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C-CONFORMAL METRIC TRANSFORMATIONS ON FINSLERIAN HYPERSURFACE

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Abstract. The purpose of the paper is to give some relation between the original Finslerian hypersurface and other C-conformal Finslerian hypersufaces. In this paper we define three types of hypersufaces, which were called a hyperplane of the 1^{st} kind, hyperplane of the 2^{nd} kind and hyperplane of the 3^{rd} kind under consideration of C-conformal metric transformation.

Key words: Finsler spaces, Finsler hypersurface, Conformal, C-conformal, Hyperplane of 1^{st} kind, 2^{nd} kind and 3^{rd} kind.

Abstrak. Tujuan dari paper ini adalah untuk memberikan beberapa kaitan antara hypersurface Finsler asal dengan hypersufaces C-konformal Finsler yang lain. Dalam tulisan ini kami mendefinisikan tiga jenis hypersufaces, yang disebut hyperplane jenis pertama, hyperplane jenis kedua dan hyperplane jenis ketiga berdasarkan transformasi metrik C-konformal.

Kata kunci: Ruang Finsler, hypersurface Finsler, konformal, C-konformal, hyperplane jenis pertama, jenis kedua dan jenis ketiga.

1. Introduction

The conformal theory and its related concepts of Finsler spaces was initiated by Knebelman in 1929. M. Hashiguchi [1] introduced a special change called C-conformal change which satisfies C-condition. The theory of Special Finsler spaces and their properties were studied by M. Matsumoto [8], C. Shibata [13] et al and authors like H. Izumi [2], S. Kikuchi [4] et al have given the condition for Finsler space to be conformally flat. C. Shibata and H. Azuma [13] have studied C-conformal

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invariant tensor of Finsler metric. The author M. Kitayama ([5], [6], [7]) have studied Finsler spaces admitting a parallel vector field and also studied Finslerian hypersurface and metric transformations. The authors H.G. Nagaraja, C.S. Bagewadi and H. Izumi [9] have published a paper on infinitesimal h-conformal motions of Finsler metric.

The authors S.K. Narasimhamurthy and C.S. Bagewadi ([10], [11]) have published a paper on C-conformal Special Finsler spaces admitting a parallel vector field and the same authors have also studied on Infinitesimal C-conformal motions of special Finsler spaces.

Throughout the paper, terminology and notations are referred to [1], [8] and [12].

2. Preliminaries

A Finsler space, we mean a triple $F^n = (M, D, L)$, where M denotes n-dimensional differentiable manifold, D is an open subset of a tangent vector bundle TM endowed with the differentiable structure induced by the differentiable manifold TM and $L : D \to R$ is a differentiable mapping having the properties

$$\begin{aligned} &i) \quad L(x,y) > 0, \quad for (x,y) \in D, \\ &ii) \quad L(x,\lambda y) = |\lambda| L(x,y), \quad for\, any\, (x,y) \in D\, and\, \lambda \in R,\; such\, that\; (x,\lambda y) \in D, \\ &iii) \quad g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad (x,y) \in D,\, is\, positive\, definite,\, where\, \dot{\partial}_i = \frac{\partial}{\partial y^i}. \end{aligned}$$

The metric tensor $g_{ij}(x,y)$ and Cartan's C-tensor C_{ijk} are given by [12]:

$$\begin{aligned} g_{ij}(x,y) &= \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad g^{ij} = (g_{ij})^{-1}, \\ C_{ijk} &= \frac{1}{2}\dot{\partial}_k g_{ij}, \quad C^i_{jk} = \frac{1}{2} g^{im}(\dot{\partial}_k g_{mj}), \end{aligned}$$

where $\dot{\partial}_j = \frac{\partial}{\partial y^i}$ and $\dot{\partial}_i = \frac{\partial}{\partial x^i}$. We use the following [12]:

$$\begin{aligned} &a) \quad l_i = \dot{\partial}_i L, \qquad l^i = y^i/L, \qquad h_{ij} = g_{ij} - l_i l_j, \\ &b) \quad \gamma^i_{jk} = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{rj} + \partial_r g_{jk}), \\ &c) \quad G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k, \quad G^i_j = \dot{\partial}_j G^i, \quad G^i_{jk} = \dot{\partial}_k G^i_j, \quad G^i_{jkl} = \dot{\partial}_l G^i_{jk}, \qquad (1) \\ &d) \quad F^i_{jk} = \frac{1}{2} g^{ir}(\delta_j g_{rk} + \delta_k g_{rj} - \delta_r g_{jk}), \\ &e) \quad N^i_j = N^i_j - y_j \sigma^i + \sigma_0 \delta^i_j + \sigma_j y^i, \end{aligned}$$

where $\delta_j = \partial_j - G^r_j \partial_r$.
The Berwald connection and the Cartan connection of F^n are given by $B\Gamma = (G^i_{jk}, N^i_j, 0)$ and $C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ respectively.

A hypersurface M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation

$$x^i = x^i(u^\alpha),$$

where u^α are Gaussian coordinates on M^{n-1} and Greek indices take values 1 to n-1. Here we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \partial x^i/\partial u^\alpha$ is of rank (n-1). The following notations are also employed [6]:

$$B^i_{\alpha\beta} = \partial x^i/\partial u^\alpha \partial u^\beta, \quad B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}, \quad B^{ij....}_{\alpha\beta....} = B^i_\alpha B^j_\beta.....$$

If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write

$$y^i = B^i_\alpha(u)v^\alpha,$$

i.e., v^α is thought of as the supporting element of M^{n-1} at a point (u^α). Since the function $\underline{L}(u, v) = L(x(u), y(u, v))$ gives rise to a Finsler matrix of M^{n-1}, we get a (n-1)-dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u, v))$.

At each point (u^α) of F^{n-1}, the unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \qquad g_{ij}N^iN^j = 1. \tag{2}$$

If (B^i_α, N_i) is the inverse matrix of (B^α_i, N^i), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad N^iB^\alpha_i = 0, \quad N^iN_i = 1,$$

and further

$$B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j.$$

Making use of the inverse $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \qquad N_i = g_{ij}N^j.$$

For the induced Cartan connections $IC\Gamma = (F^\alpha_{\beta\gamma}, N^\alpha_\beta, C^\alpha_{\beta\gamma})$ on F^{n-1}, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature tensor H_α are given by

$$i) \quad H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^{jk}_{\alpha\beta}) + M_\alpha H_\beta, \tag{3}$$

$$ii) \quad H_\alpha = N_i(B^i_{0\alpha} + N^i_jB^j_\alpha),$$

respectively, where $M_\alpha = C_{ijk}B^i_\alpha N^jN^k$ and $B^i_{0\alpha} = B^i_{\beta\alpha}v^\beta$. Transvecting $H_{\alpha\beta}$ by v^β, we get $H_{0\alpha} = H_{\beta\alpha}v^\beta = H_\alpha$.

Further more we have to put

$$M_{\alpha\beta} = C_{ijk}B^{ij}_{\alpha\beta}N^k. \tag{4}$$

3. C-Conformal Finsler Space

We shall consider conformal change of a Finsler metric formed by $L \to \overline{L} = e^{\sigma(x)}L$, where σ is conformal factor depends on the point x only and under this change we have another Finsler space $\overline{F}^n = (M^n, \overline{L})$ on the same underlying manifold M^n.

M. Hashiguchi [1] introduced the special change named C-conformal change which is by definition, a non-homothetic conformal change satisfying

$$C^i_{jk}\sigma^j = 0, \tag{5}$$

where $C^i_{jk} = g^{im}(\dot{\partial}_j g_{km})/2$, $\sigma^i = g^{im}\sigma_m$, $\sigma_m = \partial\sigma/\partial x^m$, $\sigma^j = g^{ij}\sigma_j$. From (1) and by symmetry of lower indices of C_{ijk}, we have

$$C_{ijk}\sigma^i = C_{jik}\sigma^i = C_{jki}\sigma^i = 0,$$

also we have

$$C^k_{ij}\sigma^i = C^k_{ij}\sigma^j = C^i_{jk}\sigma^k = 0.$$

In the following the quantity with bar will be defined in C-conformal Finsler space $\overline{F}^n$, and the quantity without bar will be defined in Finsler space F^n.
Under the C-conformal change, we have the following [2], [13]:

$$\begin{aligned}
&a) \quad \overline{g}_{ij} = (\overline{L}/L)^2 g_{ij}, \quad \overline{g}^{ij} = (L/\overline{L})^2 g^{ij},\\
&b) \quad \overline{y_i} = (\overline{L}/L)^2 y_i,\\
&c) \quad \overline{C}_{ijk} = C_{ijk}, \quad \overline{C}^i_{jk} = e^{2\sigma}C^i_{jk}, \quad \overline{C}_i = e^{-2\sigma}C_i,\\
&d) \quad \overline{\gamma}^i_{jk} = \gamma^i_{jk} + (\sigma_j\delta^i_k + \sigma_k\delta^i_j - g_{jk}\sigma^i),\\
&e) \quad \overline{G}^i = G^i - \frac{1}{2}L^2\sigma^i + \sigma_0 y^i,\\
&f) \quad \overline{G}^i_{jk} = G^i_{jk} - g_{jk}\sigma^i + \sigma_k\delta^i_j + \sigma_j\delta^i_k,\\
&g) \quad \overline{N}^i_j = N^i_j - y_j\sigma^i + \sigma_0\delta^i_j + \sigma_j y^i,\\
&h) \quad \overline{F}^i_{jk} = F^i_{jk} - g_{jk}\sigma^i + \sigma_k\delta^i_j + \sigma_j\delta^i_k + \sigma_0 C^i_{jk}.
\end{aligned} \tag{6}$$

4. Hypersurface Given by a C-Conformal Change

We now consider a Finsler hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of the Finsler space F^n and another Finsler hypersurface $\overline{F}^{n-1} = (M^{n-1}, \overline{\underline{L}}(u,v))$ of the Finsler space F^n given by the C-conformal change.

Let $N^i(u,v)$ be a unit normal vector at each point of the F^{n-1}, and as component of n-1 linearly independent tangent vectors of F^{n-1} and they are invariant under the C-conformal change. Thus we shall show that a unit normal vector $\overline{N}^i(u,v)$ of $\overline{F}^{n-1}$ is uniquely determined by

$$\overline{g}_{ij}B^i_\alpha\overline{N}^j = 0, \qquad \overline{g}_{ij}\overline{N}^i\overline{N}^j = 1. \tag{7}$$

By means of (2) and (6), we get

$$\overline{g}_{ij}(\pm e^{-\sigma}N^i)(\pm e^{-\sigma}N^j) = 1.$$

Therefore we can put

$$\overline{N}^i = e^{-\sigma}N^i,$$

where we have chosen the sign '+' in order to fix an orientation. It is obvious that $\overline{N}_i(u,v)$ satisfies (2), hence we obtain:

Lemma 4.1. *For a field of linear frame* $(B^i_1, B^i_2, ...B^i_{n-1}, N^i)$ *of* F^n*, there exists a field of linear frame* $(B^i_1, B^i_2, ...B^i_{n-1}, \overline{N}^i = e^{-\sigma}N^i)$ *of the* $\overline{F}^n$ *given by the C-conformal change such that (7) satisfied along* $\overline{F}^{n-1}$*.*

The quantities $\overline{B}^\alpha_i$ are uniquely defined along $\overline{F}^{n-1}$ by

$$\overline{B}^\alpha_i = \overline{g}^{\alpha\beta}\overline{g}_{ij}B^j_\beta,$$

where $(\overline{g}^{\alpha\beta})$ is the inverse metric of $(\overline{g}_{\alpha\beta})$. If $(\overline{B}^\alpha_i, \overline{N}^i)$ is the inverse vector of $(\overline{B}^i_\alpha, \overline{N}_i)$, then we have

$$B^i_\alpha \overline{B}^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha \overline{N}_i = 0, \quad \overline{N}^i \overline{B}^\alpha_i = 0, \quad \overline{N}^i \overline{N}_i = 1,$$

and also

$$B^i_\alpha \overline{B}^\alpha_j + \overline{N}^i \overline{N}_j = \delta^i_j.$$

Also we get $\overline{N}_i = \overline{g}_{ij}\overline{N}^j$, that is

$$\overline{N}_i = e^\sigma N_i. \tag{8}$$

We have from (6(e)),

$$D^i = \overline{G}^i - G^i = \sigma_0 y^i - \frac{L^2}{2}\sigma^i, \quad where \ \ \sigma_0 = \sigma_r y^r. \tag{9}$$

Differentiating (9) by y^j and from (6(f)), we obtain

$$\begin{aligned} D^i_j &= D^i_{(j)}, \\ &= \overline{G}^i_j - G^i_j, \\ &= \overline{N}^i_j - N^i_j, \\ &= -y_j\sigma^i + \sigma_0\delta^i_j + \sigma_j y^i, \end{aligned}$$

where $D^i_{(j)} = \dot{\partial}_j D^i$. From (9), we have

$$N_i D^i = \sigma_0 N_i y^i - \frac{L^2}{2}N_i\sigma^i.$$

We assume that $N_i\sigma^i = 0$. i.e., $\sigma^i(x)$ is tangential to F^{n-1} and using the condition $N_i y^i = 0$, then we have

$$N_i D^i = 0. \tag{10}$$

Differentiating (10) by y^j, we have

$$N_i D^i_{(j)} + D^i (N_i)_{(j)} = 0,$$
$$N_i D^i_j + D^i (\dot{\partial}_j N_i) = 0.$$

Transvecting above equation by B^j_α, we get

$$N_i D^i_j B^j_\alpha + D^i (\dot{\partial}_j N_i) B^j_\alpha = 0,$$
$$N_i D^i_j B^j_\alpha = 0, \tag{11}$$

where we used

$$B^j_\alpha (\dot{\partial}_j N_i) = M_\alpha N_i = C_{ijk} B^j_\alpha N^i N^k N_i = 0.$$

Definition 4.1. *If each path of the hypersurface F^{n-1} with respect to the induced connection is also a path of the ambient space F^n, then F^{n-1} is called a 'hyperplane of the 1^{st}kind'.*

A hyperplane of the 1^{st}*kind* is characterized by $H_\alpha = 0$.
From (3(ii)) and using (8), we have

$$\overline{H}_\alpha = \overline{N}_i (B^i_{0\alpha} + \overline{N}^i_j B^j_\alpha).$$

Thus

$$\begin{aligned}
\overline{H}_\alpha - e^\sigma H_\alpha &= \overline{N}_i (B^i_{0\alpha} + \overline{N}^i_j B^j_\alpha) - e^\sigma N_i (B^i_{0\alpha} + N^i_j B^j_\alpha),\\
&= e^\sigma (N_i B^i_{0\alpha} + N_i \overline{N}^i_j B^j_\alpha) - e^\sigma (N_i B^i_{0\alpha} + N_i N^i_j B^j_\alpha),\\
&= e^\sigma N_i (\overline{N}^i_j - N^i_j) B^j_\alpha,\\
&= e^\sigma N_i D^i_j B^j_\alpha.
\end{aligned}$$

Thus we have

$$\overline{H}_\alpha = e^\sigma (H_\alpha + N_i D^i_j B^j_\alpha).$$

Thus from (11), we obtained

$$\overline{H}_\alpha = e^\sigma H_\alpha.$$

Hence we state the following:

Theorem 4.1. *A Finsler hypersurface F^{n-1} is a hyperplane of 1^{st} kind if and only if C-conformal Finsler hypersurface $\overline{F}^{n-1}$ is a hyperplane of 1^{st} kind, provided $N_i \sigma^i = 0$, i.e., $\sigma^i(x)$ is tangential to F^{n-1}.*

Now from (6(h)), the so called difference tensor D^i_{jk} has the following form

$$\begin{aligned}
D^i_{jk} &= \overline{F}^i_{jk} - F^i_{jk},\\
&= -g_{ij}\sigma^i + \sigma_k \delta^i_j + \sigma_j \delta^i_k + \sigma_0 C^i_{jk}.
\end{aligned}$$

Contracting above equation by N_i and B^j_α, we get

$$\begin{aligned}
N_i D^i_{jk} B^j_\alpha &= -N_i g_{jk} \sigma^i B^j_\alpha + \sigma_k N_i \delta^i_j B^j_\alpha + \sigma_j N_i \delta^i_k B^j_\alpha + \sigma_0 C^i_{jk} N_i B^i_\alpha,\\
&= 0.
\end{aligned}$$

Where we use $\sigma_0 = \sigma_i y^i$ and equation (5). Thus we state the following:

Lemma 4.2. *Assuming that $\sigma_i(x)$ is tangential to F^{n-1}, then the tensor $N_i D^i_{jk} B^j_\alpha$ is vanishes if and only if it satisfies (5).*

Definition 4.2. *If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the ambient space F^n, then F^{n-1} is called a 'hyperplane of the 2^{nd} kind'.*

A hyperplane of the 2^{nd} kind is characterized by $H_{\alpha\beta} = 0$.
From (3(i)), we have

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk} B^{jk}_{\alpha\beta}) + M_\alpha H_\beta. \tag{12}$$

Under the C-conformal change, (12) can be written as

$$\overline{H}_{\alpha\beta} = \overline{N}_i(B^i_{\alpha\beta} + \overline{F}^i_{jk} B^{jk}_{\alpha\beta}) + \overline{M}_\alpha \overline{H}_\beta. \tag{13}$$

Using equations (12) and (13), we get

$$\begin{aligned} \overline{H}_{\alpha\beta} - e^\sigma H_{\alpha\beta} &= [\overline{N}_i(B^i_{\alpha\beta} + \overline{F}^i_{jk} B^{jk}_{\alpha\beta}) + \overline{M}_\alpha \overline{H}_\beta] \\ &\quad - e^\sigma N_i(B^i_{\alpha\beta} + F^i_{jk} B^{jk}_{\alpha\beta}) - e^\sigma M_\alpha H_\beta, \end{aligned} \tag{14}$$

using $\overline{M}_\alpha = M_\alpha$ and $\overline{H}_\alpha = e^\sigma H_\alpha$, we have

$$\overline{H}_{\alpha\beta} - e^\sigma H_{\alpha\beta} = e^\sigma N_i(\overline{F}^i_{jk} - F^i_{jk}) B^{jk}_{\alpha\beta},$$

that implies

$$\overline{H}_{\alpha\beta} - e^\sigma H_{\alpha\beta} = e^\sigma (N_i D^i_{jk} B^{jk}_{\alpha\beta}). \tag{15}$$

Thus by virtue of lemma (4.1), therefore we state the following:

Theorem 4.2. *A Finsler hypersurface F^{n-1} is a hyperplane of the 2^{nd} kind if and only if the C-conformal Finsler hypersurface $\overline{F}^{n-1}$ is a hyperplane of the 2^{nd} kind, provided $\sigma_i(x)$ is tangential to F^{n-1}.*

Definition 4.3. *If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, then F^{n-1} is called a 'hyperplane of the 3^{rd} kind'.*

A hyperplane of the 3^{rd} kind is characterized by $H_{\alpha\beta} = M_{\alpha\beta} = 0$.
From (4), under C-conformal change the tensor $M_{\alpha\beta}$ can be written as

$$\begin{aligned} \overline{M}_{\alpha\beta} &= \overline{C}_{ijk} \overline{B}^{ij}_{\alpha\beta} \overline{N}^k, \\ &= e^{-\sigma} C_{ijk} B^{ij}_{\alpha\beta} N^k, \\ &= e^{-\sigma} M_{\alpha\beta}. \end{aligned} \tag{16}$$

By characterization of hyperplane of the 3^{rd} kind and (15), we have $\overline{H}_{\alpha\beta} = \overline{M}_{\alpha\beta} = 0$.
Thus by virtue of lemma (4.1), we state the following:

Theorem 4.3. *A Finsler hypersurface F^{n-1} is a hyperplane of the 3^{rd} kind if and only if C-conformal Finsler hypersurface $\overline{F}^{n-1}$ is a hyperplane of the 3^{rd} kind, provided $\sigma_i(x)$ is tangential to F^{n-1}.*

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