

THE ANALYTICAL SOLUTIONS OF EUROPEAN OPTIONS ON SHARES PRICING MODELS

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Abstract: The Black-Scholes options formula is the breakthrough in valuating options prices. However, the formula is heavily based on several assumptions that are not realistic in practice. The extensions of the assumptions are needed to make options pricing model more realistic. This paper has reviewed the relaxation of the formula to European options on shares with the focus on its analytical solutions. The assumptions that are relaxed are non-dividends assumption, constant interest rate, constant volatility, and continuous time.

Keywords: Options, Options valuations, Analytical solutions, Black-Scholes formula.

Abstrak: Rumus opsi saham Black-Scholes merupakan terobosan dalam penentuan nilai suatu wahana keuangan derivatif opsi saham. Namun demikian, rumus ini didasari beberapa asumsi yang dalam praktiknya tidak realistis. Pengembangan asumsi tersebut diperlukan agar model penilaian harga opsi saham lebih realistis. Tulisan ini membahas relaksasi asumsi dalam rumus Black-Scholes terhadap opsi Eropa pada saham yang berfokus pada solusi analitis. Relaksasi asumsi yang dibahas merupakan asumsi tanpa dividen, suku bunga konstan, volatilitas tetap, dan waktu yang kontinu.

Kata kunci: Opsi, penilaian opsi saham, solusi analitis, rumus Black-Scholes.

Options, along with Forwards and Futures, are derivative instruments in which their values depend on the value of underlying assets. Options are also considered as contingent claims because the future payoff of the assets is contingent on the outcome of some uncertain event. There are two classes of option: put and call options. A call (put) options is a contract where the holder has the right, not the obligation, to exercise the option i.e. to buy (sell) the underlying assets for predetermined price at predetermined future date. In terms of types of options, a European option can only be exercised at the maturity date, while an American option can be exercised any time up to the maturity date.

Having rejected by *The Journal of Political Economy* and *The Review of Economics and Statistics*, the paper of Black and Scholes (1973), titled *The Pricing of Options and Corporate Liabilities* and published by the former journal after resubmitting for the second time, has been considered as the breakthrough in valuation of options in particular and other derivatives in general. The paper, along with the paper of Merton (1973) that focused on the underlying principles on option pricing model, brought the authors - Myron S. Scholes and Robert C. Merton- to the Nobel Prize in Economics in 1997. The model for European options proposed in the paper has been well known as the Black-Scholes (BS) formula.

The BS formula is an analytical solution of the BS partial differential equation (PDE):

$$\frac{\delta f}{\delta t} + rS \frac{\delta f}{\delta S} + \frac{1}{2} \sigma^2 S^2 \frac{\delta^2 f}{\delta S^2} = rf \quad [1]$$

This PDE holds for all different derivatives f with underlying assets S , and it has many solutions. Typically there are three solutions of the BS PDE: (1) analytical solution, (2) analytical approximation, and (3) numerical procedure. Analytical approximations are usually used in valuating American options because their boundary conditions are more complex than European options so that it is difficult to find the exact analytical formulas. The most well-known and important example of numerical procedures is binomial trees (Cox *et al.* 1979) that requires no specific assumptions. On the other hand, analytic solutions will result in exact formulas like the BS formula. The main advantage of the exact formula is that it is easy to use because we just need to plug in the required and known variables into them to valuating an option.

The BS model is derived especially for an option on equity or shares that pay no dividends. Formally, the BS model is built on several assumptions: (1) the risk-free interest rate is constant over time, (2) the stock price follows a random walk in continuous time i.e. lognormal distribution, (3) there are no dividends, (4) there are no transaction costs or taxes, (5) the securities are divisible, and (6) the short selling is allowed. The last three assumptions are quite general in finance and are also used in other models such as the Capital Asset Pricing Model (CAPM); they are based on the perfect capital market condition.

The extension of the model to American options has also had intensive interest. However, the valuation of the American type is considered more difficult than their European type counterpart. The nature of American options that could be exercised earlier than expiration time makes it difficult to find the boundary conditions and the optimal value. Furthermore, the research following the BS model could be classified into three main groups: application of the BS model to other than financial options; empirical testing of the model; and the relaxation of the assumptions of the model (Merton 1998). This paper will focus on the latter, especially on options on shares, and concentrate on the analytical solutions of the European type options with the reasons mentioned before. However, this paper is not intended to give the full derivation of the solutions or to be a complete historical review. Instead, the objective is to understand the ideas behind the development and the extension of the BS model.

BLACK-SCHOLES MODEL

The BS model assumes that share price, S , has a lognormal distribution. Therefore $\ln S$ is normally distributed. In fact, a log normal distribution is a case of a generalized Wiener process. Then a change in the share price could be modeled by:

$$dS = \mu S dt + \sigma S dz \quad [2]$$

or in terms of the percentage of return:

$$\frac{dS}{S} = \mu dt + \sigma dz \quad [3]$$

The process states that the change in share price is a function of two components – a drift and a stochastic component. The first component is its expected rate of return μ per unit time (dt), and the second component is volatility of the share price σ where dz is a Wiener process.

By expanding the process [2] into an Ito's process where the drift and stochastic components as a function of S and t :

$$dS = \mu(S, t) dt + \sigma(S, t) dz$$

and using Ito's lemma, it can be shown that an option which is a contingent claims could be modeled by:

$$df = \left(\frac{df}{dS} \mu S + \frac{df}{dt} + \frac{d^2 f}{2dS^2} \sigma^2 S^2 \right) dt + \frac{df}{dS} \sigma S dz \quad [4]$$

The BS formula derivation is basically based on no-arbitrage argument: if the option is correctly priced, no one could exploit sure profits by taking position in options and the underlying assets. This in turn allows the construction of a riskless portfolio by taking a long position in shares and a short position in the corresponding derivative instrument. Whatever the price of the underlying shares, the value of the portfolio will be known and unchanged in the future.

Let the portfolio consist of the short position in an option and the long position in a fraction of the share, $\delta f / \delta S$, with the notation $\delta(\cdot)$ meaning changes in short interval of time, then the value of the portfolio, Π , is

$$\Pi = -f + \frac{\delta f}{\delta S} S \quad [5]$$

and the change of Π ($\delta \Pi$) in the time interval δt is

$$\delta \Pi = -\delta f + \frac{\delta f}{\delta S} \delta S \quad [6]$$

Substitute [4] into [6] yields

$$\delta \Pi = \left(-\frac{\delta f}{\delta t} - \frac{\delta^2 f}{2\delta S^2} \sigma^2 S^2 \right) \delta t \quad [7]$$

However, under risk-neutral assumption, the portfolio must earn risk-free interest rate then

$$\delta\Pi = r\Pi\delta t \quad [8]$$

Therefore, using [7] and [5], equation [8] will yield the BS PDE as in equation [1] mentioned in the introduction section above:

$$-\left(\frac{\delta f}{\delta t} + \frac{\delta^2 f}{2\delta S^2}\sigma^2 S^2\right)\delta t = r(-f + \frac{\delta f}{\delta S}S)\delta t - \frac{\delta f}{\delta t} + rS\frac{\delta f}{\delta S} + \frac{1}{2}\sigma^2 S^2\frac{\delta^2 f}{\delta S^2} = rf$$

To solve [1], the boundary condition is needed. Subject to the boundary condition for a European call option, the expected value of the option at maturity is $E[(\max(S_T - K, 0))]$, where K is the exercise price. Again under risk-neutral world, the value of the call, c , is the expected value discounted at risk-free rate:

$$c = e^{-rT} E[\max(0, S_T - K)]$$

In case of a European put option, $p = e^{-rT} E[\max(0, K - S_T)]$. By defining $g(S_t)$ as the probability density function of S_t then

$$c = e^{-rT} \int_K^{\infty} (S_T - K)g(S_T)dS_T \quad [9]$$

The solution of [7] also satisfies the BS PDE is

$$c = SN(d_1) - Ke^{-rT}N(d_2) \quad [10]$$

where for a put option the solution is

$$p = Ke^{-rT}N(-d_2) - SN(-d_1) \quad [11]$$

$$\text{with } d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

where

- c = the value of an European call option as a function of the stock price S and time to maturity T ;
- p = the value of an European put option;
- $N(.)$ = the cumulative normal density function;
- K = the exercise price;
- T = the time to maturity;
- r = the risk-free (short-term) interest rate;
- σ^2 = the variance rate of return on the stock.

DIVIDENDS

In terms of options on shares, the non-dividend assumption in the original BS model is questionable. A share traded on an exchange usually entitles a dividend payment. Merton (1973) relaxed this assumption straight away. However, it is important to define the type of a dividend - either a known dollar income (discrete dividend) or a known yield (continuous dividend) - before taking it into account to the formula. Merton (1973) has dealt with continuous dividends.

The idea behind modified formulas is quite logical. The subtracting of the dividend from the share price is based on the well-known fact that the price should be dropped by the dividend at the ex-dividend date. Therefore, the share price should be adjusted before incorporated into valuating the corresponding options.

Let $D = \rho S$, where D is the dividend per share and its magnitude is defined as a fraction ($\rho > 0$) of a share's price (S). Merton (1973) found that the BS PDE could be

$$-\frac{\delta f}{\delta t} + (rS - D)\frac{\delta f}{\delta S} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 f}{\delta S^2} = rf$$

and the solutions of the PDE are

$$c = e^{-\rho T} SN(d_1) - Ke^{-rT} N(d_2) \tag{12}$$

$$p = Ke^{-rT} N(-d_2) - Se^{-\rho T} N(-d_1) \tag{13}$$

where $d_1 = \frac{\ln(S/K) + (r - \rho + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

Where the dividend is a known income and let $PV(D)$ be the present value of the dividends, the modified BS models on discrete dividends (Hull 2003) are:

$$c = (S - PV(D))N(d_1) - Ke^{-rT} N(d_2) \tag{14}$$

$$p = Ke^{-rT} N(-d_2) - (S - PV(D))N(-d_1) \tag{15}$$

where

$$d_1 = \frac{\ln((S - PV(D))/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

STOCHASTIC INTEREST RATE

When providing an alternative derivation of the BS formula, Merton (1973) assumed stochastic interest rate. However, Merton (1973) did not explicitly derive the analytical formula for the options model with stochastic interest rate. Instead, he proposed the general valuation formula (Equation 36 in Merton 1973: 166) where the BS formula is a special case of the general formula with constant interest rate. Using the Merton's approach and an additional assumption on interest rate process, Rabinovitch (1989) derived the explicit valuation formula. The idea to incorporate stochastic interest rate is captured into bond valuation where it is well known that the bond price is a function of interest rate and time to maturity.

Rabinovitch (1989) assumed that short-term interest rates follow a mean-reverting Ornstein-Uhlenbeck process. If r is the yield-to-maturity on a bond that pays one dollar in next instant, then r could be described by the Ornstein-Uhlenbeck process:

$$dr = q(m - r)dt + vdw \quad [16]$$

where $q(m-r)$ is the expected rate of return per unit time (dt), v is the volatility of the interest rate, and dw is a Wiener process. A parameter ρ can be defined as the correlation between the unanticipated changes in the interest rate and return of the share.

By utilizing Vasicek type interest rate, the current price of pure discount bond with time to maturity T , $P(T)$, can be written as

$$P(T) = A \exp(-rB)$$

where

$$B = (1 - \exp[-qT]) / q$$

$$A = \exp[k(B-T) - (vB/2)^2/q]$$

$$k = m + v\lambda q - (v/q)^2/2$$

$$\lambda = (\gamma - r) / \delta \quad \text{and } \gamma \text{ and } \delta^2 \text{ are bond's expected return and variance;}$$

The analytical solution of the Merton's general valuation formula is given by:

$$c = SN(d_1) - KP(T)N(d_2) \quad [17]$$

$$\text{where } d_1 = \frac{\ln[(S/K)P(T)] + \frac{T'}{2}}{\sqrt{T'}}, d_2 = d_1 - \sqrt{T'} \text{ and}$$

$$T' = \sigma^2 T + (T - 2B + (1 - \exp[-2qT]) / 2q)(v/q)^2 - 2\rho\sigma(T - B)v/q$$

Kim (2002) examined the other specific stochastic interest rate model other than mentioned above. There are two interesting findings of Kim (2002): none of the models outperforms another model and the performance of the stochastic models are not better than the original (constant interest rate) BS model.

NON-CONSTANT VOLATILITY

Compared to relaxation of other assumptions, non constant volatility has got more attention and intensively done. As mentioned in Hull (2003), this is because of the difficulty to calculating volatility needed as an input in the BS model. Rather than following a predictable pattern, volatility follows a stochastic process.

Theodorakakos (2001) classified two approaches used to incorporate non-constant volatility into options pricing model: deterministic volatility and stochastic volatility approaches. Deterministic approach assumes volatility as a deterministic function; while stochastic approach assumes volatility follows a stochastic process. Essentially, the development of non-constant volatility model is another form of not assuming log normal distribution that requires constant volatility in the model as in equation [2] [3].

Constant elasticity of variance (CEV) model of Cox and Ross (1976) is an example of the deterministic approach. This model, which assumes a change in the share price that pays dividend at yield q in a short interval of time (dS), could be modeled by:

$$dS = (r - q)Sdt + \sigma S^\alpha dz \quad [18]$$

where α is a positive constant and interpreted as correlation between the volatility and the share price. If the volatility is independent from the share price ($\alpha=1$), process [18] is equal to process [2] where $\mu=(r-q)$.

The call and put option formula based on CEV model depend on the value of α . Using the notation in Hull (2003), the formula for European call and put options when $0<\alpha<1$ are

$$c = S_0 e^{-qT} [1 - \chi^2(a, b + 2, c)] - K e^{-rT} \chi^2(c, b, a) \tag{19}$$

$$p = K e^{-rT} [1 - \chi^2(c, b, a)] - S_0 e^{-qT} \chi^2(a, b + 2, c) \tag{20}$$

and when $\alpha>1$:

$$c = S_0 e^{-qT} [1 - \chi^2(c, -b, a)] - K e^{-rT} \chi^2(a, 2 - b, c) \tag{21}$$

$$p = K e^{-rT} [1 - \chi^2(a, 2 - b, c)] - S_0 e^{-qT} \chi^2(c, -b, a) \tag{22}$$

where

$$a = \frac{[K e^{-(r-q)T}]^{2(1-\alpha)}}{(1-\alpha)^2 v} \quad b = \frac{1}{1-\alpha} \quad c = \frac{S^{2(1-\alpha)}}{(1-\alpha)^2 v} \quad \text{and} \quad v = \frac{\sigma^2}{2(r-q)(\alpha-1)} [e^{2(r-q)(\alpha-1)T} - 1]$$

$\chi^2(z, k, v)$ is the noncentral χ^2 distribution cumulative probability with non centrality parameters v and k degrees of freedom is less than z .

The most well known papers on stochastic volatility models are Hull & White (1987), Stein & Stein (1991) and Heston (1993). In stochastic volatility model, it needs another stochastic process to describe volatility. As addition of a stochastic process of share price (as equation [2]), Hull & White (1987) also defined the volatility distributes according to the following specific stochastic process:

$$d\sigma^2 = \phi\sigma^2 dt + \xi\sigma^2 dw \tag{23}$$

where ϕ and ξ may depend on σ and t but not on S . Stein & Stein (1991) and Heston (1993), on the other hand, define other alternative processes of the volatility respectively as follow:

$$d\sigma = -\gamma(\sigma - \theta)dt + kdw \tag{24}$$

$$d\sigma = \gamma(\theta - \sigma)dt + \sigma^{3/2} dw \tag{25}$$

where γ , θ and k are fixed constants; δx and δw are two independent Wiener processes. The different between stochastic volatility process offered by Heston (1993) and the other two is in terms of incorporating the correlation between share price and volatility: he allows the relationship, the others assume no relationship.

Hull & White's formula for a call option is given by:

$$c = \int_0^\infty C(\bar{V}) h(\bar{V} | \sigma_t^2) d\bar{V} \tag{26}$$

where \bar{V} is the average value of the variance rate, $C(\bar{V})$ is the call option price resulted form the BS formula (Equation [10]), replacing σ^2 with \bar{v} in calculating d_1

and d_2 , and $h(\bar{V} | \sigma_i^2)$ is the probability density function of \bar{V} in a risk-neutral world. Formula for a put is not explicitly given, but as said in their conclusion, the put formula could be getting through put-call parity relationship.

The Stein & Stein's formula of stochastic volatility process is quite complex and can be referred in Stein & Stein (1991). However, Heston's formula is given by:

$$c = S_0 P_1 - Ke^{-rT} P_2 \quad [27]$$

with the "risk-neutralized" probability P_1 and P_2 described as:

$$P_i = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \text{Re} \left[\frac{e^{-i\phi \ln[K]} f_i(x, \sigma, T; \phi)}{i\phi} \right] d\phi \quad i=1,2 \quad [28]$$

Again, the detail definition of input variables in calculating P_1 and P_2 may be seen in Heston (1993).

DISCONTINUOUS TIME

One of the BS assumptions is that trading is done continuously. It means that trading is done 24 hours a day and seven days a week without interrupting. In practice, this assumption is obviously not true. Merton (1976) is the first researcher that deals with this matter. The model proposed by him is well-known as the (random) jump diffusion model. The word 'jump' describes the discontinuity because trading 'jumps' from one point in time to another such as from Friday afternoon to the next Monday morning.

By defining λ as average number of jumps per year and k as average jump size measured as a percentage of the share price, the stochastic process of jump diffusion model is described as:

$$dS = (\mu - \lambda k) S dt + \sigma S dz + S dp \quad [29]$$

where λk is the average growth rate in the share from the jumps, dp is the Poisson process generating the jumps that independent to the process dz .

Hull (2003) stressed one important particular case of Merton's jump diffusion model where the logarithm of the size of the percentage jump is normally distributed with the standard deviation of s . With this case, call (put) options price derived by Merton's model, c' (p'), is a function of call (put) options price when derived by BS model with dividend yield q and modified variance rate and risk-free rate:

$$c' = \sum_{n=0}^{\infty} \frac{e^{-\lambda'T} (\lambda'T)^n}{n!} c \quad [30]$$

$$p' = \sum_{n=0}^{\infty} \frac{e^{-\lambda'T} (\lambda'T)^n}{n!} p \quad [31]$$

where $\lambda' = \lambda(1+k)$.

The modified variance and the risk-free rate are $\sigma^2 + \frac{ns^2}{T}$ and $r - \lambda k + \frac{n\gamma}{T}$ respectively, where $\gamma = \ln(1+k)$.

MIXED ASSUMPTIONS

Above assumptions have been discussed so far are treated independently to each other. However, the relaxation of several assumptions could be done at the same time such as (1) the stochastic volatility and stochastic interest rate model, (2) the stochastic volatility and jump-diffusion model, and (3) the stochastic volatility, stochastic interest rate and jump diffusion model. For example, Bakshi & Chen (1997) derived the first model, while Bates (1996) derived the second, and Bakshi *et al.* (1997) derived the latter.

Even though Bates (1996) originally derived the formula for options on exchange rate, the formula is also applicable in valuation options on shares. Using the fact that the forward price is a function of current share price and risk-free interest rate, the BS formula modified for stochastic volatility and jump-diffusion model for a European call option is approximated by:

$$c = e^{-r(T+\Delta t_1)} [S e^{rT} P_1 - KP_2] \tag{32}$$

where Δt_1 is the time between the last trading day and the delivery day. Again, the function of P_1 and P_2 are the same as Equation [28].

Using two-factor term structure of interest rates that is the instantaneous interest rate as a function of the risk aversion level and the current state of the economy, and using the fact that the stochastic volatility is driven by both the systematic and the idiosyncratic state variables, Bakshi & Chen (1997) found that the call options price is

$$c = S e^{-\eta} P_1 - KP(T)P_2 \tag{33}$$

Equation [32] looks similar to Equation [27] (and to some extent, to Equation [17] except that the first term in right-hand side equation is modified by the dividend adjustment function $e^{-\eta}$. The function of P_1 and P_2 are the same as Equation [28], while $P(T)$ is the same as Vasicek type interest rate discussed in section 4. S is described as stochastic process in Equation [34] below with $\lambda=0$. It is important to notice that the stochastic volatility process assumed by Bakshi & Chen (1997) is different to that of Hull & White (1987), Stein & Stein (1991) and Heston (1993) in the sense that the latter only use one systematic risk source. Furthermore, Bakshi & Chen’s model does not assume the perfect correlation between interest rate and the share price where it is observable in practice.

The model of stochastic volatility and interest rate and jump-diffusion of Bakshi *et al.* (1997) is the development of the model of stochastic volatility and interest rate of Bakshi & Chen (1997). The stochastic process of share price and the volatility used in both model are:

$$dS = (R(t) - \lambda \mu_j) S dt + \sigma S dz + J(t) S dp \tag{34}$$

$$d\sigma^2 = [\theta_v - \kappa_v \sigma^2] dt + \sigma_v \sigma dw \tag{35}$$

where $R(t)$ is the time- t instantaneous spot interest rate, λ is the frequency of jumps per year, $J(t)$ is the percentage jump size that is lognormally distributed with mean μ , dp is the Poisson process generating the jumps that independent to the process dz and dw , and θ , κ , σ , are the speed of adjustment, long-run mean and variance coefficient of σ . For non dividend paying share the modified BS formula incorporating stochastic volatility and interest rate and jump-diffusion of Bakshi *et al.* (1997) is

$$c = SP_1 - KP(T)P_2 \quad [36]$$

As mentioned before, the difference between S in Equation [33] and [36] is that the former is a special case of stochastic process of [34] with $\lambda=0$. In fact, the stochastic process in [34] and [35] are general processes in which the BS model, the stochastic interest rate BS model, and the stochastic volatility BS model are special cases of the processes.

DO THE MODIFIED MODELS PERFORM BETTER THAN THE ORIGINAL BS MODEL?

Bakshi *et al.* (1997) questioned if we gain from the extension of the BS model by relaxing some assumptions as discussed above. If there is any gain, they also questioned if the gain form a realistic feature compared to additional complexity incorporated by the modified models. To answer the research questions, Bakshi *et al.* (1997) use three criterions to evaluate the modified models: (1) the consistency of the implied structural parameters with the implied-volatility time series and interest-rate time series, (2) out-of-sample pricing errors, and (3) hedging errors. The modified models evaluated by Bakshi *et al.* (1997) are: (1) the BS model, (2) the stochastic interest rate model (SI), (3) stochastic volatility model (SV), (4) stochastic volatility and stochastic interest rate model (SVSI), (5) stochastic volatility random-jump model (SVJ), and as well as their proposed generalized model in the same paper, (6) stochastic-volatility stochastic-interest rate and random jumps model (SVSI-J).

By using 38,749 S&P 500 call options prices from June 1988 and May 1991, they found that, in terms of the consistency of the parameters, all models were misspecified: the SVJ was the least and the BS was the most misspecified. Meanwhile, in terms of the out-of-sample pricing errors, they showed that the BS was the highest and the SVJ was the lowest. Finally, in terms of hedging errors especially in single instrument hedges, the SV was the lowest and the SVJ was the second lowest. Overall, they concluded that the modified BS model with stochastic volatility and random jumps was the best model.

CONCLUSION

The Black-Scholes options formula (Black & Scholes 1973 and Merton 1973) is the breakthrough in valuating options prices. However, the BS formula is heavily based on several assumptions that are not realistic in practice. The extensions of the assumptions are needed to make options pricing model more realistic. This paper has reviewed the relaxation of European options on shares

that offer analytical solutions rather than numerical procedures and analytical approximations that are more complicated and more suitable to American options counterpart.

Merton (1973) incorporated dividend into the BS formula, and also introduced the general valuation formula for non-constant interest rate. Rabinovitch (1989) derived the explicit formula based on the Merton's general formula. However, Kim (2002) showed that the modified BS formulas incorporating stochastic interest rate were not better than the original constant interest rate BS formula. In fact, the relaxing assumption of constant volatility has got the most intention and intensively done compared to other assumptions. Cox and Ross (1976) offered the constant elasticity of variance model, while Hull & White (1987), Stein & Stein (1991) and Heston (1993) offered stochastic volatility models. In the latter models, volatility is described as another stochastic process besides a stochastic process for share price. Merton (1976), furthermore, introduced the concept of random jumps to incorporate discontinuous time instead of continuous time in trading shares by assuming the jumps as a Poisson process.

The relaxations of the BS assumptions may be treated jointly one to another. Bakshi & Chen (1997) offered the stochastic-volatility and stochastic-interest rate model, Bates (1996) proposed the stochastic-volatility and random jump model, and Bakshi *et al.* (1997) combined them all and offered the generalized model: the stochastic-volatility stochastic-interest rate and random jumps model. Bakshi *et al.* (1997), however, found and concluded that the modified BS model with stochastic volatility and random jumps is the best model compared to other modified models.

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**APPENDIX
EXTENSIONS' SUMMARY AND FORMULAS**

Assumptions	Sources	Formula
Original Black-Scholes option formula	Black & Scholes (1973)	Equation (10) (11)
Dividend		
1. Continuous dividend	Merton (1973)	Equation (12) (13)
2. Discrete dividend	Hull (2003)	Equation (14) (15)
Stochastic interest rate	Merton (1973) Rabinovitch (1989)	Equation (17)
Non-constant volatility		
1. Deterministic volatility	Cox & Ross (1976)	Equation (19) (20)
2. Stochastic volatility	Hull & White (1987) Stein & Stein (1991) Heston (1993)	Equation (26) - Equation (27)
Discontinuous time	Merton (1976)	Equation (30) (31)
Stochastic-volatility and stochastic-interest-rate	Bakshi & Chen (1997)	Equation (33)
Stochastic-volatility and random jumps	Bates (1996)	Equation [32]
Stochastic-volatility, stochastic-interest-rate, and random jumps	Bakshi <i>et al.</i> (1997)	Equation [36]