

ON THE NORMALITY OF SOME CAYLEY DIGRAPHS WITH VALENCY 2

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Abstract. We call a Cayley digraph $\Gamma = \text{Cay}(G, S)$ normal for G if $R(G)$, the right regular representation of G , is a normal subgroup of the full automorphism group $\text{Aut}(\Gamma)$ of Γ . In this paper we determine the normality of Cayley digraphs of valency 2 on the groups of order pq and also on non-abelian finite groups G such that every proper subgroup of G is abelian.

Key words: Cayley digraph, normal Cayley digraph, automorphism group.

Abstrak. Digraf Cayley $\Gamma = \text{Cay}(G, S)$ dinyatakan normal untuk G jika $R(G)$, representasi reguler kanan G , adalah subgrup normal dari grup automorfisma penuh $\text{Aut}(\Gamma)$ dari Γ . Dalam paper ini kami menentukan normalitas digraf Cayley bervalensi 2 pada grup berorde pq dan juga grup hingga non-Abelian G sedemikian sehingga setiap subgrup sejati dari G adalah Abelian.

Kata kunci: Digraf Cayley, digraf Cayley normal, grup automorfisma.

1. Introduction

Let G be a finite group, and S a subset of G such that $1_G \notin S$. The Cayley digraph $\Gamma = \text{Cay}(G, S)$ of G with respect to S is defined to have vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{(g, sg) \mid g \in G, s \in S\}$. From the definition the following obvious facts are basic for Cayley digraphs. (1) the automorphism group $\text{Aut}(\Gamma)$ of Γ contains $R(G)$, the right regular representation of G , as a subgroup; (2) Γ is connected if and only if S generates the group G ; (3) Γ is undirected if and only if $S = S^{-1}$. A Cayley digraph $\Gamma = \text{Cay}(G, S)$ is called normal if the subgroup $R(G)$ is

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a normal subgroup of the automorphism group of Γ .

The concept of normality of Cayley digraphs is known to be important for the study of arc-transitive graphs and half-transitive graphs (e.g. see [10]). In general, it is known to be difficult to determine the normality of Cayley digraphs. The only groups for which a complete classification of normal Cayley digraphs is available, are the cyclic groups of prime order and the groups of order twice a prime [1, 4]. Since Wang, Wang and Xu [9] obtained all normal disconnected Cayley digraphs of finite groups, we always suppose in this paper that the Cayley digraph $\text{Cay}(G, S)$ is connected. That is, S is a generating subset of G .

A subset S of G is said to be a CI-subset (CI stands for Cayley isomorphism) if whenever $\text{Cay}(G, T)$ is isomorphic to $\text{Cay}(G, S)$, there is an automorphism α of G such that $S^\alpha = T$. (Then we call the corresponding graph $\text{Cay}(G, S)$ a CI-graph.) Let G be a finite group and let S be a minimal generating set of G . Then Xu [10, Problem 6] asked the question:

- (1) Are S and $S \cup S^{-1}$ CI-subsets?
- (2) Are the corresponding Cayley digraphs and graphs normal?

For abelian groups, Feng and Gao [5] proved that if the Sylow 2-subgroup of G is cyclic, then the answers to both questions (1) and (2) are positive; otherwise, they are negative in general. More details can be found in [3, 6, 7].

Let X and Y be two graphs. The direct product $X \times Y$ is defined as the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X \times Y)$, $[u, v]$ is an edge in $X \times Y$ whenever $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$ or $y_1 = y_2$ and $[x_1, x_2] \in E(X)$. Two graphs are called relatively prime if they have no nontrivial common direct factor. The lexicographic product $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X[Y])$, $[u, v]$ is an edge in $X[Y]$ whenever $[x_1, x_2] \in E(X)$ or $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$. Let $V(Y) = \{y_1, y_2, \dots, y_n\}$. Then there is a natural embedding of nX in $X[Y]$, where for $1 \leq i \leq n$, the i th copy of X is the sub-graph induced on the vertex subset $\{(x, y_i) | x \in V(X)\}$ in $X[Y]$. The deleted lexicographic product $X[Y] - nX$ is the graph obtained by deleting all the edges of (this natural embedding of) nX from $X[Y]$.

In the first theorem we determine all non-normal connected Cayley digraphs of valency 2 on the groups of order pq , where p and q are prime numbers. From elementary group theory we know that up to isomorphism there are two groups of order pq ($p < q$) defined as: $G_1 = \mathbb{Z}_{pq}$, and $G_2 = \langle x, y \mid x^p = y^q = 1, x^{-1}yx = y^r \rangle$ where r, p, q satisfy $r^p \equiv 1 \pmod{q}$, $r \not\equiv 1 \pmod{q}$, $(r, p) = 1$. If $p \nmid q - 1$, then the second case can not occur. Moreover when $p = q$, then G is isomorphic to \mathbb{Z}_{p^2} or $\mathbb{Z}_p \times \mathbb{Z}_p$.

Theorem 1.1. *Let G be a group of order pq where p, q are distinct primes, and S a two-element generating subset of G . If $\Gamma = \text{Cay}(G, S)$ is not normal, then $\Gamma = \text{Cay}(G, S) \cong C_q[2K_1]$ and $\text{Aut}(\Gamma) \cong C_2 \wr C_q$, where C_q is a directed cyclic of length q .*

In the following theorem we determine the normality of Cayley digraphs of valency 2 on nonabelian finite groups G such that every proper subgroup of G is abelian. Note that Frattini subgroup $\Phi(G)$ is the intersection of maximal subgroups of group G .

Theorem 1.2. *Let G be a finite minimal non-abelian group, such that $2 \nmid |G|$ and $G/\Phi(G)$ is abelian. Moreover suppose that $\Gamma = \text{Cay}(G, S)$ be a 2-valent connected Cayley digraph of G with respect to $S = \{e, f\}$. Then Γ is normal.*

2. Primary Analysis

Let G be a finite group and let $\Gamma = \text{Cay}(G, S)$ be a connected Cayley (di)graph of valency two. Denote by $(g, z_1g, z_2z_1g, \dots, z_{n-1}z_{n-2}\dots z_1g, z_nz_{n-1}\dots z_1g)$ a cycle of length n in Γ , where $z_i \in S$ ($1 \leq i \leq n$). Obviously $z_{i+l}\dots z_{i+1}z_i \neq 1$ ($1 \leq i \leq i+l \leq n$), except $z_nz_{n-1}\dots z_1 = 1$. For simplicity we use $C_g(z_nz_{n-1}\dots z_1) (= C_g(1))$ to denote this cycle.

We give some results which will be used later in the proofs of Theorems 1.1 and 1.2.

Proposition 2.1. *[2, Proposition 2.5] Let $\Gamma = \text{Cay}(G, S)$ and $\Gamma' = \text{Cay}(G, S^\alpha)$ where $\alpha \in \text{Aut}(G)$. Then Γ is normal if and only if Γ' is normal.*

Proposition 2.2. *[10, Proposition 1.5] Let $\Gamma = \text{Cay}(G, S)$ and $A = \text{Aut}(\Gamma)$. Then Γ is normal if and only if $A_1 = \text{Aut}(G, S)$, where A_1 is the stabilizer of 1_G in A , and $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$.*

Proposition 2.3. *[3, Lemma 2.2] Let G be a finite group and $S = \{e, f\}$ be a two-element generating subset of G not containing the identity 1_G . Set $\Gamma = \text{Cay}(G, S)$ and $A = \text{Aut}(\Gamma)$. If A_1^* denotes the subgroup of A which fixes $1_G, e$ and f , then we have the following:*

- (i) $A_1^* = 1$ if and only if $\Gamma = \text{Cay}(G, S)$ is normal.
- (ii) Let $o(e^{-1}f) = m$ be odd. Suppose that $\langle d \rangle \subseteq \langle e^{-1}f \rangle$ or $\langle e^{-1}f \rangle \subseteq \langle d \rangle$, where $d = f(e^{-1}f)^{(m-1)/2}$. Then $\Gamma = \text{Cay}(G, S)$ is normal.
- (iii) Suppose $\alpha \in A_1^*$. If $o(e^{-1}f) = m$ is odd, then α fixes at least $\langle d \rangle \langle e^{-1}f \rangle$ point-wise, where $d = f(e^{-1}f)^{(m-1)/2}$.
- (iv) If one of e and f has order 2, then $\Gamma = \text{Cay}(G, S)$ is normal.

A finite group G is called minimal non-abelian if G is not abelian, but every proper subgroup of G is abelian. Now we have the following result.

Proposition 2.4. [8] *Let G be a finite minimal non-abelian group. Then one of the following holds:*

- (1) G is a minimal non-abelian p -group;
- (2) G is a semi-direct product of an elementary abelian p -group P of order p^α by a cyclic group $Q = \langle b \rangle$ of order q^β , where p, q are distinct primes and the action of b on P is an automorphism of P of order q .

Proposition 2.5. [2, Theorem 1.1] *Let G be a finite abelian group and S a generating subset of G not containing the identity 1. Assume S satisfies the condition*

$$x, y, u, v \in S \text{ with } 1 \neq xy = uv, \text{ implies } \{x, y\} = \{u, v\}.$$

(1)

Then the Cayley (di)graph $\text{Cay}(G, S)$ is normal.

3. Proof of Theorem 1.1

Note that if $S = S^{-1}$, then $\Gamma = \text{Cay}(G, S) \cong C_n$ (the cycle of length n), and obviously Γ is normal. Therefore we may let $S \neq S^{-1}$.

Now we complete the proof of Theorem 1.1. Let G be a group of order pq , where p and q are primes, and $S = \{e, f\}$ be a generating subset for G with $e \neq f$ and $1 \notin S$.

Assume first that p does not divide $q - 1$, so G is cyclic, and $G \cong \mathbb{Z}_{pq}$. In view of Proposition 2.5, we assume S does not satisfy the condition (1). Thus there are (not necessarily distinct) elements x, y, u, v in S with $1 \neq xy = uv$ and $\{x, y\} \neq \{u, v\}$. If one of e and f has order 2, then by Proposition 2.3 (iv), $\Gamma = \text{Cay}(G, S)$ is normal. Thus $e^2 \neq 1$ and similarly $f^2 \neq 1$. If, say, $\{x, y\} = \{e, f\}$ then since $\{x, y\} \neq \{u, v\}$ we must have $\{u, v\} = \{e\}$ or $\{f\}$ but this implies that $xy \neq uv$, a contradiction. It follows that $x = y = e$, say, and $u = v = f$ and $e^2 = f^2 \neq 1$. Since $e \neq f$, one of these elements, say e , must have even order, so p , say, is equal to 2. Now G is isomorphic to $\mathbb{Z}_{2q} = \langle a \rangle$. So one may let $S = \{a^i, a^{q+i}\}$ where $(i, 2q) = 1$. Since $a \mapsto a^i$, can be extended to an automorphism of G , so by Proposition 2.1, we can assume that $S = \{a, a^{q+1}\}$ and $\Gamma = \text{Cay}(G, S) \cong C_q[2K_1]$. In this case $\sigma : a^i \mapsto a^{q+i}$ ($1 \leq i < 2q, i \neq q$) such that $\sigma(1) = (1)$, $\sigma(a^q) = (a^q)$ is not in $\text{Aut}(G, S)$ but in A_1 . So by Proposition 2.2 Γ is not normal. Also $\text{Aut}(\Gamma) \cong C_2 \wr C_q$.

Now let G be nonabelian. Thus we may assume that p divides $q - 1$ and G is isomorphic to $\langle x, y \mid x^p = y^q = 1, x^{-1}yx = y^r \rangle$, where r, p , and q satisfy $r^p \equiv 1 \pmod{q}$, $r \not\equiv 1 \pmod{q}$, and $(r, p) = 1$. By simply checking, we can see that, the elements of order p are $\{x^i y^j\}$ ($1 \leq i \leq p - 1, 0 \leq j \leq q - 1$), and the elements of order q are y^i , where $1 \leq i \leq q - 1$, thus we will see easily that e and f cannot be of order q simultaneously. Thus we can suppose that $o(e) = p$ and $o(f) = p$ or q . If $o(e) = p$ and $o(f) = q$, we have $e = x^\alpha y^\beta$, ($1 \leq \alpha \leq p - 1, 0 \leq \beta \leq q - 1$) and $f = y^\delta$, $1 \leq \delta \leq q - 1$. We claim that there is only one directed cycle of length p through every vertex g of Γ . We can suppose $g = 1$ because of the transitivity of

Γ .

Obviously there is a directed cycle of length p through 1: $1 \mapsto e \mapsto e^2 \mapsto \dots \mapsto e^{p-1} \mapsto 1$. Let $C_1(d_p d_{p-1} \dots d_1)$ be another directed cycle of length p through 1, where $d_i = e$ or f ($1 < i < p$). Let n be the number of e appearing in the product $d_p \dots d_1$. Obviously $1 \leq n < p$, and $d_p \dots d_1 = 1$. Therefore $x^{\alpha n} y^t = 1$ (for some integer t). It is clear that for any $g \in G$, g can be expressed in the unique form: $x^i y^j$ ($0 \leq i < p$, $0 \leq j < q$). Thus $n \equiv 0 \pmod{p}$, a contradiction to $1 \leq n < p$. Suppose $\sigma \in A_1^*$. Since there is only one directed cycle of length p through f : $f \mapsto ef \mapsto e^2 f \mapsto \dots \mapsto e^p f = f$, we have that σ fixes $\Gamma_1(e)$ and $\Gamma_1(f)$ pointwise. From the connectivity of Γ we have $\sigma = 1$, so $A_1^* = 1$. By Proposition 2.3 (i) $\Gamma = \text{Cay}(G, S)$ is normal.

Now suppose that $o(e) = p$, $o(f) = p$. Then $e = x^i y^j$ ($1 \leq i < p$, $0 \leq j < q$). Since σ , defined by: $x^i y^j \mapsto x, y \mapsto y$ can be extended to automorphism of G , by Proposition 2.1 one may let $e = x$. If $f = x^\alpha y^\beta$ ($1 \leq \alpha < p$, $0 \leq \beta < q$) so $e^{-1} f = x^{\alpha-1} y^\beta$ ($0 \leq \alpha-1 < p$, $0 \leq \beta < q$). Suppose that $\alpha-1 = 0$, therefore $e^{-1} f = y^\beta$ and $o(e^{-1} f) = q$. We can also suppose that $\langle d \rangle \not\subseteq \langle e^{-1} f \rangle$ by Proposition 2.3 (ii), where $d = f(e^{-1} f)^{(q-1)/2}$. Thus $\langle d \rangle \langle e^{-1} f \rangle = G$. By Proposition 2.3 (iii) A_1^* fixes $G = \langle d \rangle \langle e^{-1} f \rangle$ and so $\Gamma = \text{Cay}(G, S)$ is normal. Now suppose that $\alpha-1 > 0$, so $o(e^{-1} f) = p$, the same as before we can suppose that $\langle d \rangle \not\subseteq \langle e^{-1} f \rangle$, where $d = f(e^{-1} f)^{(p-1)/2}$. Thus $G = \langle d \rangle \langle e^{-1} f \rangle$ and A_1^* fixes $G = \langle d \rangle \langle e^{-1} f \rangle$ and so $\Gamma = \text{Cay}(G, S)$ is normal. The result now follows.

4. Proof of Theorem 1.2

Recall that we use A to denote the automorphism group of the Cayley graph Γ , and A_1 to denote the group of all automorphisms of Γ that fix identity 1_G of G . Also a finite group G is called minimal non-abelian if G is not abelian, but every proper subgroup of G is abelian.

We now prove the Theorem 1.2. By Proposition 2.4 we have two cases. First suppose that G is a p -group (p is odd), in this case we assume G has the smallest order such that $\text{Cay}(G, S)$ is not a normal 2-valent connected Cayley digraph. Note that $A = R(G)A_1$ and $R(G) \cap A_1 = 1_G$. Let N be a nontrivial minimal normal subgroup of A . Since A_1 is a 2-group, A is solvable and hence N is an elementary abelian q -group, with $q=2$ or p . If N is transitive on $V(\Gamma)$ then N is regular on $V(\Gamma)$. Thus $|N| = |G| = |R(G)|$. Therefore $R(G) = N$ and $R(G)$ is normal in A which is a contradiction. Thus N is not transitive on $V(\Gamma)$. Let m be the length of an N -orbit on $V(\Gamma)$, then m divides $|N|$. Note that $N \triangleleft A$ and A acts transitively on $V(\Gamma)$, so m divides $|G|$. Therefore $q=p$ and $N < R(G)$.

Let $\Sigma = \{T_1, T_2, \dots, T_{p^\alpha}\}$ be all the orbits of N on $V(\Gamma)$. Consider the quotient digraph Γ_N defined by $V(\Gamma) = \Sigma$ and $(T_i, T_j) \in E(\Gamma_N)$ if and only if there exist $v_i \in T_i$, $v_j \in T_j$ such that $(v_i, v_j) \in E(\Gamma)$. Since the quotient group $R(G)/N$ acts regularly on Σ , therefore Γ_N must be a connected Cayley digraph of G/N . So $\Gamma_N \cong \text{Cay}(G/N, SN/N)$ ($|S \cap N| \leq 1$). If $|S \cap N| = 1$, then either e or f has order p . Without loss of generality we may assume that $o(e) = p$. Now we have two cases:

(1) $o(f)=p$ and (2) $o(f) \neq p$.

We know that there is a directed p -path through 1: $1_G \mapsto e \mapsto e^2 \mapsto \dots \mapsto e^p$. Suppose that $1_G \mapsto d_1 \mapsto d_2d_1 \mapsto \dots \mapsto d_pd_{p-1}\dots d_1$ is another p -path through 1 such that $d_pd_{p-1}\dots d_1=1$ and $d_i=f$ or e . Let r, t be the numbers of e and f appearing in $\{d_1, d_2, \dots, d_p\}$ respectively. Since G is a non-cyclic p -group, $G/\Phi(G)$ is an elementary p -group where $\Phi(G)$ is the Frattini subgroup of G , and $G/\Phi(G)=\langle \Phi(G)e \rangle \times \langle \Phi(G)f \rangle$. We have $\Phi(G)d_p\Phi(G)d_{p-1}\dots\Phi(G)d_1=\Phi(G)$, so $(\Phi(G)e)^r(\Phi(G)f)^t=\Phi(G)$, and hence $r = t = 0 \pmod{p}$. Note that $r + t=p$ and $0 \leq r, t \leq p$, therefore we have that either $r = p$ and $t = 0$ or $r = 0$ and $t = p$. Now if $o(f)=p$ then $A_1^*=1_G$ and Γ is normal by Proposition 2.3 (i). If $o(f) > p$, similarly there is a unique directed p -cycle through f : $f \mapsto ef \mapsto e^2f \mapsto \dots \mapsto f$ and consequently $A_1^*=1_G$, and Γ is normal.

Now suppose that $|S \cap N|=0$ and G/N is non-abelian. By the minimality of $|G|$, $\text{Cay}(G/N, SN/N)$ is normal. Therefore $R(G)/N \triangleleft A/N$. Since $R(G)$ is the full preimage of $R(G)/N$ under $A \mapsto A/N$, $R(G)$ is normal in A , which is a contradiction. Now suppose that G/N is abelian. Suppose $\sigma \in A/N$, such that $\sigma(1) = 1$, $\sigma(Ne) = Ne$, and $\sigma(Nf) = Nf$. It is easy to show that σ fixes $\Gamma_1(e)$, and $\Gamma_1(f)$ pointwise. Since Γ is connected, $\sigma=1$. Therefore by Proposition 2.3 (i) Γ_N is normal, and so under $A \mapsto A/N$, Γ is normal. This is a contradiction.

Now let $|G|=p^\alpha q^\beta$, where P and Q are Sylow p -subgroups and Sylow q -subgroup of G , respectively such that $P \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \dots \times \mathbb{Z}_p$, and Q is a cyclic group. We consider the quotient group $R(G)/P$, and suppose that $\Sigma = \{B_1, B_2, \dots, B_n\}$ be all the orbits of P on $V(\Gamma)$. $R(G)/P$ acts on Σ regularly, therefore the quotient graph Γ_P is isomorphic to $\text{Cay}(G/P, SP/P)$ ($|S \cap P| \leq 1$). Since $G/\Phi(G)$ is abelian with the same reason as before one can show Γ is normal, which is a contradiction and the proof of Theorem 1.2 is complete.

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