A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (0,1/3)\bigcup(1/2,1)$ and $a \neq b$.

If $\alpha \in (1/3,1/2)$, then (36) implies

$$g_4''(t) > 0 \tag{38}$$

for $t > 1$. Therefore, $L_{-1\alpha}(a,b) > A^{\alpha}(a,b)H^{1-\alpha}(a,b)$ for $\alpha \in (1/3, 1/2)$ and $a \neq b$ follows from (31)-(35) and (38).

Finally, we prove that the parameters $-1/\alpha$ and $6\alpha - 5$ in either case are best possible.

Firstly, we show that the parameter $6\alpha - 5$ in either case is best possible. We divide the proof into seven cases.

Case 1. $\alpha = 2/3$. For any $\epsilon > 0$ and $x > 0$ one has

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{1+\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{1+\epsilon} \\ = & [A^{2/3}(1,1+x)H^{1/3}(1,1+x)]^{1+\epsilon} - [L_{-1-\epsilon}(1,1+x)]^{1+\epsilon} \\ = & \frac{f_1(x)}{(1+x)^{\epsilon} - 1}, \end{aligned} \tag{39}$$

where $f_1(x) = [(1+x)^{\epsilon} - 1](1+x)^{(1+\epsilon)/3}(1+x/2)^{(1+\epsilon)/3} - \epsilon x(1+x)^{\epsilon}$.

Letting $x \to 0$ and making use of Taylor expansion we get

$$f_1(x) = \frac{\epsilon^2(1+\epsilon)}{24}x^3 + o(x^3). \tag{40}$$

Equations (39) and (40) imply that for $\epsilon > 0$ there exists $\delta_1 = \delta_1(\epsilon) > 0$, such that $L_{6\alpha-5-\epsilon}(1,1+x) < A^{\alpha}(1,1+x)H^{(1-\alpha)}(1,1+x)$ for $\alpha = 2/3$ and $x \in (0, \delta_1)$.

Case 2. $\alpha = 5/6$. For any $\epsilon \in (0,1)$ and $x > 0$ we have

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{\epsilon} \\ = & [A^{5/6}(1,1+x)H^{1/6}(1,1+x)]^{\epsilon} - [L_{-\epsilon}(1,1+x)]^{\epsilon} \\ = & \frac{f_2(x)}{(1+x)^{1-\epsilon} - 1}, \end{aligned} \tag{41}$$

where $f_2(x) = [(1+x)^{1-\epsilon} - 1](1+x)^{\epsilon/6}(1+x/2)^{2\epsilon/3} - (1-\epsilon)x$.

Letting $x \to 0$ and making using of Taylor expansion we get

$$f_2(x) = \frac{\epsilon(1-\epsilon)}{24}x^3 + o(x^3). \tag{42}$$

Equations (41) and (42) show that for $\epsilon \in (0,1)$ there exists $\delta_2 = \delta_2(\epsilon) > 0$, such that $L_{6\alpha-5-\epsilon}(1,1+x) < A^{\alpha}(1,1+x)H^{(1-\alpha)}(1,1+x)$ for $\alpha = 5/6$ and $x \in (0, \delta_2)$.

Case 3. $\alpha \in (0, 1/3)$. For any $\epsilon > 0$ and $x > 0$ one has

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{5+\epsilon-6\alpha} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{5+\epsilon-6\alpha} \\ = & \frac{f_3(x)}{[(1+x)^{4+\epsilon-6\alpha} - 1](1+\frac{x}{2})^{(1-2\alpha)(5+\epsilon-6\alpha)}}, \end{aligned} \tag{43}$$

where $f_3(x) = [(1+x)^{4+\epsilon-6\alpha} - 1](1+x)^{(1-\alpha)(5+\epsilon-6\alpha)} - (4+\epsilon-6\alpha)x(1+x)^{4+\epsilon-6\alpha}(1+x/2)^{(1-2\alpha)(5+\epsilon-6\alpha)}$.

Letting $x \to 0$ and making use of Taylor expansion we get

$$f_3(x) = \frac{\epsilon(4+\epsilon-6\alpha)(5+\epsilon-6\alpha)}{24}x^3 + o(x^3). \tag{44}$$

Equations (43) and (44) imply that for any $\alpha \in (0, 1/3)$ and $\epsilon > 0$ there exists $\delta_3 = \delta_3(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5-\epsilon}(1, 1+x) < A^{\alpha}(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_3)$.

Case 4. $\alpha \in (1/3, 1/2)$. For any $\epsilon \in (0, 4-6\alpha)$ and $x > 0$ we get

$$\begin{aligned} & [L_{6\alpha-5+\epsilon}(1,1+x)]^{5-6\alpha-\epsilon} - [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{5-6\alpha-\epsilon} \\ = & \frac{f_4(x)}{[(1+x)^{4-6\alpha-\epsilon}-1](1+\frac{x}{2})^{(1-2\alpha)(5-6\alpha-\epsilon)}}, \end{aligned} \tag{45}$$

where $f_4(x) = (4-6\alpha-\epsilon)x(1+x)^{4-6\alpha-\epsilon}(1+x/2)^{(1-2\alpha)(5-6\alpha-\epsilon)} - [(1+x)^{4-6\alpha-\epsilon} - 1](1+x)^{(1-\alpha)(5-6\alpha-\epsilon)}$

Letting $x \to 0$ and making using Taylor expansion one has

$$f_4(x) = \frac{\epsilon(4+\epsilon-6\alpha)(5+\epsilon-6\alpha)}{24}x^3 + o(x^3). \tag{46}$$

Equations (45) and (46) imply that for any $\alpha \in (1/3, 1/2)$ and $\epsilon \in (0, 4-6\alpha)$ there exists $\delta_4 = \delta_4(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5+\epsilon}(1, 1+x) > A^{\alpha}(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_4)$.

Case 5. $\alpha \in (1/2, 2/3)$. For any $\epsilon > 0$ and $x > 0$ we have

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{5-6\alpha+\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{5-6\alpha+\epsilon} \\ = & \frac{f_5(x)}{(1+x)^{4-6\alpha+\epsilon}-1}, \end{aligned} \tag{47}$$

where $f_5(x) = [(1+x)^{4-6\alpha+\epsilon} - 1](1+x/2)^{(2\alpha-1)(5-6\alpha+\epsilon)}(1+x)^{(1-\alpha)(5-6\alpha+\epsilon)} - (4-6\alpha+\epsilon)x(1+x)^{4-6\alpha+\epsilon}$.

Letting $x \to 0$ and making use of Taylor expansion we get

$$f_5(x) = \frac{\epsilon(4+\epsilon-6\alpha)^2(5+\epsilon-6\alpha)}{24}x^3 + o(x^3). \tag{48}$$

Equations (47) and (48) imply that for any $\alpha \in (1/2, 2/3)$ and $\epsilon > 0$ there exists $\delta_5 = \delta_5(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5-\epsilon}(1, 1+x) < A^{\alpha}(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_5)$.

Case 6. $\alpha \in (2/3, 5/6)$. For any $\epsilon \in (0, 6\alpha-4)$ and $x > 0$ one has

$$\begin{aligned} & [A^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)]^{5-6\alpha+\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{5-6\alpha+\epsilon} \\ = & \frac{f_6(x)}{(1+x)^{6\alpha-4-\epsilon}-1}, \end{aligned} \tag{49}$$

where $f_6(x) = [(1+x)^{6\alpha-4-\epsilon} - 1](1+x/2)^{(2\alpha-1)(5-6\alpha+\epsilon)}(1+x)^{(1-\alpha)(5-6\alpha+\epsilon)} - (6\alpha-4-\epsilon)x$.

Letting $x \to 0$ and making using Taylor expansion we obtain

$$f_6(x) = \frac{\epsilon(5+\epsilon-6\alpha)(6\alpha-4-\epsilon)}{24}x^3 + o(x^3). \tag{50}$$

Equations (49) and (50) imply that for any $\alpha \in (2/3, 5/6)$ and $\epsilon \in (0, 6\alpha-4)$ there exists $\delta_6 = \delta_6(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5-\epsilon}(1, 1+x) < A^\alpha(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_6)$.

Case 7. $\alpha \in (5/6, 1)$. For any $\epsilon \in (0, 6\alpha-5)$ and $x > 0$ we have

$$\begin{aligned} & [A^\alpha(1,1+x)H^{1-\alpha}(1,1+x)]^{6\alpha-5-\epsilon} - [L_{6\alpha-5-\epsilon}(1,1+x)]^{6\alpha-5-\epsilon} \\ = \quad & \frac{f_7(x)}{(6\alpha-4-\epsilon)x}, \end{aligned} \tag{51}$$

where $f_7(x) = (6\alpha-4-\epsilon)x(1+x/2)^{(2\alpha-1)(6\alpha-5-\epsilon)}(1+x)^{(1-\alpha)(6\alpha-5-\epsilon)} - [(1+x)^{6\alpha-4-\epsilon} - 1]$.

Letting $x \to 0$ and making use of Taylor expansion we get

$$f_7(x) = \frac{\epsilon(6\alpha-5-\epsilon)(6\alpha-4-\epsilon)}{24}x^3 + o(x^3). \tag{52}$$

Equations (51) and (52) imply that for any $\alpha \in (5/6, 1)$ and $\epsilon \in (0, 6\alpha-5)$ there exists $\delta_7 = \delta_7(\epsilon, \alpha) > 0$, such that $L_{6\alpha-5-\epsilon}(1, 1+x) < A^\alpha(1, 1+x)H^{(1-\alpha)}(1, 1+x)$ for $x \in (0, \delta_7)$.

Secondly, we prove that the parameter $-1/\alpha$ in either case is the best possible. The proof is divided into two cases.

Case A. $\alpha \in (0, 1/3)\bigcup(1/2, 1)$. For any $\epsilon \in (0, 1/\alpha - 1)$ and $t > 0$, we have

$$\begin{aligned} & L_{1/\alpha+\epsilon}(1,t) - A^\alpha(1,t)H^{1-\alpha}(1,t) \\ = \quad & t^{\frac{\alpha}{1-\epsilon\alpha}}\{[\frac{(\frac{1}{\alpha}-1-\epsilon)(1-\frac{1}{t})}{1-t^{-(\frac{1}{\alpha}-1-\epsilon)}}]^{\frac{\alpha}{1-\epsilon\alpha}} - 2^{1-2\alpha}t^{-\frac{\epsilon\alpha^2}{1-\epsilon\alpha}}(1+\frac{1}{t})^{2\alpha-1}\} \end{aligned} \tag{53}$$

and

$$\begin{aligned} & \lim_{t\to+\infty}\{[\frac{(\frac{1}{\alpha}-1-\epsilon)(1-\frac{1}{t})}{1-t^{-(\frac{1}{\alpha}-1-\epsilon)}}]^{\frac{\alpha}{1-\epsilon\alpha}} - 2^{1-2\alpha}t^{-\frac{\epsilon\alpha^2}{1-\epsilon\alpha}}(1+\frac{1}{t})^{2\alpha-1}\} \\ = \quad & (\frac{1}{\alpha}-1-\epsilon)^{\frac{\alpha}{1-\epsilon\alpha}} > 0. \end{aligned} \tag{54}$$

Equation (53) and inequality (54) imply that for any $\alpha \in (0, 1/3)\bigcup(1/2, 1)$ and $\epsilon \in (0, 1/\alpha - 1)$ there exists $T_1 = T_1(\epsilon, \alpha) > 1$, such that $L_{-1/\alpha+\epsilon}(1, t) > A^\alpha(1, t)H^{1-\alpha}(1, t)$ for $t \in (T_1, \infty)$.

Case B. $\alpha \in (1/3, 1/2)$. For any $\epsilon > 0$ and $t > 0$, we have

$$\begin{aligned} & A^\alpha(1,t)H^{1-\alpha}(1,t) - L_{-1/\alpha-\epsilon}(1,t) \\ = \quad & t^\alpha\{2^{1-2\alpha}(1+\frac{1}{t})^{2\alpha-1} - t^{-\frac{\epsilon\alpha^2}{1+\epsilon\alpha}}[\frac{(\frac{1}{\alpha}-1+\epsilon)(1-\frac{1}{t})}{1-t^{-(\frac{1}{\alpha}-1+\epsilon)}}]^{\frac{\alpha}{1+\epsilon\alpha}}\} \end{aligned} \tag{55}$$

and

$$\begin{aligned}&\lim_{t\to+\infty}\{2^{1-2\alpha}(1+\frac{1}{t})^{2\alpha-1}-t^{-\frac{\epsilon\alpha^2}{1+\epsilon\alpha}}[\frac{(\frac{1}{\alpha}-1+\epsilon)(1-\frac{1}{t})}{1-t^{-(\frac{1}{\alpha}-1+\epsilon)}}]^{\frac{\alpha}{1+\epsilon\alpha}}\}\\ =\ & 2^{1-\alpha}>0. \qquad (56)\end{aligned}$$

From (55) and (56) we clearly see that for any $\alpha \in (1/3, 1/2)$ and $\epsilon > 0$ there exists $T_2 = T_2(\epsilon, \alpha) > 1$, such that $L_{-1/\alpha-\epsilon}(1,t) < A^{\alpha}(1,t)H^{1-\alpha}(1,t)$ for $t \in (T_2, \infty)$.

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References

[1] Abuhany, A. A. K., Salem, S. R. and Salman, I. M., "On Steffensen's integral inequality with applications", *J. Rajasthan Acad. Phys. Sci.* **5** (2006), 1-12.

[2] Alzer, H., "Ungleichungen für $(e/a)^a(b/e)^b$", *Elem. Math.* **40** (1985), 120-123.

[3] Alzer, H., "Ungleichungen für Mittelwerte", *Arch. Math.* **47**(1986), 422-426.

[4] Alzer, H. and Qiu, S.-L., "Inequalities for means in two variables", *Arch. Math.* **80** (2003), 201-215.

[5] Bullen, P. S., Mitrinović, D. S. and Vasić, P. M., *Means and Their Inequalities*, D. Reidel Pubishing Co., 1988.

[6] Burk, F., "The geometric, logarithmic and arithmic mean inequality", *Amer. Math. Monthly* **94** (1987), 527-528.

[7] Carlson, B. C., "The logarithmic mean", *Amer. Math. Monthly* **79** (1972), 615-618.

[8] Chen, Ch.-P., "The monotonicity of the ratio between generalized logarithmic means", *J. Math. Anal. Appl.* **345** (2008), 86-89.

[9] Chen, Ch.-P. and Qi, F., "Monotonicity properties for generalized logarithmic means", *Aust. J. Math. Anal. Appl.* **1** (2004), Article 2, 1-4.

[10] Kahlig, P. and Matkowski, J., "Functional equations involving the logarithmic mean", *Z. Angew. Math. Mech.* **76** (1996), 385–390.

[11] Leach, E. B. and Sholander, M. C., "Extended mean values II", *J. Math. Anal. Appl.* **92** (1983), 207-223.

[12] Lin, T. P., "The power mean and the logarithmic mean", *Amer. Math. Monthly* **81** (1974), 879-883.

[13] Mond, B., Pearce, C. E. M. and Pečarić, J., "The logarithmic mean is a mean", *Math. Commun.* **2** (1997), 35-39.

[14] Pearce, C. E. M. and Pečarić, J., "Some theorems of Jensen type for generalized logarithmic means", *Rev. Roumaine Math. Pures Appl.* **40** (1995), 789-795.

[15] Pittenger, A. O., "Inequalities between arithmetic and logarithmic means", *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **678-715** (1980), 15-18.

[16] Pittenger, A. O., "The symmetric, logarithmic and power means", *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **678-715** (1980), 19-23.

[17] Pittenger, A. O., "The logarithmic mean in n variables", *Amer. Math. Monthly* **92** (1985), 99-104.

[18] Pólya, G. and Szegö, G., *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951.

[19] Qi, F., Chen Sh.-X. and Chen, Ch.-P., "Monotonicity of ratio between the generalized logarithmic means", *Math. Inequal. Appl.* **10** (2007), 559-564.

[20] Qi, F., Li, X.-A. and Chen, Sh.-X., "Refinements, extensions and generalizations of the second Kershaw's double inequality", *Math. Inequal. Appl.* **11** (2008), 457-465.

[21] Sándor, J., "A note on some inequalities for means", *Arch. Math.* **56** (1991), 471-473.

[22] Shi, H.-N., "Schur-convex functions related to Hadamard-type inequalities", *J. Math. Inequal.* **1** (2007), 127-136.

[23] Stolarsky, K. B., "The power and generalized logarithmic means", *Amer. Math. Monthly* **87** (1980), 545-548.