Electronic Journal of Graph Theory and Applications **4** (2) (2016), 223–230

Electronic Journal of
Graph Theory and Applications

Graphs with coloring redundant edges

Bart Demoen[a], Phuong-Lan Nguyen[b]

[a]*KU Leuven, Belgium*
[b]*UCO, Angers, France*

bart.demoen@cs.kuleuven.be, nguyen@uco.fr

Abstract

A graph edge is d-coloring redundant if the removal of the edge does not change the set of d-colorings of the graph. Graphs that are too sparse or too dense do not have coloring redundant edges. Tight upper and lower bounds on the number of edges in a graph in order for the graph to have a coloring redundant edge are proven. Two constructions link the class of graphs with a coloring redundant edge to the K_4-free graphs and to the uniquely colorable graphs. The structure of graphs with a coloring redundant edge is explored.

Keywords: coloring of graphs and hypergraphs, extremal problems
Mathematics Subject Classification : 05C15, 05C35
DOI:10.5614/ejgta.2016.4.2.9

1. Preliminaries

As usual in graph coloring (see for instance [3]), we focus on simple connected graphs; $\chi(G)$ denotes the chromatic number of a graph G, i.e. the smallest number of colors needed to color G. For convenience, we number the colors from 1 upwards. We use $col(v)$ to denote the color of node v in a particular coloring. $G(V, E)$ denotes a graph with node set V and edge set E. We denote by G_{ab} the graph $G(V, E \setminus \{(a, b)\}$, and by G^{ab} the graph $G(V \cup \{a, b\}, E \cup \{(a, b)\})$. In the sequel, larger or smaller graph has to be understood in terms of the number of the edges.

We make use of *complete d-partite graphs* denoted by $K_{a_1,a_2,\ldots,a_d}$ with (for convenience) $a_i \geq a_{i+1}$. The *Turán graphs* $T(n, d)$, introduced in [6], can be characterized as any $K_{a_1,a_2,\ldots,a_d}$ for which

Received: 31 August 2015, Revised: 18 September 2016, Accepted: 29 September 2016.

$(a_1 - a_d) \leq 1$ and $n = \sum_{i=1}^{d} a_i$. An alternative characterization is that $T(n,d)$ is the largest d-partite graph with n nodes.

Definition 1. An edge (a,b) in a connected graph G is d**-coloring redundant** (d-CR) if G is d-colorable and every d-coloring of G_{ab} assigns different colors to a and b.

The set of graphs with n nodes and a d-coloring redundant edge is denoted by $GCRE(n,d)$: we are in particular interested in the size (the number of edges) of the graphs for combinations of n and d. The removal of a d-CR edge (a,b) of G does not change the set of d-colorings, or otherwise said: any d-coloring of G_{ab} is a d-coloring of G.

Lemma 1.1. *If $G \in GCRE(n,d)$ then $d = \chi(G)$.*

Proof. From the definition it follows that $d \geq \chi(G)$. Suppose $d > \chi(G)$. Let (a,b) be a d-CR edge. Let C denote a color number larger than $\chi(G)$. Consider a d-coloring of G constructed as follows: first construct a $\chi(G)$-coloring of G_{ab}, and then change $col(a)$ and $col(b)$ into C. This results in a d-coloring of G_{ab} in which a and b have the same color, which contradicts the choice of (a,b). Hence the lemma follows. □

This lemma allows us to drop the reference to the chromatic number d, and simply say $G \in GCRE$, or (a,b) is a CR edge. We state two lemma's without a proof. The first one says that the removal of a CR edge does not change the chromatic number.

Lemma 1.2. *If G has a CR edge (a,b), then $\chi(G) = \chi(G_{ab})$.* □

Lemma 1.2 has an analogue in which an edge is added, and which is useful while constructing larger $GCRE$.

Lemma 1.3. *If $G \in GCRE(n,d)$, then either $G^{ab} \in GCRE(n,d)$ or $\chi(G) < \chi(G^{ab})$.* □

From Lemma 1.3, it follows that if G' is a subgraph of a connected graph G with $\chi(G') = \chi(G)$ and $G' \in GCRE$, then also $G \in GCRE$.

Lemma 1.4. *For every $k \geq 1$, $G = K_{k,1,1,\ldots,1}$ is not in $GCRE$.*

Proof. Let the natural partition of G's nodes be $\{x_1, \ldots, x_k\}, \{a_2\}, \{a_3\} \ldots, \{a_d\}$. An edge (a_i, a_j) (with $i \neq j$) is not CR, because $G_{a_i a_j}$ can be colored with less than d colors (see Lemma 1.2). No edge (x_i, a_j) is CR, because $G_{x_i a_j}$ can be colored while giving the same color to x_i and a_j. □

Theorem 1.1. *Let $\{A_1, ..., A_d\}$ be a partition of n nodes so that $|A_i| = a_i, a_i \geq a_{i+1}$ and $\sum a_i = n$. Then $G = K_{a_1,a_2,\ldots,a_d} \in GCRE(n,d)$ if and only if $a_2 \geq 2$. The only CR edges are the edges between A_i and A_j for which $a_i > 1$ and $a_j > 1$.*

Proof. An edge between the nodes in A_i and A_j for which $a_i = a_j = 1$ cannot be CR because without this edge, the node in A_i can have the same color as the node in A_j. Similarly, if $a_i = 1$ and $a_j > 1$, the removal of an edge between A_i and A_j allows a d-coloring with the same color for the two involved nodes, so such edges cannot be CR.

This leaves edges between A_i and A_j ($i < j$) each with at least two nodes. Name these selected nodes $v_{i,1}, v_{i,2}$ (both in A_i) and $v_{j,1}, v_{j,2}$ (in A_j). Let v_k be nodes selected from A_k for $k \notin \{i, j\}$. G contains as a subgraph the d-clique with nodes $\{v_1, v_2, ..., v_{i-1}, v_{i,1}, v_{i+1}, ...v_{j-1}, v_{j,1}, v_{j+1}...v_d\}$ and the d-clique with nodes
$\{v_1, v_2, ..., v_{i-1}, v_{i,2}, v_{i+1}, ...v_{j-1}, v_{j,1}, v_{j+1}...v_d\}$.

As a consequence, the nodes $v_{i,1}$ and $v_{i,2}$ have the same color in any d-coloring, which implies that the edges $(v_{i,1}, v_{j,2})$ and $(v_{i,2}, v_{j,2})$ are coloring redundant. Figure 1 exemplifies the situation for $d = 4,\ i = 2,\ j = 3$: a dashed line between two node sets means that all nodes of one set are connected by an edge to all nodes of the other set.

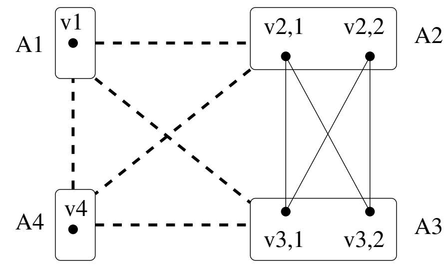

Figure 1: $K_{\_,2,2,\_}$.

By symmetry, all edges between sets with at least two nodes are CR, and no other edges are. □

Since $T(n, d)$ is a complete d-partite graph, we can conclude that $T(n, d) \in GCRE(n, d)$ if $(n \geq d + 2)$ and $d \geq 2$.

2. The results

The next subsections explore the size and the structure of elements of $GCRE(n, d)$ for all values of n and d.

2.1. *Maximal GCRE(n,d)*

Theorem 2.1. *$GCRE(n, d) = \emptyset$ for $n < d + 2$, and the maximal elements of $GCRE(n, d)$ with $n \geq d + 2$ are the Turán graphs $T(n, d)$.*

Proof. A largest element - i.e. one with the highest number of edges - $G \in GCRE(n, d)$ has the following properties:

- it is d-partite (because $\chi(G) = d$)
- adding any new edge results in a graph with chromatic number $(d + 1)$ (see Lemma 1.3) because G is maximal

It follows that such a largest graph G equals a $K_{a_1,...,a_d}$. A $K_{a_1,...,a_d}$ which gives the maximal number of edges under the restriction that $\sum_{i=1}^{d} a_i = n$ is the Turán graph $T(n, d)$. Lemma 1.4 implies that $T(n, d)$ is indeed in $GCRE(n, d)$ for $n \geq d + 2$, and that otherwise $GCRE(n, d) = \emptyset$. □

As a conclusion, we can state that up to isomorphism, there is only one largest element in $GCRE(n, d)$, and its number of edges is $\lfloor \frac{(d-1)n^2}{2d} \rfloor$.

2.2. Minimal GCRE(n,d)

For a graph G with edge (a, b), we denote by $G_{a=b}$ the graph in which (a, b) is contracted.

Lemma 2.1. *Let $G \in GCRE(n, d)$ with e edges, and let (a, b) be one of its CR edges. Let e be the number of edges in G. Then*

- *the number of nodes in $G_{a=b}$ equals $(n - 1)$*
- *the number of edges in $G_{a=b}$ is at most $(e - 1)$*
- *$G_{a=b}$ is connected*
- *$\chi(G_{a=b}) = d + 1$*

Proof. The first three are trivial to prove. The last one can be proved as follows: suppose $G_{a=b}$ can be colored with d colors, than this coloring can be lifted to a coloring of G_{ab} in which a and b have the same colors, which contradicts the fact that (a, b) is CR. $G_{a=b}$ can be $(d + 1)$-colored as follows: color G with d colors, then assign to both a and b the $(d + 1)^{th}$ color and contract (a, b). □

Theorem 2.2. *Let e be the number of edges of $G \in GCRE(n, d)$. It follows that $n + \frac{d^2-d-2}{2} \leq e$. Moreover, there exist $G \in GCRE(n, d)$ for which equality holds.*

Proof. For any connected graph with n' nodes, e' edges and chromatic number d', the following inequality holds: $\frac{d'(d'-1)}{2} + (n' - d') \leq e'$.

Let (a, b) be a CR edge of G. Then by using Lemma 2.1 for $G_{a=b}$, we can substitute e', d', n' by $(e - 1), (d + 1), (n - 1)$ and derive:

$$\frac{d(d+1)}{2} + (n - 1) - (d + 1) \leq (e - 1) \text{ or equivalently } n + \frac{d^2-d-2}{2} \leq e$$

To prove the second part of the theorem, we establish one particular example: name the n nodes $v_1, v_2, ..., v_n$. Connect the nodes as in Figure 2, i.e. the edges and their counts are

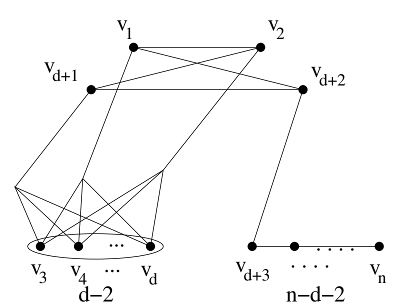

Figure 2: A $GCRE(n, d)$ with $n + \frac{d^2-d-2}{2} = e$.

- $(v_1, v_2), (v_1, v_{d+2}), (v_{d+1}, v_2), (v_{d+1}, v_{d+2})$: 4 edges
- the ellipse represents a clique between the nodes $\{v_3, v_4, ...v_d\}$: $(d - 2)(d - 3)/2$ edges
- the multi-edges represent (v_k, x) for $k = 3, 4, \ldots, d$ and $x \in \{v_1, v_2, v_{d+1}\}$: $3(d - 2)$ edges

- the nodes $v_{d+2}...v_n$ are connected amongst each other as (v_i, v_{i+1}) for $i = (d+2), (d+3), \ldots, (n-1)$: $(n-d-2)$ edges

The graph is connected, has n nodes and its number of edges is $4 + (d-2)(d-3)/2 + 3(d-2) + (n-d-2) = n + \frac{d^2-d-2}{2}$. Clearly, its chromatic number is d. Finally, the edge (v_1, v_{d+2}) is CR because the d-cliques $\{v_1, v_2, v_3, ..., v_d\}$ and $\{v_{d+1}, v_2, v_3, ..., v_d\}$ force v_1 and v_{d+1} to have the same color in any d-coloring. The reasoning is similar to the one in Theorem 1.1. □

2.3. Intermediate GCRE(n,d)

Theorem 2.3. *For all $n, d, e : d \geq 2, n \geq d+2, n + \frac{d^2-d-2}{2} \leq e \leq \lfloor \frac{(d-1)n^2}{2d} \rfloor$, there exists a $G \in GCRE(n, d)$ such that G has exactly e edges.*

Proof. Consider the graph constructed in Theorem 2.2. Define the sets $V_i, i = 1..d$ such that $V_i = \{v_{i+kd} \mid k = 0..\lfloor \frac{n-i}{d} \rfloor\}$: these sets form an equitable partition of the nodes. The graph does not contain any edge between nodes of the same V_i. Add one by one as many edges as possible between nodes in different V_i. This keeps the chromatic number equal to d, and from Lemma 1.3, all intermediate graphs are in $GCRE(n, d)$. Thanks to the choice of the partition, when the maximal amount of edges is added, the result is $T(n, d)$. □

2.4. The structure of graphs in $GCRE$

Let $G_{\setminus\{a,b\}}$ denote the graph G from which a and b and all their edges are removed. We use δ_X for the degree of a node in a graph X.

Lemma 2.2. *If G is $GCRE(n, d)$ and (a, b) is CR in G, then $\chi(G_{\setminus\{a,b\}}) = d$.*

Proof. Suppose that there exists a $(d-1)$-coloring of $G_{\setminus\{a,b\}}$, then assign the d^{th} color to a and b, and get a d-coloring of G_{ab} in which a and b have the same color, but this contradicts the choice of (a, b). □

Lemma 2.3. *Let G be connected, with $\chi(G) = d$, n nodes and the edge (a, b). (a, b) is CR in G if and only if every d-coloring col of G_{ab} satisfies*

$$|col(N_a) \cup col(N_b)| = d \text{ (or equivalently } \overline{col(N_a)} \cap \overline{col(N_b)} = \emptyset$$

where N_a (resp. N_b) is the set of neighbors of a (resp. b) in G_{ab}, and $\overline{A}$ means the complement with respect to the d available colors.

Proof. $(\Rightarrow)$ Suppose that for some d-coloring of G_{ab}, $\overline{col(N_a)} \cap \overline{col(N_b)}$ contains at least one color, say C. One can then change the color of a and b to C, contradicting the choice of (a, b).
$(\Leftarrow)$ Suppose a d-coloring of G_{ab} exists that gives the same color C to a and b. That implies that $C \in \overline{col(N_a)} \cap \overline{col(N_b)}$, which violates the assumption. □

Lemma 2.3 implies that for a CR edge (a, b) in G, $|N_a \cup N_b| \geq d$, and consequently $\delta_{G_{ab}}(a) + \delta_{G_{ab}}(b) \geq |N_a \cup N_b| \geq d$.

The general structure of $G \in GCRE(n,d)$ with CR edge (a,b) becomes more clear now. As in Figure 3, G consists of a subgraph with the same chromatic number as G. The vertices a and b are connected to that subgraph with at least d neighbors. In the figure, the graph within the rectangle is $G_{\setminus\{a,b\}}$. $G_{\setminus\{a,b\}}$ does not need to be connected, but in some sense, one component is enough.

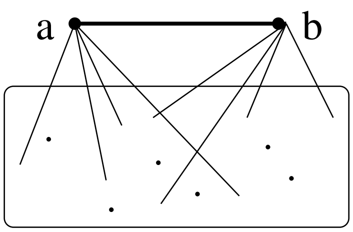

Figure 3: General structure of a $GCRE(n,d)$.

Lemma 2.4. *If G is minimal in $GCRE(n,d)$ and (a,b) is CR in G, then $G_{\setminus\{a,b\}}$ is connected.*

Proof. Let the components of $G_{\setminus\{a,b\}}$ be $S_1, ...S_k$. Denote by G_i the subgraph of G induced by the vertices a, b and the nodes of S_i. Suppose (a,b) is not CR in any G_i, then each $G_{i_{ab}}$ has a d-coloring C_i in which $col_i(a) = col_i(b)$. Rename the colors so that $col_i(a) = col_j(a)$ for all i, j: this results in a d-coloring of G_{ab} in which a and b have the same color, which is impossible. □

3. K_4-free, and uniquely colorable $GCRE(n,d)$

K_4-free $GCRE(n,d)$. We show a general construction of a $GCRE(n,d)$ without a 4-clique for every $d > 3$. The basis for this construction is Mycielski's Theorem [5] and the so called *Iterated Mycielskians* M_i which are a sequence of triangle-free graphs with chromatic number i. We start from such a graph M_i with $i = \chi(M_i) \geq 3$ and let m be the number of its nodes. Construct the graph G that has M_i as a subgraph and the following additional nodes and edges (as shown in Figure 4):

- three new nodes named a, b, x
- (a,z) and (b,z) for all $z \in M_i$
- (a,x) and (b,x)

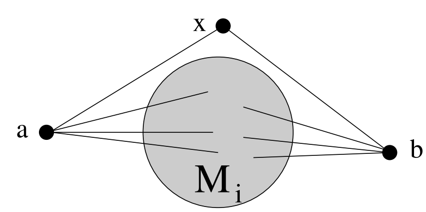

Figure 4: A K_4-free $GCRE(n,d)$ from M_i.

We now prove that G has no 4-clique and $G \in GCRE(m+3,d)$ with $d = i+1$:

- $\boldsymbol{\chi(G) = d}$: G clearly has a d-coloring, as a $(d-1)$-coloring of M_i can be extended to G by giving a and b both the d^{th} color, and giving x any color different from that. Now suppose that G had a $(d-1)$-coloring: the restriction to M_i could use only $(d-2)$ colors, since $col(a)$ must differ from all colors in M_i. So, G has no $(d-1)$-coloring and $\chi(G) = d$

- **edges** (a,x) **and** (b,x) **are CR**: indeed, a and b have the same color in every d-coloring, so $(col(a) \neq col(x)) \Leftrightarrow (col(b) \neq col(x))$, which proves each of the two edges is CR

- ***G* has no 4-clique**: suppose G has a 4-clique $C4$; since M_i is triangle-free, $C4$ contains at least one of a,b or x; x cannot be in $C4$ because it has degree 2; a and b cannot be both in $C4$ because there is no edge between them; so assume that $a \in C4$; the restriction of $C4$ to M_i would then be a 3-clique; this contradicts the fact that M_i is triangle-free

The construction applied to M_3 results in the graph in Figure 5: it has 17 edges and no 4-clique. It is not a minimal $GCRE(8,4)$, but it is one of the three minimal $GCRE(8,4)$ without a 4-clique.

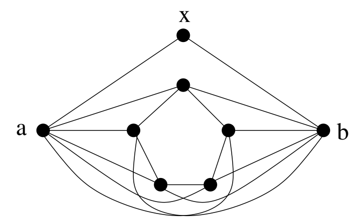

Figure 5: K_4-free $GCRE(8,4)$.

A natural question in this context is: do triangle-free $GCRE(n,d)$ exist? Via a computer experiment, we found the graph in Figure 6: it is the only triangle-free graph in $GCRE(n,d)$ for all $n \leq 9$ and $d \geq 3$. The two dashed edges are CR.

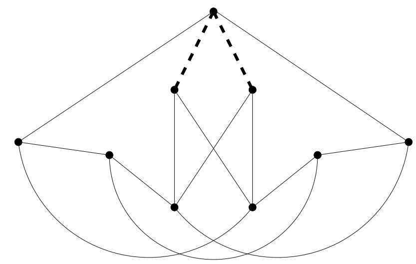

Figure 6: A triangle-free $GCRE(9,3)$.

Uniquely colorable $GCRE$. Any complete d-partite graph is uniquely colorable and Theorem 1.1 shows that infinitely many are also a $GCRE$. We give a general construction that turns every uniquely colorable graph into a uniquely colorable $GCRE$, without ending up necessarily with a complete d-partite graph.

In a uniquely colorable graph $G(V,E)$ with chromatic number d, one can partition V in subsets $V_1, V_2, ..., V_d$ such that in every d-coloring, $\{col(v) \mid v \in V_i\}$ is a singleton for each i. From $G(V,E)$, we construct a new uniquely colorable graph U whose nodes are $V \cup \{a,b\}$ (a and b are two new nodes) and whose edges consist of $E \cup \{(a,b)\} \cup \{(x,a) \mid x \in \bigcup_{i=1}^{d-1} V_i\} \cup \{(x,b) \mid x \in \bigcup_{i=2}^{d} V_i\}$.

One can check that $\chi(U) = d$, (a,b) is CR, and that U is uniquely colorable. The latter is a consequence of Theorem 4 from [4]: the partition according to the colors in any coloring is $V_1 \cup \{b\}, V_2, ..., V_d \cup \{a\}$. The additional edge (a,b) retains the uniqueness of the coloring.

Figure 7 shows the construction starting from the uniquely colorable graph with the full lines: it is not in $GCRE$. The added edges are the dashed lines. The result is one edge short of $K_{3,3}$, and it is $GCRE$.

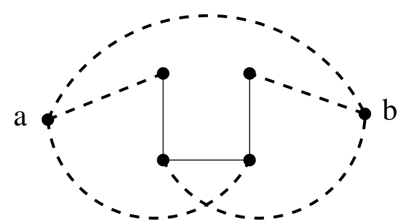

Figure 7: A uniquely colorable $GCRE$.

4. Discussion and Future Work

The motivation for this work comes from the study of redundant disequalities in the context of constraint programming: when transposed to the graph coloring context, a set of disequalities corresponds to the constraint graph with edges between disequal variables (the nodes), and a redundant disequality (one implied by the others) corresponds to a CR edge. In [1] and [2], the redundant disequalities were fully classified for the Latin Square problem and for Sudoku. It seemed worthwhile to explore the graph context and this resulted in the current work. To sum up our results:

- the maximal number of edges in a $GCRE(n, d)$ is attained by the Turán graph $T(n, d)$; the number of edges equals $\lfloor \frac{(d-1)n^2}{2d} \rfloor$
- the minimal number of edges in any $GCRE$ with n nodes and chromatic number d equals $n + \frac{d^2-d-2}{2}$
- for each e, such that $n+\frac{d^2-d-2}{2} \leq e \leq \lfloor \frac{(d-1)n^2}{2d} \rfloor$, there exists a graph $G(V, E) \in GCRE(n, d)$ such that $e = |E|$

This work has focussed solely on the existence of at least one CR edge. Ultimately, we want to understand graphs with *many* CR edges, and quantify that understanding. We would also like to develop (polynomial) algorithms that (approximately) *complete* the graph, i.e. to add as many CR edges as possible: this should benefit solving constraint satisfaction problems by typical constraint solvers. The observation in Lemma 2.3 could be of great value there. Finally, the extension of our work to list coloring is interesting, because it corresponds to constraint satisfaction problems in which the variables have different domains.

Acknowledgments

Most of this research was carried out while the first author was a guest at the Institut de Mathématiques Appliquées, Angers in France. This research was partly sponsored by the Research Foundation Flanders (FWO) through project *WOG: Declarative Methods in Computer Science*. We are also grateful to Gunnar Brinkmann for sharing his knowledge on graphs.

References

[1] B. Demoen and M. Garcia de la Banda. Redundant disequalities in the latin square problem. *Constraints* **18** (4) (2013), 471–477.

[2] B. Demoen and M. Garcia de la Banda. Redundant Sudoku rules. *Theory and Practice of Logic Programming* **14** (3), (2014), 363–377.

[3] R. Diestel. *Graph Theory, Graduate Texts in Mathematics, Volume 173*. Springer-Verlag, Heidelberg, 2010. URL `http://diestel-graph-theory.com/`.

[4] F. Harary, S. Hedetniemi, and R. Robinson. Uniquely colorable graphs. *Journal of Combinatorial Theory* **6** (3), (1969), 264–270.

[5] J. Mycielski. Sur le coloriage des graphes. *Colloq. Math.* **3** (1957), 161–162.

[6] P. Turán. On an extremal problem in graph theory (in Hungarian). *Matematiko Fizicki Lapok*, **48** (1941), 436–452.