ROTA-BAXTER OPERATORS ON DIHEDRAL QUANDLES

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ABSTRACT

Rota-Baxter operators on racks and quandles were introduced by Bardakov and Bovdi. The main goal of this paper is to study Rota-Baxter operators on dihedral quandles.

Let $R_2 = \{a_0, a_1\}$ be a 2-element dihedral quandle. Any mapping

$$B: R_2 \to R_2$$

is a Rota-Baxter operator on R_2. This raises the following questions:

Question. What can be said about Rota-Baxter operators on an arbitrary dihedral quandle R_n ?

Keywords: Rota-Baxter operators, dihedral quandle, binary operation, automorphisms

Introduction

Rota--Baxter operators for commutative algebras first appear in the paper of G. Baxter [7].
For basic results and the main properties of Rota--Baxter algebras see [10]
In [10] was define the Rota--Baxter operator on groups.
A group G with a Rota--Baxter operator B is called a Rota--Baxter group. In the paper [11] it was proved, that if (G, B) is a Rota--Baxter Lie group, then the tangent map of B at the identity is a Rota-Baxter operator of weight 1 on the Lie algebra of the Lie group G
In [9] was proved that Rota-Baxter operators on G as on a Hopf algebra are in one-two-one correspondence with Rota-Baxter operators of weight 1 on G.

2.1. **Quandles.** A quandle is a non-empty set Q with a binary operation $(x, y) \to (x * y)$ satisfying the following axioms:

(Q1) $x * x = x$ for all $x \in Q$;

(Q2) for any $x, y \epsilon Q$ there exists a unique $z \in Q$ such that $x = z * y$;

(Q3) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$

An algebraic system satisfying only (Q2) and (Q3) is called a k . Many interesting examples of quandles come from groups.

- G is a group, then the binary operation $a * b = b^{-1}ab$ turns G into the quandle $\mathrm{Conj}(G)$ called the Conjugation quandl of G.
- A group G with the binary operation $a * b = ba^{-1}b$ turns the set G into the quandle $\mathrm{Core}(G)$ called the core quandle of G. In particular, if $G = Z_n$ the cyclic group of order n then it is called the dihedral quandle and denoted by R_n.
- Let G be a group and $\varphi \in \mathrm{Aut}(G)$ Then the set G with binary operation $a * b = \varphi(ab^{-1})b$ forms a quandle $\mathrm{Alex}(G, \varphi)$ referred as the generalized Alexander quandle of G with respect to φ.

A quandle Q is called trivial if $x * y = x$ for all $x, y \in Q$. Unlike groups, a trivial quandle can have arbitrary number of elements. We denote the n element trivial quandle by T_n and an arbitrary trivial quandle by T. Notice that the axioms $(Q2)$ and $(Q3)$ are equivalent to the map $S_x: Q \to Q$ given by

$$S_x(y) = y * x$$

being an automorphism of Q for each $x \in Q$. These automorphisms are called inner automorphisms , and the group generated by all such automorphisms is denoted by $\mathrm{inn}(X)$. A quandle is said to be connected if it admits a transitive action by its group of inner automorphisms. For example, dihedral quandles of odd order are connected, whereas that of even order are disconnected. A quandle X is called involutary if $S_x^2 = \mathrm{id}_Q$ for each $x \in Q$. For example, all core quandles are involutary. A quandle (resp. rack) Q is called commutative if $x * y = y * x$ for all $x, y \in Q$. . The dihedral quandle R_3 is commutative and no trivial quandle with more than one element is commutative.

3. Rota-Baxter operators on racks and quandles

3.1. Definition and simple properties.

Definition 3.1. (V. G. Bardakov) Let $(X, *)$ be a rack. A map $B: X \to X$ is said to be a $\mathrm{Rota-Baxter\ operator}$ if the following identity holds:

$B(u) * B(v) = B((u * B(v)) * v), \quad (u, v \in X)$

The triple $(X, *, B)$ is called a $\mathrm{Rota-Baxter\ rack}$.

For example, let $(X, *)$ be a quandle and $p \in X$ be a fixed element. The map $B(x) = p$ for any $x \in X$ is a Rota--Baxter operator.

Further, from the definition follows

Proposition 3.2 Let T be a trivial quandle. Any map $B: T \to T$ is a Rota--Baxter operator on T.

Proposition 3.3 If $B: X \to X$ is a RB-operator on a rack X then its image is a subrack.

We can define an algebraic operation $\circ: X \to X$ by the rule

$u \circ v = (u * B(v)) * v \qquad u, v \in X$

and formulate the next question:

Under which conditions this operation $\circ$ defines a rack (quandle) operation on X ?

An answer on this question gives

Proposition 3.4 Let $(X, *)$ be a quandle. Additionally, assume that

$a)$ for any $a, b \in X$ there exists unique $x \in X$ such that

$(a * B(x)) * x = b$

$b)$ for any $u, v, w \in X$ holds

$$((u * B(v)) * B(w)) * w = ((u * B(w)) * w) * B((v * B(w)) * w)$$

Then $(X, \circ)$ is a rack.

Moreover, if the equality $u * (u * B(u)) = u$ holds for any $u \in X$, then $(X, \circ)$ is a quandle.

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3.2 Rota--Baxter operators on dihedral quandle

1) Let $R_2 = \{a_0, a_1\}$ be the 2-element dihedral quandle that is a quandle with the multiplication

$a_o * a_0 = a_0,\ \ a_o * a_1 = a_0,\ \ a_1 * a_0 = a_1,\ a_1 * a_1 = a_1,$

We see that it is a trivial quandle. From Proposition 3.2 follows that any map $B : R_2 \to R_2$ is a Rota--Baxter operator.

2) Let $R_3 = \{a_0, a_1, a_2\}$ be the 3-element dihedral quandle. Let $B : R_3 \to R_3$ be a map,

$B(a_0) = a_{i_0},\ \ B(a_1) = a_{i_1},\ \ B(a_2) = a_{i_2}.$

In this case we will write $B = B_{\{i_o, i_1, i_2\}}$. Also, we will denote $|B| = |B_{\{i_o, i_1, i_2\}}|$ is the cardinality of the set

.

As follows from Proposition if $i_o = i_1 = i_2$, then B is a RB-operator on R_3. Hence, if $|B| = 1$ then it is a RB-operator. Suppose that $|B| = 2$. In this case we have 18 operators of this type:

$B_{\{011\}},\ B_{\{022\}},\ B_{\{100\}},\ B_{\{122\}},\ B_{\{200\}},\ B_{\{211\}},\ B_{\{101\}},\ B_{\{202\}},\ B_{\{010\}}$

$B_{\{212\}},\ B_{\{020\}},\ B_{\{121\}},\ B_{\{110\}},\ B_{\{220\}},\ B_{\{001\}},\ B_{\{221\}},\ B_{\{002\}},\ B_{\{112\}}$

If $|B| = 3$ then B is a permutation of R_3 and we have 6 such operators. It is need to check which from these maps are RB-operators.

Let as consider $B = B_{011}$. If it is a RB-operator, then

$B(a_0) * B(a_1) = B((a_0 * B(a_1)) * a_1)$

It is easy to see that the left hand side is equal to a_2, a_2 but the right hand side is equal to a_0 Hence, is not a RB-operator.

It is need to check all other operators.

Let as consider $B = B_{\{120\}}$ In this case $|B| = 3$. If it is a RB-operator, then $B(a_0) * B(a_0) = B((a_0 * B(a_0)) * a_0)$ It is easy to see that the left hand side is equal to a a_1 but the right hand side is equal to a_2. Hence, $B_{\{120\}}$ is not a RB-operator.

It is need to check all other maps $B = B_{\{i_o, i_1, i_2\}}$.

Proposition 3.5

Let $n \geq 3$ and $B : R_n \to R_n$ --mapping, Then

1) If $|B| = 1$, the mapping B is Rota-Baxter operator on R_n

2) If $|B| = 2$ then the mapping B is a Rota-Baxter operator on R_n if and only if n is even and for some k such that $0 \leq k \leq n/2 - 1$ the mapping B can be defined by one of the following formulas:

$$B(a_s)=\begin{cases} a_k, & s=2i, \\ a_{\frac{n}{2}+k}, & s=2i+1, \end{cases} \qquad B(a_s)=\begin{cases} a_{\frac{n}{2}+k}, & s=2i, \\ a_k, & s=2i+1, \end{cases}$$

for all $0 \le s \le n-1$

Proposition 3.6

Let p be an odd prime number. The mapping $B : R_p \to R_p$ is a Rota--Baxter operator if and only if .

Hypothesis

Let $n \ge 3$ and $B : R_n \to R_n$. If $\le 3 \le |\operatorname{Im} B| \le n$, then the mapping B is not a Rota-Baxter operator on R_n .

References

1. C. Bai, L. Guo, Y. Sheng, and R. Tang, Post-groups, (Lie-)Butcher groups and the Yang-Baxter equation, Mathematische Annalen, 2023, 1-41.
2. V. G. Bardakov, M. V. Neshchadim, M. K. Yadav, Computing skew left braces of small orders, International Journal of Algebra and Computation, 30, no. 4 (2020), 839--851.
3. V.~G.~Bardakov, M.~V.~Neshchadim, M.~K.~Yadav, On λ-homomorphic skew braces, J. Pure Appl. Algebra, 226, no. 6 (2022).
4. V. G. Bardakov, V. Gubarev, Rota--Baxter groups, skew left braces, and the Yang--Baxter equation, J. Algebra, 596 (2022), 328--351.
5. V. G. Bardakov, V. Gubarev, Rota--Baxter operators on groups, Proc. Indian Acad. Sci. (Math. Sci.), 133, no. 4 (2023).
6. Valeriy G. Bardakov, Inder Bir S. Passi, and Mahender Singh, \textit{Quandle rings}, J. Algebra Appl. 18 (2019), no. 8, 1950157, 23 pp, https://doi.org/10.1142/S0219498819501573.
7. G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, Pacific J. Math. {\bf 10} (1960) 731--742.
8. S.L. de Bragan\c{c}a, Finite Dimensional Baxter Algebras, Stud. Appl. Math. (1) {\bf 54} (1975) 75--89.
9. M. Goncharov, Rota--Baxter operators on cocommutative Hopf algebras, arXiv:2011.14390, 14~p.
10. 2.L. Guo, An Introduction to Rota--Baxter Algebra. Surveys of Modern Mathematics, vol. International Press, Somerville (MA, USA); Higher education press, Beijing, 2012.
11. L. Guo, H. Lang, and Yu. Sheng, Integration and geometrization of Rota--Baxter Lie algebras, Adv. Math., 387 (2021).
12. J. Jiang, Y. Sheng and C. Zhu, Lie theory and cohomology of relative Rota--Baxter operators, arXiv:2108.02627v3.
13. J.-H. Lu, M. Yan and Y.-C. Zhu, \textit{On the set-theoretical Yang-Baxter equation.} Duke Math. J. (1) 104 (2000), 1--18.
14. N. Rathee, M. Singh, Relative Rota--Baxter groups and skew left braces, arXiv:2305.00922v1.