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Note on Parity Factors of Regular Graphs

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Abstract

In this paper, we obtain a sufficient condition for the existence of parity factors in a regular graph in terms of edge-connectivity. Moreover, we also show that our condition is sharp.

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1. Preliminaries

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of a graph G is called the *order* of G and is denoted by n. The number of edges of G is called the *size* of G and is denoted by e. For a vertex v of graph G, the number of edges of G incident to v is called the *degree* of v in G and is denoted by $d_G(v)$. For two subsets $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges of G joining S to T.

Let H be a function associating a subset of $\mathbb{Z}$ to each vertex of G. A spanning subgraph F of graph G is called an H-*factor* of G if

$$d_F(x) \in H(x) \qquad \text{for every vertex } x \in V(G). \tag{1}$$

For a spanning subgraph F of G and for a vertex v of G, define

$$\delta(H; F, v) = \min\{|d_F(v) - i \mid i \in H_v\},$$

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and let $\delta(H;F) = \sum_{x \in V(G)} \delta(H;F,x)$. Thus a spanning subgraph F is an H-factor if and only if $\delta(H;F) = 0$. Let

$$\delta_H(G) = \min\{\delta(H;F) \mid F \text{ are spanning subgraphs of } G\}.$$

A spanning subgraph F is called H-optimal if $\delta(H;F) = \delta_H(G)$. The H-factor problem is to determine the value $\delta_H(G)$. An integer h is called a *gap* of $H(v)$ if $h \notin H(v)$ but $H(v)$ contains an element less than h and an element greater than h. Lovász [11] gave a structural description on the H-factor problem in the case where $H(v)$ has no two consecutive gaps for all $v \in V(G)$ and showed that the problem is NP-complete without this restriction. Moreover, he also conjectured that the decision problem of determining whether a graph has an H-factor is polynomial in the case where $H(v)$ has no two consecutive gaps for all $v \in V(G)$. Cornuéjols [5] proved the conjecture.

Let therefore $g, f : V \to Z^+$ such that $g(v) \le f(v)$ and $g(v) \equiv f(v) \pmod 2$ for every $v \in V$. Then a spanning subgraph F of G is called a (g,f)-parity-factor, if $g(v) \le d_F(v) \le f(v)$ and $d_F(v) \equiv f(v) \pmod 2$ for all $v \in V$. Clearly, a (g,f)-parity-factor is a special kind of H-factor and it has been shown that the decision problem of determining whether a graph has a (g,f)-parity factor is polynomial.

Let a, b be two integers such that $1 \le a \le b$ and $a \equiv b \pmod 2$. If $g(v) = a$ and $f(v) = b$ for all $v \in V(G)$, then a (g,f)-parity-factor is called an (a,b)*-parity factor*. Let $n \ge 1$ be odd. If $a = 1$ and $b = n$, then an (a,b)-parity factor is called a $(1,n)$*-odd factor*. There is also a special case of the (g,f)-factor problem which is called the *even factor problem*, i.e., the problem with $g(v) = 2$, $f(v) \ge |V(G)|$ and $f(v) \equiv g(v) \pmod 2$ for all $v \in V(G)$.

Fleischner gave a sufficient condition for a graph to have an even factor in terms of edge connectivtiy.

Theorem 1.1 (Fleischner,[8]; Lovász, [12])**.** *If G is a bridgeless graph with $\delta(G) \ge 3$, then G has an even factor.*

For a general graph G and an integer k, a spanning subgraph F such that

$$d_F(x) = k \ \text{ for all } x \in V(G)$$

is called a k*-factor*. In fact, a k-factor is also a (k,k)-parity factor.

The first investigation of the $(1,n)$-odd factor problem is due to Amahashi [2], who gave a Tutte type characterization for graphs having a global odd factor.

Theorem 1.2 (Amahashi)**.** *Let n be an odd integer. A graph G has a $(1,n)$-odd factor if and only if*

$$o(G-S) \le n\,|S| \qquad \textit{for all subsets } S \subset V(G). \tag{2}$$

For general odd value functions h, Cui and Kano [6] established a Tutte type of theorem.

Theorem 1.3 (Cui and Kano, [6])**.** *Let $h : V(G) \to N$ be odd value function. A graph G has a $(1,h)$-odd factor if and only if*

$$o(G-S) \le h(S) \qquad \textit{for all subsets } S \subset V(G). \tag{3}$$

Now there are many results on consecutive factors (i.e. (g, f)-factor). But the research progress on non-consecutive factors is slow. In non-consecutive factor problems, (g, f)-parity factors have many similar properties with k-factors. So we believe that many results on k-factors can be extended to (g, f)-factor. In this paper, we will extend a result on k-factors of regular graphs to the (g, f)-parity-factors.

Now let us recall one of the classical results due to Petersen.

Theorem 1.4 (Petersen [13]). *Let r and k be integers such that $1 \leq k \leq r$. Every $2r$-regular graph has a $2k$-factor.*

Considering the edge-connectivity, Gallai [7] proved the following result.

Theorem 1.5 (Gallai [7]). *Let r and k be integers such that $1 \leq k < r$, and G an m-edge-connected r-regular graph, where $m \geq 1$. If one of the following conditions holds, then G has a k-factor.*

(i) *r is even, k is odd, $|G|$ is even, and $\frac{r}{m} \leq k \leq r(1 - \frac{1}{m})$;*

(ii) *r is odd, k is even and $2 \leq k \leq r(1 - \frac{1}{m})$;*

(iii) *r and k are both odd and $\frac{r}{m} \leq k$.*

Bollobás, Satio and Wormald [3] improved above the result.

Theorem 1.6 (Bollobás, Saito and Wormald). *Let r and k be integers such that $1 \leq k < r$, and G be an m-edge-connected r-regular graph, where $m \geq 1$ is a positive integer. Let $m^* \in \{m, m+1\}$ such that $m^* \equiv 1 \pmod 2$. If one of the the following conditions holds, then G has a k-factor.*

(i) *r is odd, k is even and $2 \leq k \leq r(1 - \frac{1}{m^*})$;*

(ii) *r and k are both odd and $\frac{r}{m^*} \leq k$.*

In this paper, we extend Theorems 1.5 and 1.6 to (a, b)-factors. The main tool in our proofs is the following theorem of Lovász (see[11]).

Theorem 1.7 (Lovász [11]). *G has a (g, f)-parity factor if and only if for all disjoint subsets S and T of $V(G)$,*

$$\delta(S,T) = f(S) + \sum_{x \in T} d_G(x) - g(T) - e_G(S,T) - \tau \geq 0,$$

where τ denotes the number of components C, called f-odd components of $G - (S \cup T)$ such that $e_G(V(C), T) + f(V(C)) \equiv 1 \pmod 2$. Moreover, $\delta(S,T) \equiv f(V(G)) \pmod 2$.

2. Main Theorem

Theorem 2.1. *Let a, b and r be integers such that $1 \leq a \leq b < r$ and $a \equiv b \pmod 2$. Let G be an m-edge-connected r-regular graph with n vertices. Let $m^* \in \{m, m+1\}$ such that $m^* \equiv 1 \pmod 2$. If one of the following conditions holds, then G has an (a, b)-parity factor.*

- (i) *r is even, a, b are odd, $|G|$ is even, $\frac{r}{m} \leq b$ and $a \leq r(1 - \frac{1}{m})$;*
- (ii) *r is odd, a, b are even and $a \leq r(1 - \frac{1}{m^*})$;*
- (iii) *r, a, b are odd and $\frac{r}{m^*} \leq b$.*

By Theorem 1.6, (ii) and (iii) are true. Now we prove (i). Let $\theta_1 = \frac{a}{r}$ and $\theta_2 = \frac{b}{r}$. Then $0 < \theta_1 \leq \theta_2 < 1$. Suppose that G contains no (a, b)-parity factors. By Theorem 1.7, there exist two disjoint subsets S and T of $V(G)$ such that $S \cup T \neq \emptyset$, and

$$-2 \geq \delta(S, T) = b|S| + \sum_{x \in T} d_G(x) - a|T| - e_G(S, T) - \tau, \tag{4}$$

where τ is the number of a-odd (i.e. b-odd) components C of $G - (S \cup T)$. Let $C_1, \cdots, C_\tau$ denote a-odd components of $G - S - T$ and $D = C_1 \cup \cdots \cup C_\tau$.

Note that

$$\begin{aligned}
-2 \geq \delta(S, T) &= b|S| + \sum_{x \in T} d_G(x) - a|T| - e_G(S, T) - \tau \\
&= b|S| + (r - a)|T| - e_G(S, T) - \tau \\
&= \theta_2 r|S| + (1 - \theta_1) r|T| - e_G(S, T) - \tau \\
&= \theta_2 \sum_{x \in S} d_G(x) + (1 - \theta_1) \sum_{x \in T} d_G(x) - e_G(S, T) - \tau \\
&\geq \theta_2 (e_G(S, T) + \sum_{i=1}^{\tau} e_G(S, C_i)) + (1 - \theta_1)(e_G(S, T) + \sum_{i=1}^{\tau} e_G(T, C_i)) - e_G(S, T) - \tau \\
&= \sum_{i=1}^{\tau} (\theta_2 e_G(S, C_i) + (1 - \theta_1) e_G(T, C_i) - 1) + (\theta_2 - \theta_1) e_G(S, T) \\
&\geq \sum_{i=1}^{\tau} (\theta_2 e_G(S, C_i) + (1 - \theta_1) e_G(T, C_i) - 1).
\end{aligned}$$

Since G is connected and $0 < \theta_1 \leq \theta_2 < 1$, so $\theta_2 e_G(S, C_i) + (1 - \theta_1) e_G(T, C_i) > 0$ for each C_i. Hence we will obtain a contradiction by showing that for every $C = C_i$, $1 \leq i \leq \tau$, we have

$$\theta_2 e_G(S, C) + (1 - \theta_1) e_G(T, C) \geq 1. \tag{5}$$

These inequalities imply

$$\begin{aligned}-2 \geq \delta_G(S,T) &\geq \sum_{i=1}^{\tau}(\theta_2 e_G(S,C_i) + (1-\theta_1)e_G(T,C_i) - 1)\\ &> \sum_{i=1}^{\tau-2}(\theta_2 e_G(S,C_i) + (1-\theta_1)e_G(T,C_i) - 1) - 2 \geq -2,\end{aligned}$$

which is impossible.

Now, we will prove the 5 is true. Since C is an a-odd component of $G-(S\cup T)$, we have

$$a|C| + e_G(T,C) \equiv 1 \text{ (mod 2)}. \tag{6}$$

Moreover, since

$$r|C| = \sum_{x\in V(C)} d_G(x) = e_G(S\cup T, C) + 2|E(C)|,$$

we have

$$r|C| = e_G(S\cup T, C) \text{ (mod 2)}. \tag{7}$$

It is obvious that the two inequalities $e_G(S,C) \geq 1$ and $e_G(T,C) \geq 1$ imply

$$\theta_2 e_G(S,C) + (1-\theta_1)e_G(T,C) \geq \theta_2 + 1 - \theta_1 = 1.$$

Hence we may assume $e_G(S,C) = 0$ or $e_G(T,C) = 0$.

We consider the condition (i). If $e_G(S,C) = 0$, then $e_G(T,C) \geq m$. Since $a \leq r(1-\frac{1}{m})$, then $\theta_1 \leq 1 - \frac{1}{m}$ and so $1 \leq (1-\theta_1)m$. By substituting $e_G(T,C) \geq m$ and $e_G(S,C) = 0$ into (5), we have

$$(1-\theta_1)e_G(T,C) \geq (1-\theta_1)m \geq 1.$$

If $e_G(T,C) = 0$, then $e_G(S,C) \geq m$. Since $\frac{r}{m} \leq b$, hence $\theta_2 m \geq 1$, and so we obtain

$$\theta_2 e_G(S,C) \geq \theta_2 m \geq 1.$$

Consequently, condition (i) guarantees (5) holds and thus (i) is true. The proof is completed. □

Remark: The edge connectivity conditions in Theorem 2.1 are sharp.

We will give the construction for condition (i) of Theorem 2.1. For (ii) and (iii), the constructions are similar. Let $r \geq 2$ be an even integer, $a, b \geq 1$ two odd integers and $2 \leq m \leq r-2$ an even integer such that $b < r/m$ or $r(1-\frac{1}{m}) < a$. Since G has an (a,b)-parity factor if and only if G has an $(r-b, r-a)$-parity factor, so we can assume $b < r/m$. Let $J(r,m)$ be the complete graph K_{r+1} from which a matching of size $m/2$ is deleted. Take r disjoint copies of $J(r,m)$. Add m new vertices and connect each of these vertices to a vertex of degree $r-1$ of $J(r,m)$. This gives an m-edge-connected r-regular graph denoted by G. Let S denote the set of m new vertices and $T = \emptyset$. Let τ denote the number of components C, which are called a-odd components of $G-(S\cup T)$ and $e_G(V(C),T) + a|C| \equiv 1 \pmod 2$. Then we have $\tau = r$, and

$$\delta(S,T) = b|S| + \sum_{x\in T} d_{G-S}(x) - a|T| - \tau(S,T) = bm - r < 0.$$

So by Theorem 1.7, G contains no (a,b)-parity factors.

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