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Algebraic and computer-based methods in the undirected degree/diameter problem – A brief survey

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Abstract

This paper discusses the most popular algebraic techniques and computational methods that have been used to construct large undirected graphs with given degree and diameter.

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1. The Degree/Diameter Problem

Some of the questions arising more frequently in graph theory have to do with the interplay among different graph parameters. One such question is the *Degree/Diameter Problem* (or DDP, for short), which can be stated as:

Problem 1 (Degree/Diameter problem for undirected graphs)**.** *Given positive integers Δ and D, find the largest possible number of vertices $N_{\Delta,D}$ of a graph of maximum degree Δ and diameter D.*

It is easy to show that an upper bound for $N_{\Delta,D}$ is

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$$M_{\Delta,D} = 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1}$$
$$= \begin{cases} 1 + \Delta \frac{(\Delta-1)^D - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2D + 1 & \text{if } \Delta = 2 \end{cases}$$

The number $M_{\Delta,D}$ is called the *Moore bound*, and a graph of order $M_{\Delta,D}$ is called a *Moore graph*. Note that such a graph is necessarily regular of degree Δ.
Moore graphs exist only for a few combinations of Δ and D. For $D = 1$ and $\Delta \geq 1$, they are the complete graphs on $\Delta + 1$ vertices. For $D \geq 2$ and $\Delta = 2$, they are the cycles on $2D + 1$ vertices. For $D = 2$, Moore graphs exist for $\Delta = 2, 3, 7$, and possibly 57 [50]. There are no other Moore graphs.
The same problem can be stated for digraphs, but this survey focuses on the undirected case only. For more information about the Degree/Diameter Problem, both in the directed and the undirected case, see [71].

All research related to the Degree-Diameter Problem can be roughly classified as belonging to one of the following categories:

- **Upper bounds**: Given that the Moore bound can only be attained for a few combinations of degree and diameter, it is interesting to know what is the actual largest number of vertices that a graph can have, for other combinations of Δ and D. In other words, this line of research seeks to obtain tighter theoretical upper bounds for all combinations of Δ and D.

- **Lower bounds**: This second line of research, of more practical importance perhaps, is related with the construction of ever larger graphs, with order as close as possible to the theoretical upper bound.

The ideal situation is that the lower and upper bounds coincide, but that situation is again very rare for the moment. Only for a few combinations of degree and diameter the order of the largest known graph matches the theoretical upper bound, while in the vast majority of cases there is a gap between the lower and the upper bound, which varies according to Δ and D.
This situation is especially evident in the case of undirected graphs, where the gap is considerably large for most combinations of Δ and D. In the case of directed graphs the situation is more favourable, since there exist constructions that yield graphs with order close to the theoretical upper bound for all combinations of Δ and D.
In this survey we review some of the most effective methods for improving the lower bounds. The current knowledge about lower bounds is summarized in [64], which contains tables for the largest known general graphs, the largest known Cayley graphs, circulant, planar, and bipartite graphs, for degrees 3 to 16 and diameters 2 to 10. For each graph family, the website also maintains tables of the best known upper bounds, and the percentage of the largest known graph orders in relation to those upper bounds.
Besides the record graphs, which are the ones shown in the tables, we are also interested in methods that provide *suboptimal* graphs, especially when they yield *infinite families*. For instance, the method described in [67] yields an infinite family of graphs of diameter 2, and even though some

of the small-degree members of the family have been superseded by larger graphs, the family still stands. Additionally, some suboptimal graphs might have other important properties, such as symmetry and fault-tolerance, which are useful in applications.
Back to lower bounds, all the efforts to construct large graphs can in turn be roughly classified into two main categories:

- **Analytic constructions**: Graph operations, like compounding and voltage assignment, provide the means for building large graphs from smaller base graphs or blocks. A judicious choice of the building blocks and operations on them can yield large graphs for certain combinations of Δ and D. This is the principle that has been followed in [44, 81] (with graph compounding), and [67] (with voltage assignment), for instance.

- **Computer-based techniques**: This second approach relies on the power of electronic computers in order to find larger graphs, either using the standard combinatorial optimization techniques (as in [100]), or graph operations, like voltage assignment, coupled with computer search (as in [59]).

In terms of the number of graphs produced, the most successful techniques are graph compounding, polarity graphs of generalized polygons, and voltage assignment, the latter in conjunction with computer search. Of the 162 entries in the current table of largest known general graphs recorded in [64], 98 have been found by voltage assignment (60%), while 19 have been found with the aid of graph compounding (12%), and 18 are somehow related to polarity graphs of generalized polygons (11%). Finally, 11 entries have been found by other computer-based search techniques (7%). Together, all these techniques account for 90% of the table. These proportions can be best visualized in Figure 1.

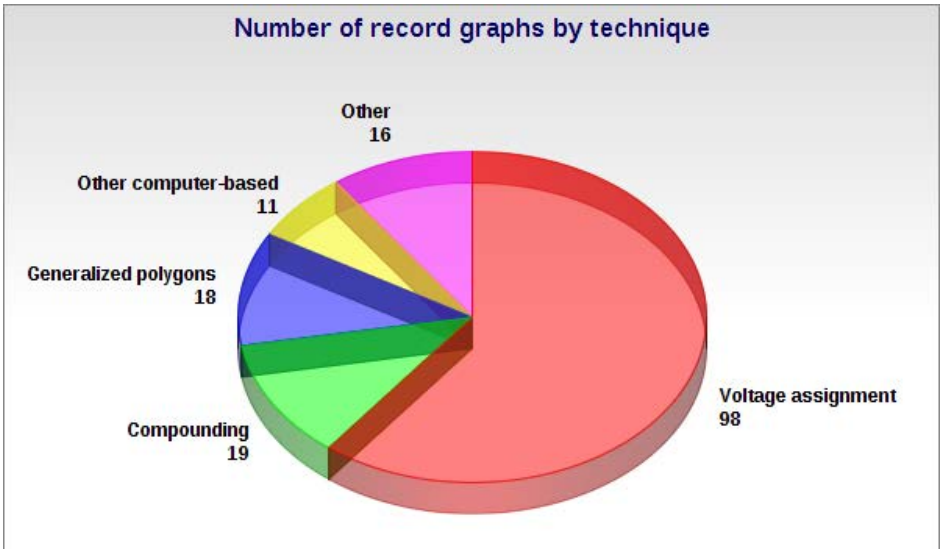

Figure 1. Number of record graphs by technique

2. Basic algorithmic techniques

In this section we deal with some computer-based techniques for solving intractable combinatorial problems, such as DDP. Computer approaches range from systematic search techniques to heuristic methods, such as genetic algorithms and simulated annealing, which are now commonplace in Combinatorial Optimization. A *combinatorial optimization problem* consists of an *objective function* to be minimized or maximized, $f : D_1 \times \cdots \times D_k \to \mathbb{R}^+$, and a set of constraints among the variables $x_i \in D_i$. A feasible solution is a $k-$tuple $(x_1, \ldots, x_k)$, such that $x_i \in D_i$ and all constraints are satisfied. The set $\mathcal{S}$ of all feasible solutions is often called the *search space*, and it is usually endowed with a *neighborhood structure*, which is a function $\mathcal{N} : \mathcal{S} \to 2^{\mathcal{S}}$ that assigns a set of neighbours to every feasible solution $s \in \mathcal{S}$.

Combinatorial optimization problems include the famous Travelling Salesman Problem (TSP – Problem ND22 of [38]), the $0-1$ Knapsack Problem (Problem MP9 of [38]), various scheduling problems, and so on. In particular, DDP can be seen as a maximization problem, where the objective function is the number of vertices of the resulting subgraph. Alternatively, we can minimize the diameter, keeping the degree and the number of vertices constant.

When feasible, an exhaustive exploration of the search space yields the optimal solution to the problem. However, many of these combinatorial optimization problems are $\mathcal{NP}-$hard, and hence, exhaustive exploration of the search space is not feasible. Approximate or heuristic methods sacrifice the guarantee of finding an optimal solution for the sake of lower computation overhead. The greatest danger for these methods is to be trapped in a *locally optimal solution*, which is an $\hat{s} \in \mathcal{S}$ such that $f(\hat{s}) \leq s$ (if we have a minimization problem), or $f(\hat{s}) \geq s$ (if we have a maximization problem), for all $s \in \mathcal{N}(\hat{s})$. Different methods are classified according to the strategy they use for coping with this problem.

In recent years, a set of strategies have been devised, which are very general in nature, and where particular problem-specific heuristics can be plugged in, in order to solve different combinatorial optimization problems. Due to their general nature, these methods are collectively called *metaheuristics*. They include greedy algorithms, basic local search, simulated annealing, tabu search, evolutionary algorithms (comprising genetic algorithms), and ant colony optimization, among others.

The complexity status of DDP is not known: no polynomial-time algorithm has been found so far to solve it, and yet it has not been proved to be $\mathcal{NP}-$hard. For the moment, approximate and heuristic methods constitute a viable option. In the remaining of this section we review some of these algorithmic techniques, especially those that have been successfully employed in DDP and related problems. W.l.o.g. we will assume that we have to *minimize* the objective function f. For more information about metaheuristics, including their taxonomy, and a framework for their analysis and design, we refer the reader to [10].

2.1. Systematic search

The basis of many systematic search procedures is the technique known as *backtracking*, which is essentially a depth-first traversal of the search space, that has been organized as a tree. The idea behind backtracking is that some portions of this search space can be pruned if we realize that they do not contain an optimal solution.

Suppose that we have a partial solution $s^{(i)} = (x_1, \ldots, x_i)$ to the optimization problem above, with $i \leq k$, and such that $x_j \in D_j$ for $j = 1, \ldots, i$, and all constraints are satisfied. If the value of the objective function in this partial solution is worse than the best value found so far, there is no need to explore any of the solutions $s = (x_1, \ldots, x_k)$ having $(x_1, \ldots, x_i)$ as a prefix. Algorithm 1 formalizes these ideas.

Algorithm 1: Backtracking

Input : The objective function f and the set of constraints.
Output: An optimal solution $\hat{s}$.

```
1  i:=1;
2  Initialize ŝ;
3  while i > 0 do
4      while there is an untried x_i s.t. s^(i) is feasible do
5          if f(s^(i)) < f(ŝ) then
6              if i = k then
7                  ŝ:=s^(k);
8              else
9                  i:=i + 1;
10             end
11         end
12     end
13     i:=i − 1;
14 end
15 Return ŝ;
```

Backtracking can be combined with more aggresive pruning techniques, or isomorph rejection sieves (e.g. [14, 56]).

2.2. Greedy algorithms

The *greedy technique* is perhaps the most straightforward and most popular heuristic for dealing with combinatorial optimization problems. It constructs the solution $s^{(k)} = (x_1, \ldots, x_k)$ in stages, adding each new component x_i one by one, and leaving the previous components fixed. The i-th component is chosen as the $x_i \in D_i$ that optimizes the partial solution $s^{(i)} = (x_1, \ldots, x_i)$. Algorithm 2 describes the technique more formally.

It is easy to see that the running time of Algorithm 2 is bounded above by $\sum_{i=1}^{k} |D_i|$, and hence polynomial in k, provided that the $|D_i|$ are polynomial in k. So simple as it looks, the greedy technique is the basis of many effient and popular algorithms, such as Dijkstra's algorithm, for computing single-source shortest paths in graphs, and Kruskal's algorithm, for constructing a minimum-weight spanning tree. Although in general the solution provided by a greedy algorithm is not optimal, it can be optimal if our problem has an underlying *matroid* or *greedoid* structure [55], such as in the two examples mentioned above.

Algorithm 2: The greedy method

Input : The objective function f and the set of constraints.
Output: A feasible solution $\hat{s}$.

```
for i := 1 to k do
    Choose x_i ∈ D_i such that s^(i) = (x_1, ..., x_i) is feasible,
    and f(s^(i)) is minimal;
end
Return ŝ = s^(k);
```

2.3. Basic local search and its derivatives

Local search starts with some initial feasible solution s, and iteratively tries to replace s by a better solution s' in the neighborhood of s, until a local minimum is reached. This process is also called *iterative improvement* for obvious reasons. The replacement of s by s' can be done in a number of ways. For instance, we can choose the best feasible solution in $\mathcal{N}(s)$, or we can take s' as the first element of $\mathcal{N}(s)$ that we come across, that is better than s, or any intermediate option.
The quality of the solutions obtained by basic local search is usually very poor, therefore a number of different strategies have been developed to reduce the risk of getting trapped in a local minimum. *Simulated annealing* is one of the first strategies in that sense. The main idea behind simulated annealing is to allow occasional moves resulting in a worse solution s' (uphill moves). This emulates a cooling process of metals, where the atoms strive to adopt a configuration of minimal potential energy (a crystal), but occasional rearrangements occur in the opposite direction.
The process is depicted in Algorithm 3. The control parameter T (the temperature) regulates the speed of convergence and the probability of accepting worse solutions at any given moment. That probability is usually computed by the formula $p(T, s', s) = \exp(\frac{f(s')-f(s)}{T})$. The temperature is decreased during the search process, thus the probability of accepting uphill moves is higher at the beginning of the process, and gradually decreases, so that the process converges to basic local search. There is a trade-off associated with temperature regulation, namely computation time versus quality of solutions. If we decrease the temperature slowly, the chances of converging to a globally optimal solution are better, whereas computation time obviously increases. On the other hand, a quick temperature decrease leads to a smaller computation time, but the algorithm is more likely to get trapped in a local minimum. Therefore, the choice of the cooling schedule is critical for the performance of the algorithm.
Note that the probability of accepting an uphill move also depends on the absolute value of that move. Small uphill moves are more likely to be accepted than large ones. A variant of simulated annealing, called *threshold accepting* avoids the computation of probabilities in Step 8 of Algorithm 3, and simply compares the difference $f(s') - f(s)$ with a given threshold, to decide whether to accept it or not.

Another metaheuristic that improves basic local search is *tabu search*. Simple tabu search keeps a list of the most recently visited solutions (tabu list), in order to avoid re-visiting them. In other words, it relies on *memory* to escape local minima and avoid cycles. The trade-off here involves the benefits provided by the tabu list, and the cost of maintaining and searching it.

Algorithm 3: Simulated annealing

Input : The objective function f, the set of constraints, and the neighborhood function $\mathcal{N}$.
Output: A feasible solution $\hat{s}$ (that is in general a local minimum of f).

```
1  s:=Generate initial solution;
2  T:= T_0;
3  while termination conditions not met do
4  |   s':=PickAtRandom(N(s));
5  |   if f(s') < f(s) then
6  |   |   s:=s';
7  |   else
8  |   |   Accept s' with probability p(T, s', s);
9  |   end
10 |   Update T;
11 end
```

2.4. Evolutionary algorithms

Evolutionary algorithms are inspired by natural evolution principles. The driving force of natural evolution is natural *selection*, survival of the fittest individuals among a diverse population. The main factors that contribute to population diversity are *recombination* and *mutation*. An evolutionary algorithm emulates this process: it starts with an initial population of feasible solutions, and then iterates the three operators (recombination, mutation, and selection) in order to obtain increasingly better populations, where "better" here means having a lower value of the objective function f. Algorithm 4 describes a general evolutionary algorithm.

Algorithm 4: General evolutionary algorithm

Input : The objective function f, and the set of constraints.
Output: A good feasible solution $\hat{s}$.

```
1  P:=Generate initial population;
2  Evaluate P;
3  while termination conditions not met do
4  |   P':=Recombine(P);
5  |   P'':=Mutate(P');
6  |   Evaluate(P'');
7  |   P:=Select(P ∪ P'');
8  end
9  Return the 'best' individual of P;
```

In a variant of evolutionary algorithms called *memetic algorithms*, a local search is applied to every individual of the population [75]. Memetic algorithms have been used recently to address various network design problems, although not DDP specifically.

2.5. Approximation algorithms

An *approximation algorithm* is any heuristic algorithm for which we can get a *performance guarantee*, i.e. we can *prove* that the solution given by the algorithm will always lie within a certain neighborhood of the optimal solution. Let us assume that we have a maximization problem, and let OPT be the value of the objective function f in the optimal solution of some problem instance, and ALG the value of f produced by the algorithm. The performance of the algorithm is measured by the ratio OPT/ALG, called the *approximation ratio*. Usually, the goal is to find a polynomial-time heuristic algorithm with the smallest approximation ratio OPT/ALG.

Combinatorial optimization problems can be classified according to the difficulty with which they can be approximated, i.e. the approximation ratio that can be reached in polynomial time, in the worst or average case. Our maximization problem is said to have a *Polynomial-Time Approximation Scheme (PTAS)* if for every $\epsilon > 0$ there exists a polynomial-time algorithm that computes a solution with worst-case approximation ratio $1 - \epsilon$ (or $1 + \epsilon$ for minimization problems).

In turn, the class of problems having a PTAS is included in the class Apx, that consists of all $\mathcal{NP}-$hard optimization problems for which there is a polynomial-time algorithm with a constant approximation ratio OPT/ALG, for all problem instances. Loosely speaking, these are the problems that can be approximated in a reasonable time.

There are a number of books that treat the design of approximation algorithms in depth, e.g. [95] and [99]. In addition to that, [20] gives a list of optimization problems, and specifies which of them have a PTAS. [93] focuses on inapproximability results, and [68] investigates the conditions when the greedy technique yields good approximations.

3. Computer-based techniques in DDP

Since the number of graphs of a given maximum degree and diameter is very large, it is a natural idea to recourse to computers for help in the task of locating the largest instances. Computer-guided search has produced some instances of large circulant graphs in [35], and in combination with algebraic techniques, such as voltage assignment, it has produced many instances of large graphs, as in [59]. However, as Δ and D grow, the search space becomes too large, even for computers, so that brute-force search is not feasible, and some heuristics are needed to guide the search or prune the search space. In this section we briefly review some heuristic algorithms that have been used for that purpose.

The first serious attempt in that direction was made by Allwright, who used 2-opting, a local search technique developed for the Travelling Salesman Problem, to find large graphs with given degree and diameter [1].

In another landmark paper, Dinneen and Hafner used computer search and clever techniques to reduce the search space [25]. Their large graphs were Cayley graphs of non-abelian groups, like semidirect products of cyclic groups, for instance.

Around the same time, Mitjana and Comellas obtained the graph of order 253 at the entry (8,3) of the table, using a threshold accepting algorithm (see Section 2.3). It is also a Cayley graph of the semidirect product of two cyclic groups. The idea had been around since 1990, at least [17].

Later, Wohlmuth found another large Cayley graph with the aid of a genetic algorithm (see Section 2.4), that today still stands as the largest Cayley graph of degree 6 and diameter 3 [100]. Let us

explain this algorithm briefly. The population consists of a set of Cayley graphs, that are subgroups of the symmetric group S_n, each one being represented by a generating set of $n-$permutations. Mutation of an individual Cayley graph is achieved by replacing one generator by the product of two different generators in the same generating set. Therefore, a subgroup of the former Cayley graph is obtained. The crossover operator exchanges generators from different generating sets. Finally, the fitness function to be minimized is defined by

$$f(x_1, x_2, x_3, x_4) = \begin{cases} 10x_2 + 10x_1 + 10 & \text{if } D \leq 2 \\ 10x_2 + x_3 & \text{if } D = 3 \\ 10x_2 - x_4 & \text{if } D \geq 3 \end{cases}$$

where x_i represents the number of vertices with distance i from any fixed vertex x (the choice of x is irrelevant, since we are working with a Cayley graph, which is vertex-transitive). That function was chosen so as to minimize the number of cycles of length ≤ 6, which correspond to large values of x_1, x_2, x_3, and to minimize the number of vertices at distance 4.

Sampels also used a genetic algorithm for constructing large Cayley graphs [85]. Some of the groups he used were specified by a finite presentation, but the large majority of them were semidirect products of cyclic groups, generated by a set of permutations.

Exoo argued that Cayley graphs are relatively rare, and therefore it makes sense to relax the symmetry conditions, and look for improvements in a larger search space. The graphs he considered have an order that is a small integral multiple of the size of their respective automorphism group. Thus, he managed to obtain several large graphs in the low-order entries of the table, with the aid of an algorithm that is a synthesis of simulated annealing and tabu search (see Section 2.3), which had been previously used to construct Ramsey colourings. The algorithm is described in [29], and the graphs are given in [30].

Finally, [69] describes a hybrid heuristic algorithm (HSAGA) to solve a relaxed variant of the Degree-Diameter Problem. HSAGA stands for 'Hybrid Simulated Annealing Genetic Algorithm'.

4. Cayley graphs

Let Γ be a group, and $S \subset \Gamma$, where $1 \notin S$.[1] The *Cayley color digraph* on Γ with connection set S is a digraph with vertex set $V = \Gamma$, and there is an arc going from g to h, labeled s (where $s \in S$), if $sg = h$. The labels are the 'colors'; if we drop them, we get the *Cayley digraph* on Γ with connection set S.

Now, if we take $S := S \cup S^{-1}$ and collapse two opposing arcs that have inverse labels, then we get the *Cayley color graph* and the *Cayley graph*, respectively, with connection set S. The symbol $Cay(\Gamma, S)$ will denote the Cayley graph on Γ with connection set S.

Cayley graphs are vertex-transitive, however the converse is not true in general. The smallest example of a vertex-transitive non-Cayley graph is the Petersen graph. The following result gives a complete characterization of Cayley graphs:

[1] Some authors require that S be a generating set of Γ.

Theorem 4.1 ([84])**.** *A graph G is a Cayley graph if, and only if, $Aut(G)$ contains a regular subgroup.*

Recall that a group Γ acting on a set X is said to be regular if for any two $x, y \in X$ there is exactly an element $\gamma \in \Gamma$ that transforms x into y.

Notable Cayley graphs include the complete graph K_n, the complete multipartite graph, and the $d-$dimensional cube Q_d. A Cayley graph on the cyclic group $\mathbb{Z}_n$ is called a *circulant graph*. Due to their symmetry properties, among others, Cayley graphs (including circulant graphs) have been widely studied as models of communication networks since the late 80's and early 90's (see [3, 19, 90, 57, 49, 102, 39], for instance).

$M_{\Delta,D} - 2$ is the best general upper bound known so far for Cayley graphs on arbitrary groups (and for vertex-transitive graphs, for that matter). Many of the largest known general graphs are Cayley graphs. Additionally, the site [64] maintains a table of the largest known Cayley graphs.

The case of abelian Cayley graphs is treated in depth in [26]. However, in order to obtain a large Cayley graph, the group should be as far as possible from abelian [32]. Indeed, this had already been noted by Dinneen and Hafner in [25], who constructed Cayley graphs of semidirect products of cyclic groups, as well as other combinations of direct and semidirect products.

If $G_1 = Cay(\Gamma_1, S_1)$ and $G_2 = Cay(\Gamma_2, S_2)$, then the Cartesian product $G_1 \square G_2$ is a Cayley graph of the group $\Gamma_1 \times \Gamma_2$ on the generating set $(S_1 \times \{1_{\Gamma_2}\}) \cup (\{1_{\Gamma_1}\} \times S_2)$, where 1_Γ denotes the identity element of Γ. On the other hand, the Cayley graph of a semidirect product of groups corresponds to the zig-zag product of their corresponding Cayley factors (see [2, 83], and [48], p. 440).

There is a tradeoff between the order of the Cayley graph and its computational complexity, and even though semidirect products of cyclic groups are not very far from abelian, they have the advantage that they are easy to generate and compute with, and they did produce good results in [25] (probably because they were the first non-abelian groups that were explored systematically).

Additional constructions of large Cayley graphs are given in [21, 87, 88, 65, 66, 89, 96, 97, 98]. Some of them also use combinations of direct and semidirect products of groups (e.g. [87, 65, 96, 97]). More information about Cayley graph constructions can be found in [71].

5. Circulant graphs

Circulant graphs are a special case of abelian Cayley graphs, namely Cayley graphs of cyclic groups. The study of circulant graphs began in 1970 with Elspas and Turner [28]. Coincidentally, it was also Elspas who had formulated the Degree/Diameter Problem back in 1964 [27], but apparently he missed the connection between both topics.

Even though the abelian property of the underlying group prevents abelian Cayley graphs in general (and circulant graphs in particular) to grow as large as their non-abelian counterparts, these graphs have been widely used as topologies for computer networks and parallel computers, due to their other nice properties. Paraphrasing [26]: "... the extra structure provided by the groups may provide compensating advantages ..., such as good routing algorithms, easy constructibility, and the ability to map common problems onto the architecture".

An undirected circulant graph $C(n; S)$ is a Cayley graph on the cyclic group $\mathbb{Z}_n$, with a symmetric connection set S (i.e. $S = S^{-1}$). Since $\mathbb{Z}_n$ is abelian, we can switch to additive notation and

rephrase the symmetry condition of the connection set as $S = -S$. Being a Cayley graph, $C(n; S)$ is vertex-transitive. The degree of $C(n; S)$ is $\Delta = |S|$, and its order is obviously n. A circulant graph can also be defined as a graph of n vertices whose adjacency matrix is circulant [27].

Regarding the degree, we distinguish two cases:

1. Even degree: $\Delta = 2t$. In that case, $S = \{\pm s_1, \ldots, \pm s_t\}$, where $1 \leq s_1 < \ldots < s_t < \frac{n}{2}$.
2. Odd degree: $\Delta = 2t + 1$. In that case, $S = \{\pm s_1, \ldots, \pm s_t, \frac{n}{2}\}$, where $1 \leq s_1 < \ldots < s_t < \frac{n}{2}$. It follows that odd degree is only possible when n is even.

$C(n; S)$ is connected if, and only if, $\gcd(n, s_1, \ldots, s_t) = 1$. If $\gcd(n, r) = 1$, then $C(n; S)$ is isomorphic to $C(n; rS)$, where multiplication is taken modulo n. In that case we say that the connection sets S and rS are multiplicatively related. It should be noted, however, that two circulant graphs may be isomorphic without their connection sets being multiplicatively related [27, 77].

Now let $N^{circ}_{\Delta,D}$ be the number of vertices of the largest circulant graph with degree Δ and diameter D. It was proved in [13] that, if $\Delta = 2t$, then

$$N^{circ}_{\Delta,D} \leq F(t, D) = \sum_{i=0}^{t} 2^i \binom{t}{i}\binom{D}{i} \tag{1}$$

This upper bound was later rediscovered by Muga [76]. The quantity $F(t, D)$ also turns out to be an upper bound for $N^{AC}_{\Delta,D}$, the order of the largest Cayley graph over an abelian group, with degree Δ and diameter D [26]. It is quite surprising that no better upper bound (yet) exists for circulant graphs, considering that they are a special case of abelian Cayley graphs.

By the way, the numbers $F(t, D)$ of Eq. 1 are known as *Delannoy numbers* (sequence A008288 of [78]), and they arise in a variety of combinatorial and geometric problems [92]. For example, they correspond to the volume of the ball of radius $D/2$ in the L^1 metric in t dimensions [26, 70, 94]. Unaware of the Delannoy connection, Stanton and Cowan had already studied these numbers back in 1970 [91], and had given several interesting formulas for them, such as:

$$F(t, D) = \sum_{i=0}^{t} \binom{t}{i}\binom{D+i}{t} = \sum_{i=0}^{t} \binom{D+i}{i}\binom{D}{t-i} \tag{2}$$

Other exact and asymptotic formulas are given in [70].

In the case of odd Δ (i.e. $\Delta = 2t + 1$) we have the generator $\frac{n}{2}$, which is its own inverse. Figure 2 provides a graphical example of that case, for $\Delta = 5$ and $D = 3$. We have denoted the generators as a, b, c, where $c = -c$. We can see in this example that the edges labeled with c duplicate every vertex, except those in the lowest level (level D). Therefore, an upper bound for the maximum number of vertices in this case is:

$$F'(t, D) = F(t, D) + F(t, D - 1) \tag{3}$$

Circulant graphs of the form $C(n; \{\pm 1, \pm s_2, \ldots, \pm s_t\})$ are called *multi-loop graphs*. In particular, for $t = 2$ and $t = 3$ they are called *double-loop graphs* and *triple-loop graphs*, respectively. According to [74], the maximum order of a triple-loop network $C(n; \{\pm 1, \pm s_2, \pm s_3\})$ is:

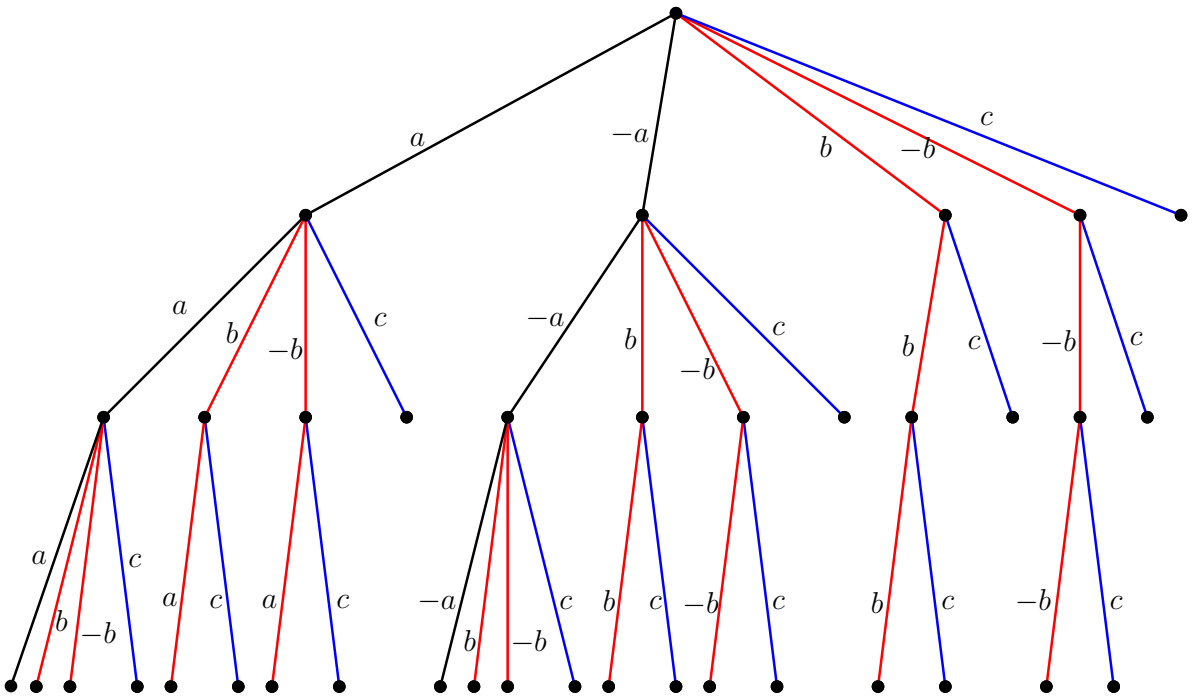

Figure 2. Tree representation of a maximal abelian Cayley graph of degree 5 and diameter 3.

$$T_{6,D} = \begin{cases} \frac{32}{27}D^3 + \frac{16}{9}D^2 + 2D + 1 & \text{if } D \equiv 0 \bmod 3 \\ 32\lfloor\frac{D}{3}\rfloor^3 + 48\lfloor\frac{D}{3}\rfloor^2 + 30\lfloor\frac{D}{3}\rfloor + 7 & \text{if } D \equiv 1 \bmod 3 \\ 32\lfloor\frac{D}{3}\rfloor^3 + 80\lfloor\frac{D}{3}\rfloor^2 + 70\lfloor\frac{D}{3}\rfloor + 21 & \text{if } D \equiv 2 \bmod 3 \end{cases} \tag{4}$$

Some important sources of large circulant graphs are:

1. For $t = 2$ an optimal circulant graph $C(n; \pm s_1, \pm s_2)$, is achieved for $s_1 = \lfloor \frac{1}{2}(\sqrt{2n-1} - 1)\rfloor$ and $s_2 = s_1 + 1$ [9, 13, 72].

2. Monakhov and Monakhova [73] used an evolutionary algorithm to find dense families of undirected circulant graphs. In particular, with the aid of this algorithm they found some families of large triple-loop graphs.

3. For larger degrees we have the construction $C(n; \pm 1, \pm s, \ldots, \pm s^{t-1})$, where $n = s^t$, and s is an odd integer, which yields good circulant graphs of diameter $\frac{t}{2}(s-1) = \frac{t}{2}n^{1/t} - \frac{t}{2}$ [101].

4. Applying the methods described in [26] for abelian Cayley graphs, Delorme and Lewis have recently obtained several circulant graphs of large order [24, 58].

5. By a systematic computer-based search, several large circulant graphs have been found in [35].

6. Voltage assignment

So far we have seen specializations of Cayley graphs, such as circulant graphs. In this section we deal with a generalization of the Cayley graph construction, called voltage assignment. In its original form, voltage assignment takes a *base digraph* and a group to obtain a new, larger digraph. By dropping arc directions, or by taking a symmetric generating set for the group, one can also construct undirected graphs.

This section is divided into two subsections, the first one devoted to the definition and basic properties of the construction, and the second one to a short review of its connection with the Degree-Diameter Problem. For more detail we refer the reader to [46, 4, 5].

6.1. Definition and main properties

Let G be a digraph, Γ a finite group, and $\alpha : A \to \Gamma$ a labelling of the arcs with elements of Γ. The labels are usually called *voltages*, and α is a *voltage assignment*. Given G, Γ, and α, a new digraph G^α is constructed as follows: $V(G^\alpha) = V^\alpha = V \times \Gamma$, and $A(G^\alpha) = A^\alpha = A \times \Gamma$. If we have an arc $(u, v) \in A$ in G, with voltage $h \in \Gamma$, then (u, g) is joined to (v, gh) in G^α, for all $g \in \Gamma$. Quite often the vertex $(v, g) \in V^\alpha$ is written v_g. The digraph G^α is a topological covering of G, and it is commonly referred to as the the *derived digraph* or the *lift* of G by Γ and α. The digraph G is called the *base digraph*. A straightforward consequence of this definition is that both the number of vertices and the number of arcs of G^α are divisible by $|\Gamma|$. Figure 3 shows an example of a voltage assignment of a dipole, with voltages in $\mathbb{Z}_3$, and the resulting derived digraph.

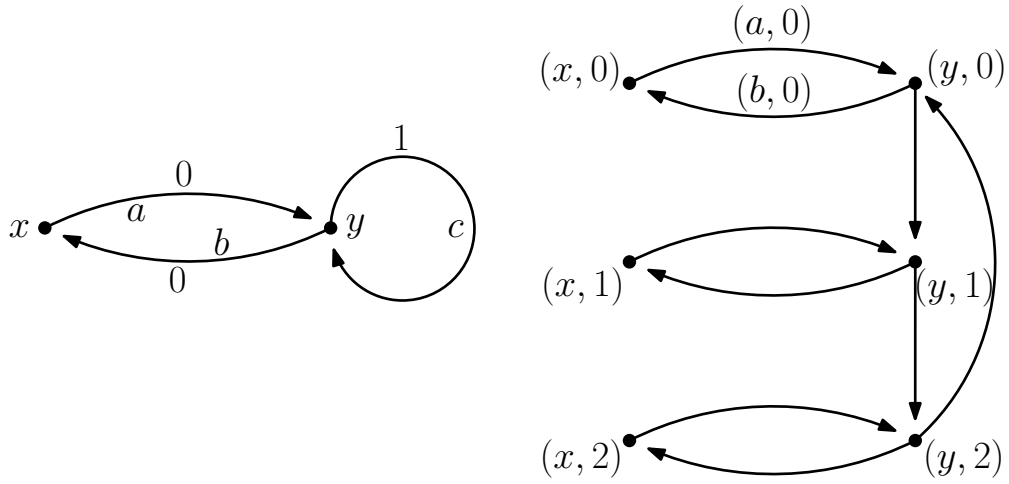

Figure 3. Voltage assignment of a dipole, with voltages in $\mathbb{Z}_3$, and the resulting derived digraph.

The function $\pi : G^\alpha \to G$ that erases the second coordinate of any vertex or arc of G^α, is called the *natural projection* of G^α onto G. The natural projection is defined for individual vertices or arcs of G^α, but it can be extended to subgraphs of G^α in two ways. The most obvious extension is the set-theoretic extension, i.e. $\pi(S) = \cup_{s \in S} \pi(s)$. Let us denote this extension by π_1. However, in some cases it is very useful to consider another extension π_2 of π, namely the one that allows

repetition of elements.[2] For example, if we have a walk W in G^α, consisting of the arcs a_g, b_h, a_i, then $\pi_2(W) = a, b, a$, while $\pi_1(W) = \{a, b\}$. That is, $\pi_2(W)$ is also a walk in G, while $\pi_1(W)$ is just a set of arcs. If W is a natural walk in G^α, then $\pi_2(W)$ is also a natural walk (in G).
A lift of the walk $W = a_1, a_2, \ldots, a_k$, in G, is a walk $W' = a'_1, a'_2, \ldots, a'_k$ in G^α, such that $\pi_2(W') = W$. We say that W *lifts* to W'. On the other hand, we say that a walk W in G *expands* to a subgraph W' of G^α if $\pi_1(W') = W$, and W' is maximal with respect to that property. W' is the *expansion*[3] of W. In some sense, the lifting operation is the 'inverse' of π_2, and the expansion is the 'inverse' of π_1.
Given a walk $W = a_1, a_2, \ldots, a_k$, on G, whose arcs have voltages $g_1, g_2, \ldots, g_k$, the *net voltage* (or simply the *voltage*) of W, denoted $\alpha(W)$, is defined as the product $g_1^{e_1} g_2^{e_2}, \ldots, g_k^{e_k}$, where $e_i = 1$ if the corresponding arc a_i is traversed in the natural direction, and $e_i = -1$ if it is traversed in reverse direction. If W is a natural walk, then all the exponents are 1. With the aid of projection π_2 we can also define the net voltage of a walk W' on G^α, just as the net voltage of $\pi_2(W')$, which is a walk on G.
If x is a vertex (resp. an arc) of G, then $\pi_1^{-1}(x) = \{(x; g) : g \in \Gamma\}$ is called the vertex (resp. arc) *fibre* over x. There are several known facts concerning fibres:

1. If $a = (u, v)$ is an arc in G, then the fibre over a is a perfect matching between the fibre over u and the fibre over v (see 2.1.2 in [46], p 60).

2. The fibre over a loop is a set of cycles (Idem).

3. If the walk W starts at vertex $u \in G$, then for each vertex u_g in the fibre over u, there is a unique lift of W that starts at u_g (hence it can be denoted W_g). Moreover, if W ends at vertex v, and has net voltage h, then W_g terminates at v_{gh} (Theorems 2.1.1 and 2.1.2 of [46], p 62).

Now, given the set of voltages $X = \{x_1, x_2, \ldots, x_k\}$ on the arcs leaving a vertex v of G, and given an arbitrary $g \in \Gamma$, define $g\alpha$ to be the voltage assignment identical to α, such that the set U is modified as follows: $X = \{gx_1, gx_2, \ldots, gx_k\}$. The resulting derived graph $G^{g\alpha}$ is isomorphic to G^α. This implies that any voltage assignment α is equivalent to a voltage assignment $g\alpha$ with the identity element of G assigned to the arcs of any spanning tree of G.
A voltage assignment a is said to be in *standard form* if a spanning tree of G is assigned the identity element of G. From the above results we conclude that for each voltage assignment α' we can find α in standard form. The relation defined by 'having the same standard form' is an equivalence relation on voltage assignments. It is possible, however, that two voltage assignments from two different equivalence classes will give rise to isomorphic lifts.

6.2. Voltage assignment in DDP

Voltage assignment has been the most successful technique so far for obtaining large graphs of given degree and diameter: It is responsible of about 60% of the largest known graphs of the table [64]. Therefore, this application of voltage assignment merits a detailed account here.

[2] Most authors do not make any distinction between π_1 and π_2, but we believe that this omission may be a source of confusion.

[3] This should not be confused with expanding graphs, or expanders.

Voltage assignment has been used in a variety of ways, ranging from ad hoc constructions, to general constructions, and in combination with computer search. For example, in a classic paper, McKay, Miller and Širáň describe a family of large vertex-transitive graphs with $\Delta = (3q - 1)/2$, where q is a prime power congruent with 1 (mod 4), though the construction generalizes to all prime powers [67]. The base graphs used in [67] are complete bipartite graphs with loops. Later, Šiagiová showed that the McKay-Miller-Širáň graphs can also be obtained as lifts of dipoles [86]. Other authors played for some time with the possibility of using voltage assignment for obtaining large graphs (e.g. [4, 5, 103]), until a breakthrough occurred in 2008, when Loz and Širáň published a large number of new record graphs found with the aid of voltage assignment and computer search [59]. Their algorithm explores a search space consisting of lifts of simple base graphs (bouquets, dipoles, etc.) by non-abelian groups. Their groups were taken among the ones suggested by Dinneen and Hafner for constructing large Cayley graphs [25], namely semidirect products of cyclic groups, such as $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$, $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes_\psi \mathbb{Z}_n$, and $(\mathbb{Z}_m \rtimes_r \mathbb{Z}_n) \rtimes (\mathbb{Z}_m \rtimes_r \mathbb{Z}_n)$. Those were precisely the groups that were used later in [62] to enlarge the table of the largest known graphs up to degree 20 [64].

Indeed, the method of [59] can be seen as a generalization of [25] inasmuch as Cayley graphs can be obtained as lifts of bouquets. The algorithm used by Loz and Širáň is a basic *random search*, combined with a number of clever *sieves*, designed to prune the search space. The algorithm starts with a base graph and a certain family of groups, and then iterates through the groups, generating random voltage assignments, and computing the diameter of the corresponding lift in each case. The sieves control the prospective voltage groups, the number of trials to be made with each group, and further computations with the derived graph.

The first sieve consists of sorting out the groups that do not meet certain criteria, like having a small center, for instance. Then, the *escalation sieve* controls the number of random trials to be made with each voltage group. Initially, it assigns a predefined number of trials to each group, and then, the number of trials is modified during the search process, according to the prospects of finding a good derived graph with the given group. Finally, the *girth sieve* abandons the computation of the diameter, if the girth is found to be smaller than a predefined threshold.

Algorithm 5 describes the basic random search structure of this procedure, to which the sieves can be plugged in. Some more details concerning the sieves and the implementation of the algorithm are given in [61], but no detailed pseudo-code description of the algorithm has been published so far.

Just as in the case of Cayley graphs, recent results show that in order to obtain good graphs by voltage assignment, the underlying group should be as far as possible from abelian [60, 32, 33]. An alternative for the use of non-abelian groups is to use base graphs that are not bouquets (e.g. dipoles). Recent progress in finding large graphs with given degree and diameter via voltage assignment includes [88, 65, 66, 16], which illustrate these alternatives. For instance, [88] uses abelian groups on the dipole, while [65] uses non-abelian groups that can be factorized as a combination of direct and semidirect products. In [16], Canale and Rodríguez improve some of the values in the table of general graphs using a computer-guided search similar to that of [59].

Finally, it is worth mentioning that the voltage assignment technique has also been used to find other types of graphs, like cages [31, 61, 63]. In particular, the paper [31] explores some interesting methods to find the appropriate voltage group and voltage assignment. That paper

Algorithm 5: Basic random search

Input : A base graph G, and a family of groups Ω.
Output: A set (possibly empty) of large graphs, that are lifts of G by $\Gamma \in \Omega$.

```
Initialize the max. number of trials, m_{d,k};
Choose a BFS spanning-tree T of G, and label the arcs of T with the generic identity
element;
foreach unexplored group Γ ∈ Ω do
    for i:= 1 to m_{d,k} do
        Generate a random voltage assignment α;
        Compute the girth and diameter of G';
        if diameter ≤ k then
            Return Γ and α;
            Break;
        end
    end
end
```

contains the seeds of a theory that was developed later in [33]. For a comprehensive account see [34].

7. The constructions by De Bruijn and Kautz, and their variants

The *de Bruijn graph of type* (t, k) [12] is a graph $G = (V, E)$, such that V is formed by all words of length k over a finite alphabet A with cardinality t, and two vertices $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k)$ are joined by an edge if either $a_i = b_{i+1}$ or $a_{i+1} = b_i$, for $1 \leq i \leq k-1$. In other words,

$$(a_1, a_2, \ldots, a_k) \sim \begin{cases} (a_0, a_1, \ldots, a_{k-1}) & \text{for any } a_0 \in A \\ (a_2, a_3, \ldots, a_{k+1}) & \text{for any } a_{k+1} \in A \end{cases}$$

For $t \geq 3$ and $k \geq 3$, the de Bruijn graph of type (t, k) has order t^k, diameter $D = k$ and maximum degree $\Delta = 2t$. That gives the following lower bound for the size of the largest graph with maximum even degree Δ, and diameter D:

$$N_{\Delta,D} \geq \left(\frac{\Delta}{2}\right)^D \tag{5}$$

Given a de Bruijn graph of type (t, k), a *Kautz graph of type* (t, k) is obtained by deleting words with two consecutive identical letters [54]. The Kautz graph is therefore an induced subgraph of the de Bruijn graph, and if $t \geq 3$ and $k \geq 3$, it has order $t(t-1)^{k-1}$, diameter k and maximum degree $2t - 2$. Again, if Δ is even we get the bound:

$$N_{\Delta,D} \geq \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1} \tag{6}$$

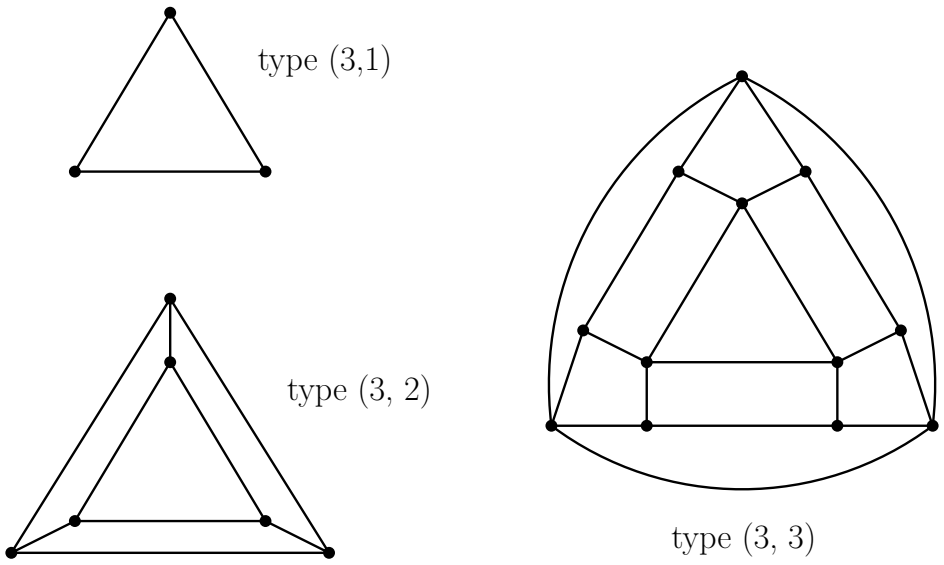

Figure 4. The first three Kautz graphs on a three-letter alphabet

which is an improvement over (5).

Undirected Kautz digraphs are obtained from directed Kautz digraphs by removing the orientation of the arcs, and removing duplicate edges. In turn, directed Kautz graphs can be obtained by line digraph iteration, starting from a complete digraph [37]. Figure 4 depicts the first three Kautz graphs on a three-letter alphabet.

A further improvement of the bound (6) was obtained by Canale and Gómez [15]. They proved that there are two constants D_0 and $\alpha < 1.59$ such that, for each $D \geq D_0$ and infinitely many values of Δ, it is possible to construct a $(\Delta, D)-$graph of order $(\frac{\Delta}{\alpha})^D$.

Imase and Ito generalized de Bruijn graphs in [52] and Kautz graphs in [53]. Both Kautz and de Bruijn graphs are also special cases of *linear congruential graphs* [79, 80]. A linear congruential graph has vertex set $V = \{0, 1, \ldots, n-1\}$. A set F of linear functions is given (called the *generators*), and every vertex $x \in V$ is joined to $f(x) \mod n$, for every $f \in F$.

Note that linear congruential graphs also constitute a generalization of circulant graphs. Some linear congruential graphs represent an improvement over 6, but tight general upper and lower bounds for their diameter are still unknown.

8. Graph compounding

As mentioned in Section 1, graph compounding is one of the most successful techniques for the construction of large graphs of given maximum degree and diameter. Let us recall that the technique consists of replacing one or more vertices of a graph with elements of a set of graphs S, and then the edges of the resulting graph are rearranged appropriately. So stated, graph compounding is not well defined as a graph operation; it is simply an ambiguous generalization of the *graph replacement product* ([48], p. 440).

Nevertheless, the idea is straightforward and easy to apply, the only drawback being the calculation of the diameter, which depends on the choice of the edges connecting the elements of S. In particular, the special case of graph replacement product has been used for a long time in network design. Some popular computer architectures, such as the Cube-Connected Cycles (shown in Fig. 5), can be defined that way.

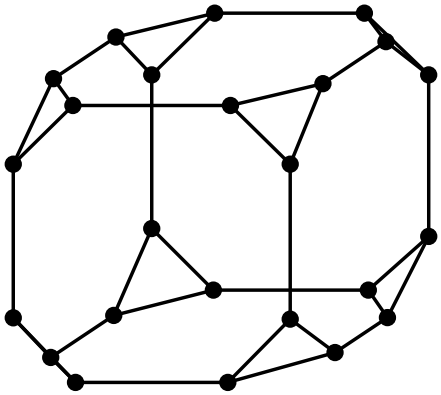

Figure 5. Cube-connected cycles of dimension 3

In the Degree/Diameter Problem, the technique was introduced by Bermond, Delorme and Quisquater [6], and later it has been systematically used by other authors, either alone or in combination with other methods (see, for example, [7, 8, 15, 18, 22, 23, 36, 41, 44, 40, 42, 45, 43, 81]).

In particular, compounding of complete graphs into bipartite Moore graphs has been used recently with good results. It was first suggested in [82], where a single vertex from a bipartite Moore graph is replaced with a suitable complete graph. In a bipartite Moore graph it is even possible to replace several vertices with copies of suitable complete graphs without increasing the diameter of the bipartite Moore graph. This idea was first used in [42]. Subsequently, using bipartite Moore graphs of diameter 6, some improvements of this approach have been achieved in [18, 43], and more recently in [44, 81].

9. Conclusion

As we have seen, there is a wide variety of algebraic and computer-based techniques to construct large graphs with given degree and diameter, whose potential is still far from exhausted. In particular, voltage assignment, combined with computer-based search, has produced the largest number of record graphs, and is likely to produce more in the future.

However, the use of voltage assignment has been so far limited to judicious choices of base graphs and voltage groups (as in [67]), or to nearly brute-force computer search (as in [59]). The powerful group-theoretic machinery has been barely exploited, and very little is known about the conditions, either necessary or sufficient, that a base graph, a voltage group, and a particular voltage assignment must satisfy, in order to obtain a graph with some desired properties (e.g. small diameter, bipartiteness, planarity, etc.).

Taking into account that even under these circumstances the voltage assignment technique has been so successful, we are totally convinced that its potential is still enormous. For example, both in voltage assignment and in Cayley graphs there is still great scope in the use of non-abelian groups, especially solvable groups with large derived length, or outright non-solvable groups (e.g. perfect groups).

The Degree/Diameter Problem also poses interesting challenges to Computer Science, the main of them being the determination of its complexity, which remains open.

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