



# Odd sum labeling of graphs obtained by duplicating any edge of some graphs

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## Abstract

An injective function  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is an odd sum labeling if the induced edge labeling  $f^*$  defined by  $f^*(uv) = f(u) + f(v)$ , for all  $uv \in E(G)$ , is bijective and  $f^*(E(G)) = \{1, 3, 5, \dots, 2q - 1\}$ . A graph is said to be an odd sum graph if it admits an odd sum labeling. In this paper we study the odd sum property of graphs obtained by duplicating any edge of some graphs.

*Keywords:* odd sum labeling, odd sum graphs

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## 1. Introduction

Throughout this paper, by a graph we mean a finite, undirected simple graph. For notations and terminology, we follow [3].

A path on  $n$  vertices is denoted by  $P_n$  and a cycle on  $n$  vertices is denoted by  $C_n$ . A ladder  $L_n, n \geq 2$ , is the graph  $P_2 \square P_n$ , the Cartesian product of the graphs  $P_2$  and  $P_n$ . The graph  $G \circ K_1$  is obtained from the graph  $G$  by attaching a new pendant vertex at each vertex of  $G$ . Duplicating of an edge  $e = uv$  of a graph  $G$  produces a new graph  $G'$  by adding an edge  $e' = u'v'$  such that  $N(u') = N(u) \cup \{v'\} - \{v\}$  and  $N(v') = N(v) \cup \{u'\} - \{u\}$  [9].

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In [7], the concept of mean labeling was introduced and further studied in [8]. An injective function  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is said to be a *mean labeling* if the induced edge labeling  $f^*$  defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd,} \end{cases}$$

is injective and  $f^*(E(G)) = \{0, 1, 2, \dots, q\}$ . A graph  $G$  is said to be an *odd mean graph* if there exists an injective function  $f$  from  $V(G)$  to  $\{0, 1, 2, \dots, 2q - 1\}$  such that the induced map  $f^*$  from  $E(G) \rightarrow \{1, 3, 5, \dots, 2q - 1\}$  defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

is a bijection [6].

In [4], an odd edge labeling of a graph is defined as follows. A labeling  $f : V(G) \rightarrow \{0, 1, 2, \dots, p - 1\}$  is called an *odd edge labeling* of  $G$  if the edge labeling  $f^*$  on  $E(G)$  defined by  $f^*(uv) = f(u) + f(v)$  for any edge  $uv \in E(G)$  is such that the edge weights are odd. Here the edge labeling is not necessarily injective.

In [1], we introduced a new concept called *odd sum labeling* of graphs. The odd sum property of subdivision of some graphs have been studied in [2] and the same was referred in [5]. An injective function  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is an odd sum labeling if the induced edge labeling  $f^*$  defined by  $f^*(uv) = f(u) + f(v)$ , for all  $uv \in E(G)$ , is bijective and  $f^*(E(G)) = \{1, 3, 5, \dots, 2q - 1\}$ . A graph is said to be an *odd sum graph* if it admits an odd sum labeling. In this paper, we study the odd sum property of graphs obtained by duplicating any edge of some graphs.

**Theorem 1.1.** [1] *Every graph having an odd cycle is not an odd sum graph.*

## 2. Main Results

**Proposition 2.1.** *Let  $G$  be a graph obtained by duplicating an edge  $e$  of a path  $P_n, n \geq 3$ . Then,  $G$  is not an odd sum graph except the case when  $e$  is a pendant edge of  $P_5$ .*

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices on the path  $P_n$ . Let  $e' = v'_i v'_{i+1}$  be the duplicating edge of  $e = v_i v_{i+1}$ , for some  $i, 1 \leq i \leq n - 1$ .

**Case 1.**  $i = 1$  or  $i = n - 1$ .

Since the graph  $G$  is isomorphic when  $i = 1$  or  $i = n - 1$ , we may take  $i = 1$ .

**Subcase (i).**  $n \equiv 0(mod 4)$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 1\}$  as follows:

$$f(v_j) = \begin{cases} 6 - j, & j = 1, 2, 3 \\ j - 2, & 4 \leq j \leq n \text{ and } j \equiv 0(mod 4) \\ j + 4, & 5 \leq j \leq n - 3 \text{ and } j \equiv 1(mod 4) \\ j + 2, & 6 \leq j \leq n - 2 \text{ and } j \equiv 2(mod 4) \\ j, & 7 \leq j \leq n - 1 \text{ and } j \equiv 3(mod 4) \end{cases}$$

and  $f(v'_j) = 2 - j, \quad j = 1, 2.$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

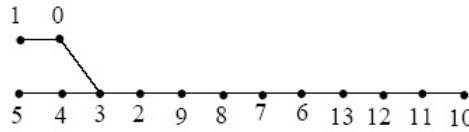


Figure 2.1. An odd sum labeling of  $G$  when  $n = 12$  and  $i = 1$ .

**Subcase (ii).**  $n \equiv 2(\text{mod } 4)$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 1\}$  as follows:

$$f(v_j) = \begin{cases} 6 - j, & j = 1, 3 \\ 2, & j = 2 \\ 4, & j = 6 \\ j + 2, & j = 4, 5 \\ j + 2, & 8 \leq j \leq n - 2 \text{ and } j \equiv 0(\text{mod } 4) \\ j, & 9 \leq j \leq n - 1 \text{ and } j \equiv 1(\text{mod } 4) \\ j - 2, & 10 \leq j \leq n \text{ and } j \equiv 2(\text{mod } 4) \\ j + 4, & 7 \leq j \leq n - 3 \text{ and } j \equiv 3(\text{mod } 4) \end{cases}$$

and  $f(v'_j) = 2 - j, \quad j = 1, 2.$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

**Subcase (iii).**  $n \equiv 1(\text{mod } 4)$  and  $n \geq 9$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 1\}$  as follows:

$$f(v_j) = \begin{cases} j - 1, & 1 \leq j \leq 3 \\ j + 3, & j = 4, 7 \\ j + 1, & j = 5, 8 \\ j - 1, & j = 6, 9 \\ j - 1, & 12 \leq j \leq n \text{ and } j \equiv 0, 1(\text{mod } 4) \\ j + 3, & 10 \leq j \leq n - 2 \text{ and } j \equiv 2, 3(\text{mod } 4) \end{cases}$$

and  $f(v'_j) = 5 - j, \quad j = 1, 2.$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 5$ , the number of edges in  $G$  is 6. By assigning the label 0 (or 1, 2) to the vertex  $v_3$ , the labels 5 and 6 will be assigned to the end vertices of any one of the edges  $v_1v_2, v_4v_5$  and  $v'_1v'_2$  in order to get the edge label 11. If so, then 9 will not be an edge label. By assigning the label 4 (or 5, 6) to the vertex  $v_3$ , the labels 0 and 1 will be assigned to the end vertices of any one of the edges  $v_1v_2, v_4v_5$  and  $v'_1v'_2$  in order to get the edge label 1. If so, then 3 will not be an edge label. By assigning the label 3 to the vertex  $v_3$ , the pair of labels 0, 1 and 5, 6 will be assigned to the end vertices of any two of the edges  $v_1v_2, v_4v_5$  and  $v'_1v'_2$  in order to get the edge label 1 and 11

respectively. This implies that the labels 2 and 4 are to be assigned to the remaining vertices which gives an even edge label 6. Hence an odd sum labeling does not exist for  $G$  when  $n = 5$ .

**Subcase (iv).**  $n \equiv 3(mod 4)$  and  $n \geq 7$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 1\}$  as follows:

$$f(v_j) = \begin{cases} j - 1, & 1 \leq j \leq 3 \\ j + 3, & 4 \leq j \leq n - 2 \text{ and } j \equiv 0, 1(mod 4) \\ j - 1, & 6 \leq j \leq n \text{ and } j \equiv 2, 3(mod 4) \end{cases}$$

and  $f(v'_j) = 5 - j, \quad j = 1, 2.$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained.

While  $n = 3$ , an odd sum labeling of  $G$  is given in Figure 2.2.

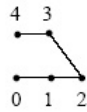


Figure 2.2. An odd sum labeling of  $G$  when  $n = 3$  and

**Case 2.**  $2 \leq i \leq n - 2.$

**Subcase (i).**  $(n - i) \equiv 0(mod 4)$  and  $n - i \geq 12.$

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 2\}$  as follows:

$$f(v_j) = \begin{cases} j - 1, & 1 \leq j \leq i - 1, j = i + 1, i + 6 \\ j + 1, & j = i, i + 2, i + 3 \\ j + 3, & j = i + 4, i + 5 \\ j + 3, & i + 7 \leq j \leq n - 6 \text{ and } j - i \equiv 0(mod 4), j = n - 2, n - 1 \\ j + 1, & i + 7 \leq j \leq n - 6 \text{ and } j - i \equiv 1(mod 4), j = n - 4, n - 3 \\ j - 1, & i + 7 \leq j \leq n - 6 \text{ and } j - i \equiv 2(mod 4), j = n \\ j + 5, & i + 7 \leq j \leq n - 6 \text{ and } j - i \equiv 3(mod 4), j = n - 5, \end{cases}$$

$f(v'_i) = i - 1$  and  $f(v'_{i+1}) = i + 6.$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

**Subcase (ii).**  $(n - i) \equiv 1(mod 4)$  and  $n - i \geq 9.$

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 2\}$  as follows:

$$f(v_j) = \begin{cases} j - 1, & 1 \leq j \leq i - 1, j = i + 1, i + 3 \\ j + 1, & j = i \\ j + 3, & j = i + 2 \\ j + 5, & i + 4 \leq j \leq n - 6 \text{ and } j - i \equiv 0(mod 4), j = n - 5 \\ j + 3, & i + 4 \leq j \leq n - 6 \text{ and } j - i \equiv 1(mod 4), j = n - 2, n - 1 \\ j + 1, & i + 4 \leq j \leq n - 6 \text{ and } j - i \equiv 2(mod 4), j = n - 4, n - 3 \\ j - 1, & i + 4 \leq j \leq n - 6 \text{ and } j - i \equiv 3(mod 4), j = n, \end{cases}$$

$f(v'_i) = i - 1$  and  $f(v'_{i+1}) = i + 4.$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

**Subcase (iii).**  $(n - i) \equiv 2(\text{mod } 4)$  and  $n - i \geq 6$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 2\}$  as follows:

$$f(v_j) = \begin{cases} j - 1, & 1 \leq j \leq i - 1, j = i + 1, i + 6 \\ j + 1, & j = i, i + 2, i + 3 \\ j + 3, & j = i + 4, i + 5 \\ j + 3, & i + 7 \leq j \leq n \text{ and } j - i \equiv 0(\text{mod } 4) \\ j + 1, & i + 7 \leq j \leq n \text{ and } j - i \equiv 1(\text{mod } 4) \\ j - 1, & i + 7 \leq j \leq n \text{ and } j - i \equiv 2(\text{mod } 4) \\ j + 5, & i + 7 \leq j \leq n \text{ and } j - i \equiv 3(\text{mod } 4), \end{cases}$$

$f(v'_i) = i - 1$  and  $f(v'_{i+1}) = i + 6$ .

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

**Subcase (iv).**  $(n - i) \equiv 3(\text{mod } 4)$  and  $n - i \geq 7$ .

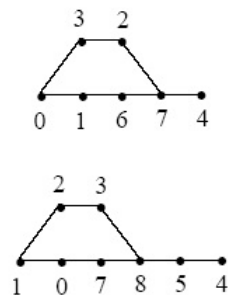
Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 2\}$  as follows:

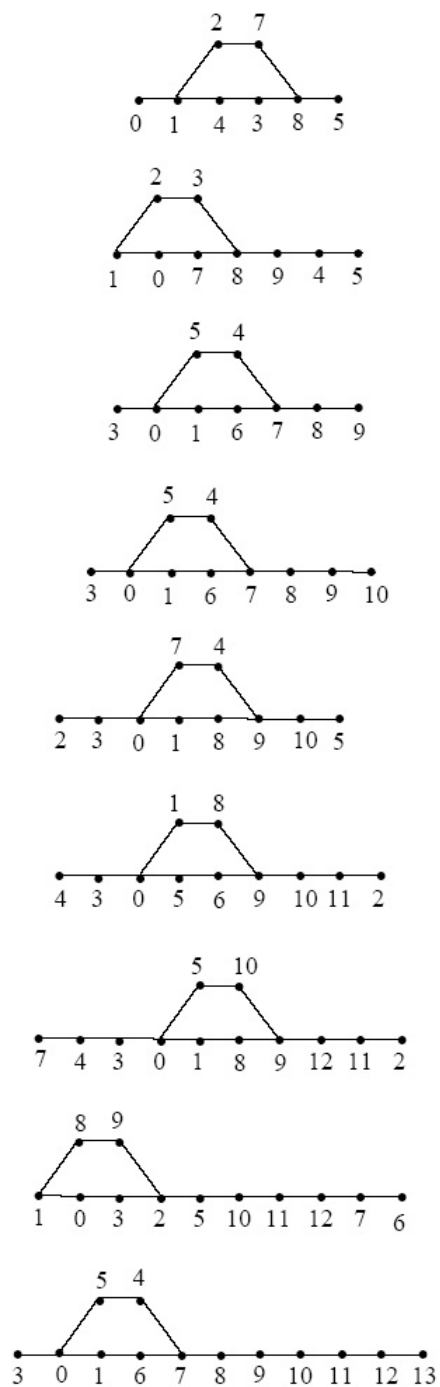
$$f(v_j) = \begin{cases} j - 1, & 1 \leq j \leq i - 1, j = i + 1, i + 3 \\ j + 1, & j = i \\ j + 3, & j = i + 2 \\ j + 5, & i + 4 \leq j \leq n \text{ and } j - i \equiv 0(\text{mod } 4) \\ j + 3, & i + 4 \leq j \leq n \text{ and } j - i \equiv 1(\text{mod } 4) \\ j + 1, & i + 4 \leq j \leq n \text{ and } j - i \equiv 2(\text{mod } 4) \\ j - 1, & i + 4 \leq j \leq n \text{ and } j - i \equiv 3(\text{mod } 4), \end{cases}$$

$f(v'_i) = i - 1$  and  $f(v'_{i+1}) = i + 4$ .

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

The graphs which are not lying under any of the sub cases of Case 2 are given with their odd sum labeling in Figure 2.3.





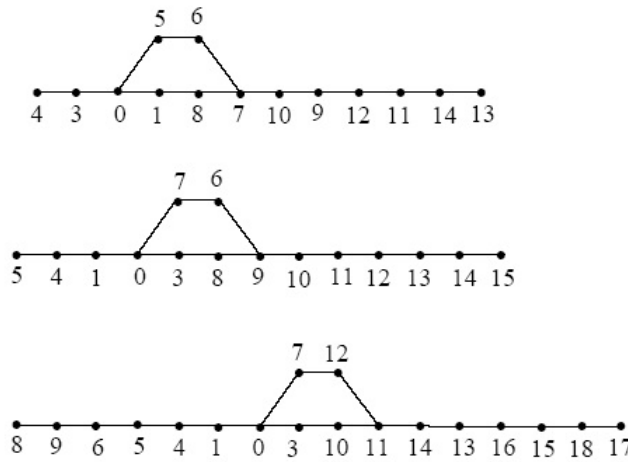


Figure 2.3. An odd sum labeling of  $G$  for various values of  $n$ .

□

**Proposition 2.2.** *Let  $G$  be the graph obtained from  $P_n \circ K_1$  by duplicating an edge, for  $n \geq 2$ . Then,  $G$  is not an odd sum graph only when  $n = 3$  and the pendant edge attached at the pendent vertex of  $P_n$  is duplicated.*

*Proof.* Let  $u_1, u_2, \dots, u_n$  be the vertices of the path of length  $n - 1$  and  $v_i, 1 \leq i \leq n$  be the pendant vertices of  $u_i, 1 \leq i \leq n$ . Let  $G$  be the graph obtained from  $P_n \circ K_1$  by duplicating an edge  $e$  (other than  $u_1v_1$  and  $u_nv_n$ ) by  $e'$ .

**Case 1.**  $e = u_iv_i$ , for some  $i, 2 \leq i \leq n - 1$ .

Let its duplication be  $e' = u'_iv'_i$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + 2\}$  as follows:

$$\text{For } i \text{ is odd, } f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & j = 1, \\ 2, & j = 2. \end{cases}$$

$$\text{For } i \text{ is even, } f(u_j) = \begin{cases} 1, & j = 1 \\ 2, & j = 2 \end{cases} \text{ and } f(v_j) = \begin{cases} 0, & j = 1, \\ 3, & j = 2. \end{cases}$$

$$f(u_j) = \begin{cases} f(u_{j-2}) + 4, & 3 \leq j \leq i - 1 \\ f(u_{j-1}) + 5, & j = i \\ f(u_{j-1}) + 1, & j = i + 1 \\ f(u_{j-2}) + 4, & i + 2 \leq j \leq n, \end{cases}$$

$$f(v_j) = \begin{cases} f(v_{j-2}) + 4, & 3 \leq j \leq i - 1 \\ f(v_{j-1}) + 5, & j = i \\ f(v_{j-1}) + 3, & j = i + 1 \\ f(v_{j-2}) + 4, & i + 2 \leq j \leq n, \end{cases}$$

$$f(u'_i) = f(u_i) - 4 \text{ and } f(v'_i) = f(u_i) - 3.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

**Case 2.**  $e = u_1v_1$  (or  $u_nv_n$ ).

Let its duplication be  $e' = u'_1v'_1$ . Assume that  $n \geq 4$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + 1\}$  as follows:

$$f(u_1) = 1, f(u_2) = 2, f(u_3) = 9, f(u_i) = \begin{cases} 2i, & 4 \leq i \leq n \text{ and } i \text{ is even} \\ 2i + 1, & 5 \leq i \leq n \text{ and } i \text{ is odd,} \end{cases}$$

$$f(v_1) = 0, f(v_2) = 5, f(u_3) = 4, f(v_4) = 7,$$

$$f(v_i) = \begin{cases} 2i, & 5 \leq i \leq n \text{ and } i \text{ is odd} \\ 2i + 1, & 6 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(u'_1) = 3 \text{ and } f(v'_1) = 6.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained.

Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 3$ , the number of edges in  $G$  is 7. By assigning the label 0 (or 1, 2, 3) to the vertex  $v_2$ , the labels 6 and 7 will be assigned to the end vertices of any one of the edges  $u_1v_1, u_3v_3$  and  $u'_1v'_1$  in order to get the edge label 13. If so, then 11 will not be an edge label. By assigning the label 4 (or 5, 6, 7) to the vertex  $v_2$ , the labels 0 and 1 will be assigned to the end vertices of any one of the edges  $u_1v_1, u_3v_3$  and  $u'_1v'_1$  in order to get the edge label 1. If so, then 3 will not be an edge label. So an odd sum labeling is not possible in  $G$  when  $n = 3$ .

When  $n = 2$ , an odd sum labeling is shown in Figure 2.4.

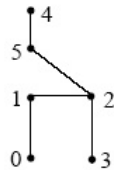


Figure 2.4. An odd sum labeling of  $G$  when  $n = 2$  and  $i = 1$ .

**Case 3.**  $e = u_iu_{i+1}$  for some  $i, 2 \leq i \leq n - 2$ .

Let its duplication be  $e' = u'_iu'_{i+1}$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + 4\}$  as follows:

$$\text{For } i \text{ is odd, } f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \end{cases} \text{ and } f(v_j) = \begin{cases} 1, & j = 1, \\ 2, & j = 2. \end{cases}$$



For  $i$  is even,  $f(u_j) = \begin{cases} 1, & j = 1 \\ 2, & j = 2 \end{cases}$  and  $f(v_j) = \begin{cases} 0, & j = 1, \\ 3, & j = 2. \end{cases}$

$$f(u_j) = \begin{cases} f(u_{j-2}) + 4, & 3 \leq j \leq i \\ f(u_{j-2}) + 8, & j = i + 1 \\ f(u_{j-2}) + 10, & j = i + 2 \\ f(u_{j-2}) + 4, & i + 3 \leq j \leq n, \end{cases}$$

$$f(v_j) = \begin{cases} f(v_{j-2}) + 4, & 3 \leq j \leq i \\ f(v_{j-2}) + 8, & j = i + 1, i + 2 \\ f(v_{j-2}) + 6, & j = i + 3 \\ f(v_{j-2}) + 4, & i + 4 \leq j \leq n, \end{cases}$$

$$f(u'_i) = f(u_i) + 4 \text{ and } f(u'_{i+1}) = f(u_{i+1}) - 2.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

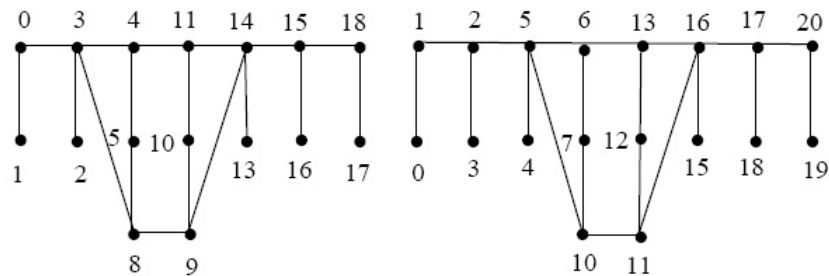


Figure 2.5. An odd sum labeling of  $G$  when  $n = 7, i = 3$  and  $n = 8, i = 4$ .

**Case 4.**  $e = u_1u_2$  (or  $u_{n-1}u_n$ ).

Let its duplication be  $e' = u'_1u'_2$ .

For  $n \geq 3$ , we define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + 3\}$  as follows:

$$f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \\ f(u_{j-2}) + 8, & j = 3, 4 \\ f(u_{j-2}) + 4, & 5 \leq j \leq n, \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 2, & j = 2 \\ f(v_{j-2}) + 8, & j = 3, 4 \\ f(v_{j-2}) + 4, & 5 \leq j \leq n, \end{cases}$$

$$f(u'_1) = 6 \text{ and } f(u'_2) = 7.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

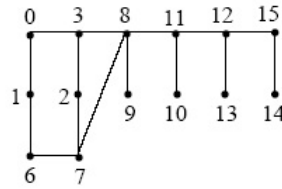


Figure 2.6. An odd sum labeling of  $G$  when  $n = 6$  and  $i = 1$ .

□

**Proposition 2.3.** Let  $G$  be the graph obtained by duplicating an edge of a cycle  $C_n, n \geq 4$ . Then,  $G$  is an odd sum graph if and only if  $n$  is even.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices on the cycle  $C_n$ . Let  $G$  be the graph obtained by duplicating an edge of  $C_n$ . By Theorem 1.1 an odd sum labeling of  $G$  does not exist when  $n$  is odd. So assume that  $n$  is even.

**Case 1.**  $n \equiv 2 \pmod{4}$  and  $n \geq 14$ .

Let  $e' = v'_2v'_3$  be the duplicating edge of  $e = v_2v_3$  in  $G$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 3\}$  as follows:

$$f(v_i) = \begin{cases} i - 1, & 1 \leq i \leq \frac{n+2}{4} \text{ and } i \text{ is odd} \\ i + 1, & \frac{n+6}{4} \leq i \leq \frac{n-4}{2} \text{ and } i \text{ is odd} \\ i + 3, & \frac{n}{2} \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ i + 1, & 2 \leq i \leq \frac{n+2}{4} \text{ and } i \text{ is even} \\ i + 3, & \frac{n+10}{4} \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v'_2) = 1 \text{ and } f(v'_3) = \frac{n + 2}{2}.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 6$  and  $10$ , the odd sum labeling of  $G$  is given in Figure 2.7.

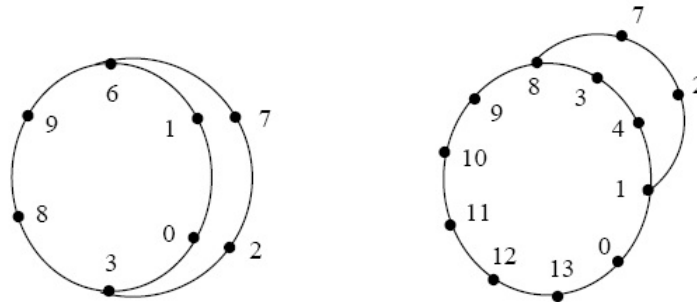


Figure 2.7. An odd sum labeling of  $G$  when  $n = 6$  and  $10$ .

**Case 2.**  $n \equiv 0 \pmod{4}$  and  $n \geq 16$ .

Let  $e' = v'_2v'_3$  be the duplicating edge of  $e = v_2v_3$  in  $G$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, n + 3\}$  as follows:

$$f(v_i) = \begin{cases} i - 1, & i = 1, 3 \\ i + 1, & 5 \leq i \leq \frac{n}{4} \text{ and } i \text{ is odd} \\ i + 3, & \frac{n+4}{4} \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ i - 1, & 2 \leq i \leq \frac{n}{4} \text{ and } i \text{ is even} \\ i + 1, & \frac{n+4}{4} \leq i \leq \frac{n-4}{2} \text{ and } i \text{ is even} \\ i + 3, & \frac{n}{2} \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v'_2) = \frac{n + 2}{2} \text{ and } f(v'_3) = 4.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 4, 8$  and  $12$ , the odd sum labeling of  $G$  is given in Figure 2.8.

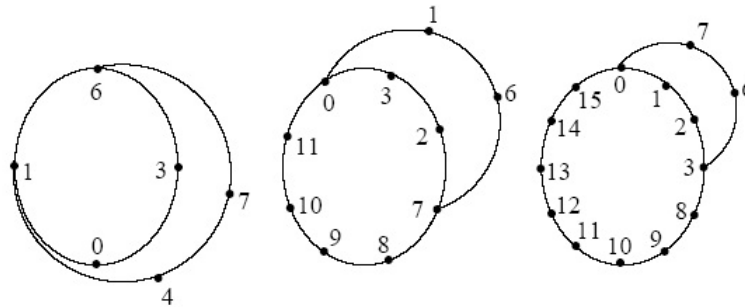


Figure 2.8. An odd sum labeling of  $G$  when  $n = 4, 8$  and  $12$ .

□

**Proposition 2.4.** Let  $G$  be a graph obtained by duplicating an edge of  $C_n \circ K_1, n \geq 4$ . Then,  $G$  is an odd sum graph if and only if  $n$  is even.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices on the cycle  $C_n$  and  $u_i, 1 \leq i \leq n$  be the pendant vertex attached at  $v_i, 1 \leq i \leq n$  in  $C_n \circ K_1$ . Let  $G$  be a graph obtained by duplicating an edge  $e$  in  $C_n \circ K_1$ . By Theorem 1.1, an odd sum labeling of  $G$  does not exist when  $n$  is odd. So assume that  $n$  is even.

**Case 1.** Let  $u'_{n-1}v'_{n-1}$  be the duplicating edge of  $u_{n-1}v_{n-1}$  in  $G$ .

**Subcase (i).**  $n \equiv 0 \pmod{4}$  and  $n \geq 8$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + 3\}$  as follows:

$$f(v_i) = \begin{cases} 2i - 2, & 1 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ 2i - 1, & 2 \leq i \leq \frac{n}{2} \text{ and } i \text{ is even} \\ 2i + 1, & \frac{n}{2} + 2 \leq i \leq n - 2 \text{ and } i \text{ is even} \\ 2i + 3, & i = n, \end{cases}$$

$$f(u_i) = \begin{cases} 2i - 1, & 1 \leq i \leq \frac{n}{2} + 1 \text{ and } i \text{ is odd} \\ 2i + 1, & \frac{n}{2} + 3 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ 2i - 2, & 2 \leq i \leq n - 2 \text{ and } i \text{ is even} \\ 2i + 2, & i = n, \end{cases}$$

$$f(u'_{n-1}) = 2n + 1 \text{ and } f(v'_{n-1}) = 2n.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

An odd sum labeling of  $G$  when  $n = 4$  is given in Figure 2.9.

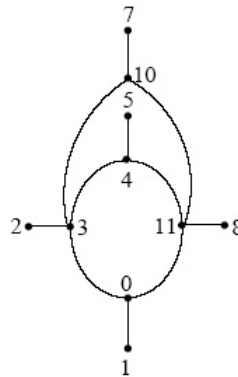


Figure 2.9. An odd sum labeling of  $G$  when  $n = 4$ .

**Subcase (ii).**  $n \equiv 2 \pmod{4}$  and  $n \geq 10$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + 3\}$  as follows:

$$f(v_i) = \begin{cases} 2i - 2, & 1 \leq i \leq \frac{n}{2} \text{ and } i \text{ is odd} \\ 2i, & \frac{n}{2} + 2 \leq i \leq n - 3 \text{ and } i \text{ is odd} \\ 2i - 2, & i = n - 1 \\ 2i - 1, & 2 \leq i \leq n - 2 \text{ and } i \text{ is even} \\ 2i + 3, & i = n, \end{cases}$$

$$f(u_i) = \begin{cases} 2i - 1, & 1 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ 2i - 2, & 2 \leq i \leq \frac{n}{2} + 1 \text{ and } i \text{ is even} \\ 2i, & \frac{n}{2} + 3 \leq i \leq n - 4 \text{ and } i \text{ is even} \\ 2i + 4, & i = n - 2 \\ 2i - 2, & i = n, \end{cases}$$

$$f(u'_{n-1}) = 2n + 1 \text{ and } f(v'_{n-1}) = 2n + 2.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 6$ , an odd sum labeling of  $G$  is given in Figure 2.10.

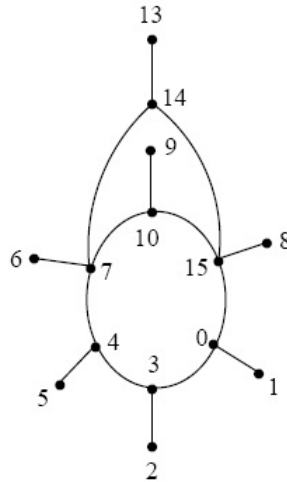


Figure 2.10. An odd sum labeling of  $G$  when  $n = 6$ .

**Case 2.** Let  $e'$  be the duplicating edge of an edge  $e$  on the cycle  $C_n$  in  $G$ .

**Subcase (i).**  $n \equiv 0(mod 4)$  and  $n \geq 4$ .

Let  $e' = v'_n v'_1$  be the duplicating edge of  $v_n v_1$  in  $G$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + 5\}$  as follows:

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 2i, & 3 \leq i \leq \frac{n}{2} - 1 \text{ and } i \text{ is odd} \\ 2i + 4, & \frac{n}{2} + 1 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ 2i + 1, & 2 \leq i \leq n - 2 \text{ and } i \text{ is even} \\ 2i + 5, & i = n, \end{cases}$$

$$f(u_i) = \begin{cases} 1, & i = 1 \\ 2i + 1, & 3 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ 2i, & 2 \leq i \leq \frac{n}{2} \text{ and } i \text{ is even} \\ 2i + 4, & \frac{n}{2} + 2 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

and  $f(v'_i) = \begin{cases} 2i, & i = 1 \\ 2i + 1, & i = n. \end{cases}$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

**Subcase (ii).**  $n \equiv 2(mod 4)$  and  $n \geq 10$ .

Let  $e' = v'_2 v'_3$  be the duplicating edge of  $v_2 v_3$  in  $G$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + 5\}$  as follows:

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 2i + 2, & 3 \leq i \leq \frac{n}{2} - 2 \text{ and } i \text{ is odd} \\ 2i + 4, & \frac{n}{2} \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ 3, & i = 2 \\ 2i + 5, & 4 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(u_i) = \begin{cases} 1, & i = 1 \\ 2i + 3, & i = 3 \\ 2i + 5, & 5 \leq i \leq n - 1 \text{ and } i \text{ is odd} \\ 2, & i = 2 \\ 2i + 2, & 4 \leq i \leq \frac{n}{2} - 1 \text{ and } i \text{ is even} \\ 2i + 4, & \frac{n}{2} + 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(v'_2) = 7 \text{ and } f(v'_3) = 6.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 6$ , an odd sum labeling of  $G$  is given in Figure 2.11.

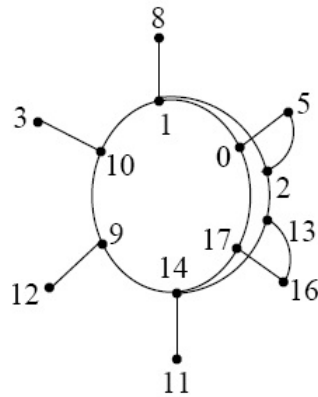


Figure 2.11. An odd sum labeling of  $G$  when  $n = 6$ .

□

**Proposition 2.5.** Let  $G$  be the graph obtained by duplicating an edge  $e$  of the ladder  $L_n, n \geq 2$ . Then,  $G$  is not an odd sum graph only when  $n = 3$  and  $e$  is the edge between the pendant vertices of the paths of length  $n - 1$  in  $L_n$ .

*Proof.* Let  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  be the vertices on the paths of length  $n - 1$  in the ladder  $L_n$ . Let  $G$  be the graph obtained by duplicating any one of the edge  $e$  of  $L_n$ .

**Case 1.**  $e'$  is the duplicating edge of an pendant edge  $e$  of the paths of length  $n - 1$ .

Let  $e' = u'_1u'_2$  be the duplicating edge of  $e = u_1u_2$ . Assume that  $n \geq 4$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 3n + 2\}$  as follows:

$$f(u_i) = \begin{cases} 1, & i = 1 \\ 3i + 4, & 2 \leq i \leq n - 1 \\ 3i + 2, & i = n, \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 3i - 1, & 2 \leq i \leq n \end{cases}$$

and  $f(u'_i) = i + 2, i = 1, 2.$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 2$  and  $3$ , an odd sum labeling of  $G$  is given in Figure 2.12.

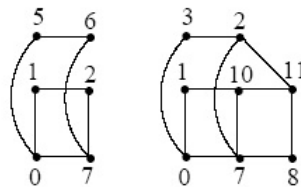


Figure 2.12. An odd sum labeling of  $G$  when  $n = 2$  and  $3$ .

**Case 2.**  $e'$  is the duplicating edge of an edge  $e = u_i u_{i+1}$  (or  $v_i v_{i+1}$ ), for some  $i, 2 \leq i \leq n - 2$ .

Let  $e' = u'_i u'_{i+1}$  be the duplicating edge of the edge  $e = u_i u_{i+1}$  for some  $i, 2 \leq i \leq n - 2$ . Assume that  $n \geq 5$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 3(n + 1)\}$  as follows:

$$f(u_j) = \begin{cases} 3j - 2, & 1 \leq j \leq i \\ 3j + 2, & j = i + 1 \\ 3j + 4, & j = i + 2 \\ 3j + 2, & i + 3 \leq j \leq n, \end{cases}$$

$$f(v_j) = \begin{cases} 3j - 3, & 1 \leq j \leq i \\ 3j + 1, & j = i + 1, i + 2 \\ 3j + 3, & i + 3 \leq j \leq n \end{cases}$$

$$\text{and } f(u'_j) = \begin{cases} 3j + 2, & j = i \\ 3j, & j = i + 1. \end{cases}$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 4$ , an odd sum labeling of  $G$  is given in Figure 2.13.

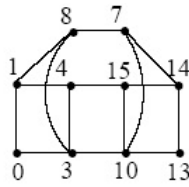


Figure 2.13. An odd sum labeling of  $G$  when  $n = 4$  and  $i = 2$ .

**Case 3.** Let  $e' = u'_1v'_1$  be the duplicating edge of the edge  $e = u_1v_1$ .

**Subcase (i).**  $n \equiv 0(mod 2)$  and  $n \geq 4$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 3n + 1\}$  as follows:

$$f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \\ 3j + 3, & 3 \leq j \leq n - 1 \text{ and } j \equiv 3(mod 4) \\ 3j + 1, & 4 \leq j \leq n \text{ and } j \equiv 0(mod 4) \\ 3j - 1, & 5 \leq j \leq n \text{ and } j \equiv 1, 2(mod 4), \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 8, & j = 2 \\ 3j, & 3 \leq j \leq n \text{ and } j \equiv 2, 3(mod 4) \\ 3j - 2, & 4 \leq j \leq n \text{ and } j \equiv 0(mod 4) \\ 3j + 4, & 5 \leq j \leq n - 1 \text{ and } j \equiv 1(mod 4), \end{cases}$$

$$f(u'_1) = 2 \text{ and } f(v'_1) = 5.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

For  $n = 2$ , the graph  $G$  is isomorphic to the graph on Figure 2.14.

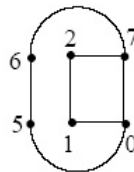


Figure 2.14. An odd sum labeling of  $G$  when  $n = 2$ .

**Subcase (ii).**  $n \equiv 1(mod 2)$  and  $n \geq 5$ .



Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 3n + 1\}$  as follows:

$$f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \\ 3j + 3, & j = 3, 4 \leq j \leq n - 1 \text{ and } j \equiv 0(\text{mod } 4) \\ 3j + 1, & 5 \leq j \leq n \text{ and } j \equiv 1(\text{mod } 4) \\ 3j - 1, & 6 \leq j \leq n \text{ and } j \equiv 2, 3(\text{mod } 4), \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 8, & j = 2 \\ 3j, & j = 3, 7 \leq j \leq n \text{ and } j \equiv 0, 3(\text{mod } 4) \\ 3j - 2, & j = 4, 5 \leq j \leq n \text{ and } j \equiv 1(\text{mod } 4) \\ 3j + 4, & 6 \leq j \leq n - 1 \text{ and } j \equiv 2(\text{mod } 4), \end{cases}$$

$f(u'_1) = 2$  and  $f(v'_1) = 5$ .

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

When  $n = 3$ , it is observed that there does not exist an odd sum labeling of  $G$ .

**Case 4.** Let  $e' = u'_i v'_i$  be the duplicating edge of the edge  $e = u_i v_i, 2 \leq i \leq \frac{n+1}{2}$ .

**Subcase (i).**  $n - i \equiv 1(\text{mod } 2)$  and  $n - i \geq 3$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 3(n + 1)\}$  as follows:

$$f(u_j) = \begin{cases} 3j - 2, & 1 \leq j \leq i - 1 \\ 3j - 4, & j = i \\ 3j + 4, & j = i + 1 \\ 3j + 2, & i + 2 \leq j \leq n \text{ and } j - i \equiv 1, 2(\text{mod } 4) \\ 3j, & i + 2 \leq j \leq n \text{ and } j - i \equiv 3(\text{mod } 4) \\ 3j + 6, & i + 2 \leq j \leq n \text{ and } j - i \equiv 0(\text{mod } 4), \end{cases}$$

$$f(v_j) = \begin{cases} 3j - 3, & 1 \leq j \leq i - 1 \\ 3j - 1, & j = i, i + 1 \\ 3j + 5, & i + 2 \leq j \leq n \text{ and } j - i \equiv 2(\text{mod } 4) \\ 3j + 3, & i + 2 \leq j \leq n \text{ and } j - i \equiv 3(\text{mod } 4) \\ 3j + 1, & i + 2 \leq j \leq n \text{ and } j - i \equiv 0, 1(\text{mod } 4), \end{cases}$$

$f(u'_i) = 3i + 4$  and  $f(v'_i) = 3i + 3$ .

From this vertex labeling, the required induced edge labeling for  $G$  will be attained.

Thus  $f$  is an odd sum labeling of  $G$ .

**Subcase (ii).**  $n - i \equiv 0(\text{mod } 2)$  and  $n - i \geq 4$ .

Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 3(n + 1)\}$  as follows:

$$f(u_j) = \begin{cases} 3j - 2, & 1 \leq j \leq i - 1 \\ 3j - 4, & j = i \\ 3j + 4, & j = i + 1 \\ 3j + 2, & j = i + 2 \\ 3j, & j = i + 3 \\ 3j, & i + 4 \leq j \leq n \text{ and } j - i \equiv 0 \pmod{4} \\ 3j + 6, & i + 4 \leq j \leq n \text{ and } j - i \equiv 1 \pmod{4} \\ 3j + 2, & i + 4 \leq j \leq n \text{ and } j - i \equiv 2, 3 \pmod{4}, \end{cases}$$

$$f(v_j) = \begin{cases} 3j - 3, & 1 \leq j \leq i - 1 \\ 3j - 1, & j = i, i + 1 \\ 3j + 5, & j = i + 2, i + 3 \\ 3j + 3, & i + 4 \leq j \leq n \text{ and } j - i \equiv 0 \pmod{4} \\ 3j + 1, & i + 4 \leq j \leq n \text{ and } j - i \equiv 1, 2 \pmod{4} \\ 3j + 5, & i + 4 \leq j \leq n \text{ and } j - i \equiv 3 \pmod{4}, \end{cases}$$

$$f(u'_i) = 3i + 4 \text{ and } f(v'_i) = 3i + 3.$$

From this vertex labeling, the required induced edge labeling for  $G$  will be attained. Thus  $f$  is an odd sum labeling of  $G$ .

An odd sum labelings of the graphs in Case 4 which do not fall on subcase (i) and (ii) are given in Figure 2.15.

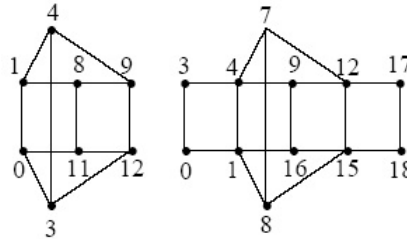


Figure 2.15. An odd sum labeling of  $G$  when  $n = 3, i = 2$  and  $n = 5, i = 3$ .

□

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