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The method of double chains for largest families with excluded subposets

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Abstract

For a given finite poset P, $La(n,P)$ denotes the largest size of a family $\mathcal{F}$ of subsets of $[n]$ not containing P as a weak subposet. We exactly determine $La(n,P)$ for infinitely many P posets. These posets are built from seven base posets using two operations. For arbitrary posets, an upper bound is given for $La(n,P)$ depending on $|P|$ and the size of the longest chain in P. To prove these theorems we introduce a new method, counting the intersections of $\mathcal{F}$ with double chains, rather than chains.

Keywords: excluded subposet, Lubell's function, double chain
Mathematics Subject Classification : 05D05

1. Introduction

Let $[n]=\{1,2,\ldots,n\}$ be a finite set. We investigate families $\mathcal{F}$ of subsets of $[n]$ avoiding certain configurations of inclusion.

Definition Let P be a finite poset, and $\mathcal{F}$ be a family of subsets of $[n]$. We say that P is contained in $\mathcal{F}$ if there is an injective mapping $f : P \to \mathcal{F}$ satisfying $a <_p b \Rightarrow f(a) \subset f(b)$ for all $a,b \in P$. $\mathcal{F}$ is called P-free if P is not contained in it.

Let $La(n,P) = \{\max|\mathcal{F}| \mid \mathcal{F} \text{ contains no } P\}$

Note that we do not want to find P as an induced subposet, so the subsets of $\mathcal{F}$ can satisfy more inclusions than the elements of the poset P.

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We are interested in determining $La(n,P)$ for as many posets as possible. The first theorem of this kind was proved by Sperner. Later it was generalized by Erdős.

Theorem 1.1 (Sperner)**.** [1] *Let $\mathcal{F}$ be a family of subsets of $[n]$, with no member of $\mathcal{F}$ being the subset of an other one. Then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} \tag{1}$$

Theorem 1.2 (Erdős)**.** [2] *Let $\mathcal{F}$ be a family of subsets of $[n]$, with no $k+1$ members of $\mathcal{F}$ satisfying $A_1 \subset A_2 \subset \cdots \subset A_{k+1}$ ($k \leq n$). Then $|\mathcal{F}|$ is at most the sum of the k biggest binomial coefficients belonging to n. The bound is sharp, since it can be achieved by choosing all subsets F with $\lfloor \frac{n-k+1}{2} \rfloor \leq |F| \leq \lfloor \frac{n+k-1}{2} \rfloor$.*

Since choosing all the subsets with certain sizes near $n/2$ is the maximal family for many excluded posets, we use the following notation.

Notation $\Sigma(n,m) = \sum_{i=\lfloor \frac{n-m+1}{2} \rfloor}^{\lfloor \frac{n+m-1}{2} \rfloor} \binom{n}{i}$ denotes the sum of the m largest binomial coefficients belonging to n.

Now we can reformulate Theorem 1.2. Let P_{k+1} be the path poset with $k+1$ elements. Then

$$La(n, P_{k+1}) = \Sigma(n,k) \tag{2}$$

We give here a proof of Theorem 1.2 to illustrate the chain method introduced by Lubell [3].

Proof. (Theorem 1.2) A chain is $n+1$ subsets of $[n]$ satisfying $L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_n$ and $|L_i| = i$ for all $i = 0,1,2,\dots n$. The number of chains is $n!$. We use double counting for the pairs (C,F) where C is a chain, $F \in C$ and $F \in \mathcal{F}$.

The number of chains going through some subset $F \in \mathcal{F}$ is $|F|!(n-|F|)!$. So the number of pairs is

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)!$$

One chain can contain at most k elements of $\mathcal{F}$, otherwise a P_{k+1} poset would be formed. So the number of pairs is at most $k \cdot n!$. It implies

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \leq k \cdot n! \tag{3}$$

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq k \tag{4}$$

Fixing $|\mathcal{F}|$, the left side takes its minimum when we choose the subsets with sizes as near to $n/2$ as possible. Choosing all $\Sigma(n,k)$ subsets with sizes $\lfloor \frac{n-k+1}{2} \rfloor \leq |F| \leq \lfloor \frac{n+k-1}{2} \rfloor$, we have equality. So we have

$$La(n, P_{k+1}) = \Sigma(n,k) \tag{5}$$

□

$La(n,P)$ is determined asymptotically for many posets, but its exact value is known for very few P. (See [4] and [5])

2. The method of double chains

The main purpose of the present paper is to exactly determine $La(n, P)$ for some posets P. Our main tool is a modification of the the chain method, double chains are used rather than chains.

Definition Let $C : L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_n$ be a chain. The double chain assigned to C is a set $D = \{L_0, L_1, \ldots, L_n, M_1, M_2, \ldots, M_{n-1}\}$, where $M_i = L_{i-1} \cup (L_{i+1} \backslash L_i)$.

Note that $|M_i| = |L_i| = i$,
$i < j \Rightarrow L_i \subset L_j,\ L_i \subset M_j,\ M_i \subset L_j$ and $i + 1 < j \Rightarrow M_i \subset M_j$.

$\{L_0, L_1, \ldots, L_n\}$ is called the primary line of D and $\{M_1, M_2, \ldots, M_{n-1}\}$ is the secondary line.

$\mathcal{D}$ denotes the set of all $n!$ double chains.

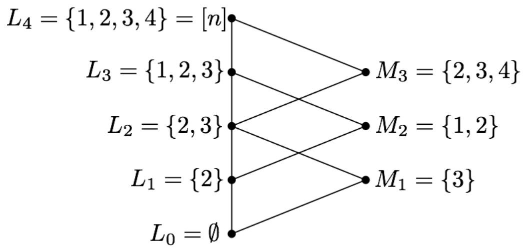

Figure 1. The double chain assigned to the chain $\emptyset \subset \{2\} \subset \{2,3\} \subset \{1,2,3\} \subset \{1,2,3,4\}$.

Lemma 2.1. *Let $\mathcal{F}$ be a family of subsets of $[n]$ $(n \geq 2)$, and let m be a positive real number. Assume that*

$$\sum_{D \in \mathcal{D}} |\mathcal{F} \cap D| \leq 2m \cdot n! \tag{6}$$

Then

$$|\mathcal{F}| \leq m \binom{n}{\lfloor n/2 \rfloor} \tag{7}$$

If m is an integer and $m \leq n - 1$, we have the following better bound:

$$|\mathcal{F}| \leq \Sigma(n, m) \tag{8}$$

Proof. First we count how many double chains contains a given subset $F \subset [n]$. $\emptyset$ and $[n]$ are contained in all $n!$ double chains. Now let $F \notin \{\emptyset, [n]\}$. F is contained in the primary line of $|F|!(n - |F|)!$ double chains. Now count the double chains containing F in the secondary line. Letting $F = M_{|F|}$, we have $|F| \cdot (n - |F|)$ possibilities to choose $L_{|F|}$, since we have to replace one element of $M_{|F|}$ with a new one. $M_{|F|}$ and $L_{|F|}$ already define $L_{|F|-1}$ and $L_{|F|+1}$. We have $(|F| - 1)!$ and $(n - |F| - 1)!$ possibilities for the first and last part of the primary line, so the number of double chains containing F in the secondary line is $|F|(n - |F|)(|F| - 1)!(n - |F| - 1)! = |F|!(n - |F|)!$. It gives a total of $2|F|!(n - |F|)!$ double chains containing F.

Let $t = |\mathcal{F} \cap \{\emptyset, [n]\}|$. Double counting the pairs (D, F) where $D \in \mathcal{D}$, $F \in D$ and $F \in \mathcal{F}$ we have

$$t \cdot n! + \sum_{F \in \mathcal{F} \setminus \{\emptyset, [n]\}} 2|F|!(n - |F|)! \leq 2m \cdot n! \tag{9}$$

$$t \cdot \frac{1}{2} + \sum_{F \in \mathcal{F} \setminus \{\emptyset, [n]\}} \frac{1}{\binom{n}{|F|}} \leq m \tag{10}$$

Since $\binom{n}{\lfloor n/2 \rfloor}$ is the biggest binomial coefficient, and $\binom{n}{\lfloor n/2 \rfloor} \geq 2$ we have

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} \leq m \tag{11}$$

It proves (7). If m is an integer, and $m \leq n - 1$, considering $|\mathcal{F}|$ fixed, the left side of (10) is minimal when we choose subsets with sizes as near to $n/2$ as possible. Choosing all $\Sigma(n, m)$ subsets with such sizes, we have equality in (10). It implies $|\mathcal{F}| \leq \Sigma(n, m)$, so (8) is proved. □

Definition The infinite double chain is an infinite poset with elements $L_i,\ i \in \mathbb{Z}$ and $M_i,\ i \in \mathbb{Z}$. The defining relations between the elements are

$$i < j \Rightarrow L_i \subset L_j,\ L_i \subset M_j,\ M_i \subset L_j$$

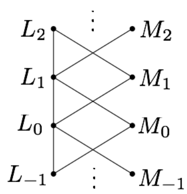

Figure 2. The infinite double chain.

Note that the poset formed by the elements of any double chain with the inclusion as relation is a subposet of the infinite double chain.

Lemma 2.2. *Let m be an integer or half of an integer and P be a finite poset. Assume that any subset of size $2m + 1$ of the infinite double chain contains P as a (not necessarily induced) subposet. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that $\mathcal{F}$ does not contain P. Then*

$$|\mathcal{F}| \leq m \binom{n}{\lfloor n/2 \rfloor} \tag{12}$$

If m is an integer and $m \leq n - 1$ we have the following better bound:

$$|\mathcal{F}| \leq \Sigma(n, m) \tag{13}$$

Proof. Since the poset formed by the elements of any double chains is a subposet of the infinite double chain,
$|\mathcal{F} \cap D| \leq 2m$ for all double chains D. There are $n!$ double chains, so

$$\sum_{D \in \mathcal{D}} |\mathcal{F} \cap D| \leq 2m \cdot n! \tag{14}$$

holds. Now we can use Lemma 2.1 and finish the proof. □

3. An upper estimate for arbitrary posets

Definition The size of the longest chain in a finite poset P is the largest integer $L(P)$ such that for some $a_1, a_2, \ldots, a_{L(P)} \in P$, $a_1 <_p a_2 <_p \cdots <_p a_{L(P)}$ holds.

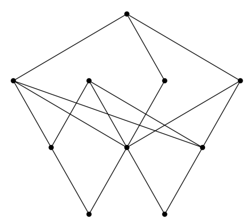

Figure 3. A poset with $|P| = 10$ elements and longest chain of length $L(P) = 4$.

Theorem 3.1. *Let P be a finite poset and let $\mathcal{F}$ be a P-free family of subsets of $[n]$. Then*

$$|\mathcal{F}| \leq \left(\frac{|P| + L(P)}{2} - 1\right)\binom{n}{\lfloor n/2 \rfloor} \tag{15}$$

If $\frac{|P|+L(P)}{2} - 1$ is an integer and $\frac{|P|+L(P)}{2} \leq n$ we have the following better bound:

$$|\mathcal{F}| \leq \Sigma\left(n, \frac{|P| + L(P)}{2} - 1\right) \tag{16}$$

Proof. We want to use Lemma 2.2 with $m = \frac{|P|+L(P)}{2} - 1$. So the only thing we have to prove is the following lemma. □

Lemma 3.2. *Let P be a finite poset. Then any subset S of size $|P| + L(P) - 1$ of the infinite double chain contains P as a (not necessarily induced) subposet.*

Proof. We prove the lemma using induction on $L(P)$. When $L(P) = 1$, we have a subset of size $|P|$ in the infinite double chain. We can choose them all, we get the poset P, since there are no relations between its elements. Assume that we already proved the lemma for all posets with longest chain of size $l - 1$, and prove it for a poset P with $L(P) = l$.

Arrange the elements of the infinite double chain as follows:

$$\ldots L_{-1}, M_{-1}, L_0, M_0, L_1, M_1, L_2, M_2 \ldots$$

Assume that P has k minimal elements, and choose the k first elements of S for them according to the above arrangement. Note that all remaining elements of S, except for at most one, are greater in the infinite double chain than all the k elements we just chose. If there is such an exception, delete that element from S. Now we have at least $|P| + L(P) - k - 2$ elements of S left, all greater than the k we chose for the minimal elements. Denote the set of these elements by S'.

Let P' be the poset obtained by P after deleting its minimal elements. It has $|P'| = |P| - k$ elements and a longest chain of size $L(P') = L(P) - 1$. By the inductive hypothesis P' is formed by some elements of S', since $|S'| \geq |P| + L(P) - k - 2 = |P'| + L(P') - 1$. Considering these elements together with the first k, they form P as a weak subposet in S. $\square$

Remark The previously known upper bound for maximal families not containing a general P as weak subposet was $\Sigma(n, |P| - 1)$. We can get it from Theorem 1.2 since P is a subposet of the path poset $P_{|P|}$. The new upper bound, $\Sigma\left(n, \frac{|P|+L(P)}{2} - 1\right)$ is better since $L(P) \leq |P|$, and equality occurs only when P is a path poset.

4. Exact results

In this section we will describe infinitely many posets for which Theorem 3.1 provides a sharp bound.

Definition For a finite poset P, $e(P)$ is the maximal m such that the family formed by all subsets of $[n]$ of size $k, k+1, \ldots, k+m-1$ is P-free for all n and k.

We will prove that $La(n, P) = \Sigma(n, e(P))$ if n is large enough for infinitely many P, verifying the following conjecture for these posets.

Conjecture [6] For every finite poset P

$$La(n, P) = e(P)\binom{n}{\lfloor n/2 \rfloor}(1 + O(1/n)) \tag{17}$$

In [6] Bukh proved the conjecture for all posets whose Hasse-diagram is a tree.

Notation

$$b(P) = \frac{|P| + L(P)}{2} - 1, \text{ the bound used in Theorem 3.1} \tag{18}$$

Lemma 4.1. *Assume that $e(P) = b(P)$ for a finite poset P and $n \geq b(P) + 1$. Then*

$$La(n, P) = \Sigma(n, e(P)) = \Sigma(n, b(P)) \tag{19}$$

Proof. The family of subsets of size $\lfloor \frac{n-e(P)+1}{2} \rfloor \leq |F| \leq \lfloor \frac{n+e(P)-1}{2} \rfloor$ has $\Sigma(n, e(P))$ elements and is P-free by the definition of $e(P)$. On the other hand, Theorem 3.1 states that a P-free family has at most $\Sigma(n, b(P))$ elements. $\square$

Now we show some posets satisfying $e(P) = b(P)$.

Definition (See figure 4).
E is the poset with one element.
The elements of the following posets are divided into levels so that a is greater than b in the poset if and only if a is in a higher level than b.
B is the butterfly poset, a poset with 2 elements on each level.
D_3 is the 3-diamond poset, a poset with respectively 1, 3 and 1 element on its levels.
Q is a poset with respectively 2, 3 and 2 elements on its levels.
R is a poset with respectively 1, 4, 4 and 1 element on its levels.
S is a poset with respectively 1, 4 and 2 elements on its levels.
S' is a poset with respectively 2, 4 and 1 element on its levels.

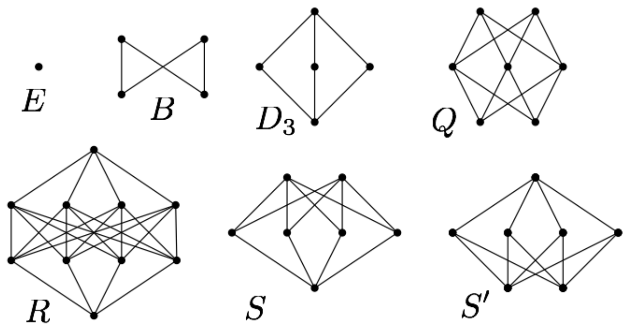

Figure 4. 7 small posets satisfying $e(P) = b(P)$.

Lemma 4.2. *For all* $P \in \{E, B, D_3, Q, R, S, S'\}$, $e(P) = b(P)$ *holds.*

Proof. $b(P)$ is an integer for all the above posets. Assume that $e(P) \geq b(P) + 1$. Then for $n \geq b(P) + 1$ there would be a P-free family $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = \Sigma(n, b(P) + 1) > \Sigma(n, b(P))$, contradicting Theorem 3.1. So $e(P) \leq b(P)$. We will show that for every poset $P \in \{E, B, D_3, Q, R, S, S'\}$ and integers n, k the family formed by all subsets of $[n]$ of size $k, k+1, \ldots, k + b(P) - 1$ is P-free. It gives us $e(P) \geq b(P)$, and completes the proof.

The statement is trivial for $P = E$ since $b(E) = 0$.

$b(B) = 2$. The set of all subsets with k and $k + 1$ elements is B-free since two subsets of size $k + 1$ can not have two different common subsets of size k.

$b(D_3) = 3$. The set of all subsets with $k, k+1$ and $k+2$ elements is D_3-free since for two subsets A, B, $|B| - |A| \leq 2$ there are at most two subsets F satisfying $A \subset F \subset B$.

$b(Q) = 4$. Assume that Q is formed by 7 subsets of size $k, k+1, k+2$ or $k+3$. There are at least 4 subsets in the lower 2 or the upper 2 levels. They should form a B poset, that is not possible.

$b(R) = 6$. Assume that R is formed by 10 subsets of size $k, k+1, \ldots, k+5$. Let A be the least, and B be the greatest subset. Let U be the union of the 5 smaller subsets. At least 3 subsets in the second level are different from U, and contained in it. Similarly, at least 3 subsets of the third level are different from U, and contain it. Since D_3 is not formed by subsets of size $m, m+1$ and $m+2$, $|A| + 6 \leq |U| + 3 \leq |B|$, a contradiction.

$b(S) = 4$. Assume that S is formed by 7 subsets of size $k, k+1, k+2$ or $k+3$. Let V be the intersection of the two elements of the top level, then $|V| \leq k+2$. V contains all elements of the middle level, and is different from at least 3 of them. These 3 elements together with the least element and V form a D_3 from subsets of size $k, k+1$ and $k+2$, and it is a contradiction.

A family is S'-free if and only if the family of the complements of its elements is S-free. It gives $e(S') = e(S) \geq b(S) = b(S')$. □

We define two ways of building posets from smaller ones, keeping the property $e(P) = b(P)$.

Definition Let P_1, P_2 posets. $P_1 \oplus P_2$ is the poset obtained by P_1 and P_2 adding the relations $a < b$ for all $a \in P_1, b \in P_2$.

Assume that P_1 has a greatest element and P_2 has a least element. $P_1 \otimes P_2$ is the poset obtained by identifying the greatest element of P_1 with the least element of P_2.

Lemma 4.3. $e(P_1 \oplus P_2) \geq e(P_1) + e(P_2) + 1$. *If* $P_1 \otimes P_2$ *is defined, then* $e(P_1 \otimes P_2) \geq e(P_1) + e(P_2)$.

Proof. In order to find a P_1, we need at least $e(P_1)+1$ levels, for a P_2, we need at least $e(P_2)+1$ levels. It follows from the properties of $\oplus$ that the lowest level of P_2 is above the highest level of P_1 in any occurrence of $P_1 \oplus P_2$, which thus needs at least $e(P_1) + 1 + e(P_2) + 1$ levels. In the case of $P_1 \otimes P_2$, the same reasoning applies, noting that highest level of P_1 and the lowest level of P_2 coincide. □

Lemma 4.4. *Assume that* P_1 *and* P_2 *are finite posets such that* $e(P_1) = b(P_1)$ *and* $e(P_2) = b(P_2)$. *Then*

$$e(P_1 \oplus P_2) = b(P_1 \oplus P_2) \tag{20}$$

Assume that P_1 *has a greatest element and* P_2 *has a least element. Then*

$$e(P_1 \otimes P_2) = b(P_1 \otimes P_2) \tag{21}$$

Proof. Note that $|P_1 \oplus P_2| = |P_1| + |P_2|$, $L(P_1 \oplus P_2) = L(P_1) + L(P_2)$, and $e(P_1 \oplus P_2) \geq e(P_1) + e(P_2) + 1$. Similarly, $|P_1 \otimes P_2| = |P_1| + |P_2| - 1$, $L(P_1 \otimes P_2) = L(P_1) + L(P_2) - 1$, and $e(P_1 \otimes P_2) \geq e(P_1) + e(P_2)$.

From the above equations and (18) we have

$$e(P_1 \oplus P_2) \geq e(P_1) + e(P_2) + 1 = b(P_1) + b(P_2) + 1 = b(P_1 \oplus P_2) \tag{22}$$

and

$$e(P_1 \otimes P_2) \geq e(P_1) + e(P_2) = b(P_1) + b(P_2) = b(P_1 \otimes P_2) \tag{23}$$

if P_1 has a greatest element and P_2 has a least element. We have already seen that $e(P) \leq b(P)$ always holds. □

The following theorem summarizes our results.

Theorem 4.5. *Let* P *be a finite poset built from the posets* E, B, D_3, Q, R, S *and* S' *using the operations* $\oplus$ *and* $\otimes$. *(See figure 5 for examples.) For* $n \geq b(P) + 1$

$$La(n, P) = \Sigma(n, b(P)) = \Sigma(n, e(P)) \tag{24}$$

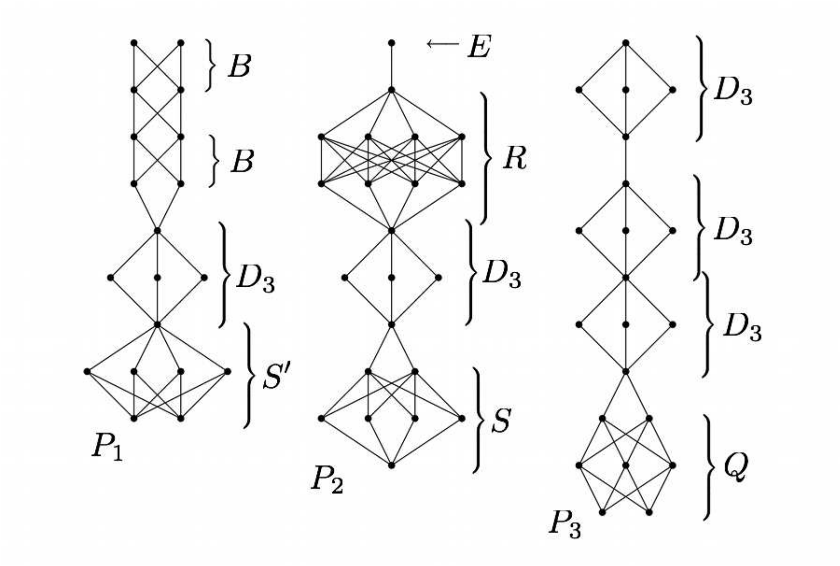

Figure 5. Posets built from E, B, D_3, Q, R, S and S' using $\oplus$ and $\otimes$. $P_1 = S' \otimes D_3 \oplus B \oplus B$, $P_2 = S \oplus D_3 \otimes R \oplus E$ and $P_3 = Q \oplus D_3 \otimes D_3 \oplus D_3$.

Proof. From Lemma 4.2 and Lemma 4.4 we have $e(P) = b(P)$. Then Lemma 4.1 proves the theorem. □

Remark Theorem 4.5 is the generalization of the theorem of Erdős (Theorem 1.2), and the following two results.

Theorem 4.6 (De Bonis, Katona, Swanepoel)**.** [7] *For* $n \geq 3$

$$La(n, B) = \Sigma(n, 2) \tag{25}$$

Theorem 4.7 (Griggs, Li, Lu)**.** (Special case of Theorem 2.5 in [8]) *For* $n \geq 2$

$$La(n, D_3) = \Sigma(n, 3) \tag{26}$$

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