



# Ideal bases in constructions defined by directed graphs

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## Abstract

The present article continues the investigation of visible ideal bases in constructions defined using directed graphs. Our main theorem establishes that, for every balanced digraph  $D$  and each idempotent semiring  $R$  with 1, the incidence semiring  $I_D(R)$  of the digraph  $D$  has a convenient visible ideal basis  $B_D(R)$ . It also shows that the elements of  $B_D(R)$  can always be used to generate two-sided ideals with the largest possible weight among the weights of all two-sided ideals in the incidence semiring.

*Keywords:* digraphs, incidence semirings, two-sided ideals, visible bases, weights of ideals.  
Mathematics Subject Classification : 05C78, 05C99, 62H30, 68T10  
DOI: 10.5614/ejgta.2015.3.1.5

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Received: 04 September 2013, Revised: 25 November 2014, Accepted: 26 February 2015.

## 1. Introduction

The investigation of semirings is an important direction of theoretical computer science (cf. [9, 27, 52, 58, 62]) and symbolic computation (cf. [57]), where many efficient semiring-based algorithms have been developed (cf. [54]). In graph theory, semirings have been applied, for example, to the investigation of trust networks defined as directed graphs [23], to the study of planar flows of directed graphs [28], and to the design of a sophisticated Python library for parallel graph computations [56].

The construction of incidence semirings of directed graphs was introduced in [7]. To illustrate the applicability and usefulness of this notion, we include a section with examples, theoretical results and open questions concerning relations between properties of digraphs and properties of their incidence semiring (see Section 3). These examples confirm that the general problem of investigating the relations of properties of digraphs and their incidence semirings is interesting.

The main theorem of this paper handles two-sided ideals in incidence semirings of digraphs (Theorem 4.1 in Section 4). The concept of an ideal is important and has been used in various research directions. Visible ideal bases were introduced in [7] by analogy with different constructions (cf. [29, 49]). Our main theorem establishes that, for each balanced digraph  $D$  and every idempotent semiring  $R$  with identity element, the incidence semiring  $I_D(R)$  always has a convenient visible ideal basis  $B_D(R)$ , and elements of  $B_D(R)$  can be used to generate two-sided ideals with the largest possible weight among the weights of all two-sided ideals in the incidence semiring (see Theorem 4.1). Complete definitions of these terms are given in the next section.

## 2. Preliminaries

We use standard terminology and refer the readers to the monographs [14, 30, 33, 34, 53, 59] and articles [13, 19, 22, 47, 46] for more information. Throughout the word ‘digraph’ means a finite directed graph without multiple parallel edges but possibly with loops, and  $D = (V, E)$  is a digraph with the set  $V$  of vertices and the set  $E$  of edges.

Following [8], we do not assume that all semirings have identity elements. This makes it possible to consider incidence semirings for larger classes of digraphs. More specifically, a *semiring* is a set  $R$  with addition  $+$  and multiplication  $\cdot$  satisfying the following conditions:

- (1)  $(R, +)$  is a commutative semigroup with zero  $0$ ,
- (2)  $(R, \cdot)$  is a semigroup,
- (3) multiplication distributes over addition,
- (4) zero  $0$  annihilates  $R$ , i.e.,  $0 \cdot R = R \cdot 0 = 0$ .

In analogy with a similar terminology of ring theory, we call every semiring with  $1$  a *semiring with identity element*. A semiring  $R$  is called an *idempotent semiring*, or a *dioid*, or a *semiring with idempotent addition*, if  $x + x = x$  for  $x \in R$ .

*Definition 1.* Let  $D$  be a digraph, and let  $R$  be a semiring. Denote by  $I_D(R)$  the set consisting of zero 0 and all finite sums  $\sum_{i=1}^n r_i(g_i, h_i)$ , where  $n \geq 1$ ,  $r_i \in R$ ,  $(g_i, h_i) \in E$ , where all empty sums are assumed to be equal to 0, and where two operations are defined: the standard addition  $+$  and the multiplication  $\cdot$  defined by the distributive law and the rule

$$(g_1, h_1) \cdot (g_2, h_2) = \begin{cases} (g_1, h_2) & \text{if } h_1 = g_2 \text{ and } (g_1, h_2) \in E, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

for all  $(g_1, h_1), (g_2, h_2) \in E$ . The set  $I_D(R)$  with two operations  $+$  and  $\cdot$  defined above will be called the *incidence construction* of the digraph  $D$  over  $R$ . If  $I_D(R)$  is a semiring, then it is called the *incidence semiring* of  $D$  over  $R$ , or the *semiring* of  $D$  over  $R$ .

Every digraph  $D$  has the incidence construction  $I_D(R)$  of  $D$  over  $R$ . However, easy examples show that for some digraphs it is not a semiring (see Proposition 3.1 in the next section). If  $I_D(R)$  is a semiring, then we say that  $D$  has an incidence semiring. Examples of well known and important classes of digraphs that possess incidence semirings are given in the next section.

Incidence semirings of digraphs are a natural generalization of incidence rings (cf. [33, §3.15], [39], [49] and [59]). If  $R$  has an identity element 1, then to simplify notation we identify every edge  $e$  in  $E$  with the element  $1e$  in  $I_D(R)$ .

This paper focuses on ideals in the incidence semiring  $I_D(R)$  of the digraph  $D$ . Let  $\mathbb{N}$  be the set of all positive integers, and let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Suppose that  $T$  is a subset of  $I_D(R)$ . An *ideal* generated by  $T$  in  $I_D(R)$  is the set

$$\text{id}(T) = \left\{ \sum_{i=1}^k \ell_i g_i r_i \mid k \in \mathbb{N}_0, g_i \in T, \ell_i, r_i \in I_D(R) \cup \mathbb{N} \right\}, \quad (2)$$

where it is assumed that the identity element 1 of  $\mathbb{N}$  acts as an identity on the whole  $I_D(R)$  too. These ideals are also called *two-sided ideals*, since they are generated by multiplying the elements of the set  $T$  from both sides. In contrast, [7] treated different types of ideals called one-sided ideals, or left ideals and right ideals. The *weight*  $\text{wt}(r)$  of an element  $r = \sum_{i=1}^n r_i(g_i, h_i) \in I_D(R)$  is the number of nonzero coefficients  $r_i$  in the sum. The *weight* of a subset  $S$  of  $I_D(R)$  is defined as the minimum weight of a nonzero element in  $S$ .

The problem of generating ideals with the largest weight was originally motivated by applications to the design of classification systems, also known as classifiers (cf. [61], Section 7.5). Let us refer the readers to a few recent examples of papers in high quality journals devoted to the applications of classifiers in security [1, 4, 20, 38, 50] and health informatics [3, 5, 6, 31, 37, 60]. The role of ideals with largest weight in the design of classifiers is very well explained in [2], where a nice diagram illustrating the classification process is given. More explanations of the role of ideals with largest weight are given, for example, in [8, 48]. These explanations and previous work show that it is essential to find ideals with the largest weight in the incidence semirings.

### 3. Examples and Open Questions

Since the notion of an incidence semiring has been introduced by the authors only recently, to illustrate the applicability and usefulness of this concept this section presents examples and

theoretical results that establish relations between properties of the graph and properties of its incidence semiring. For example, the first proposition of this section shows that the balanced property of the digraph is equivalent to the associative law in the incidence construction over the digraph. The digraph  $D$  is said to be *balanced* if, for all  $g_1, g_2, g_3, g_4 \in V$  with  $(g_1, g_2), (g_2, g_3), (g_3, g_4), (g_1, g_4) \in E$ , the following equivalence holds:

$$(g_1, g_3) \in E \Leftrightarrow (g_2, g_4) \in E,$$

see [33, §3.15]. The multiplication in  $R$  is said to be *associative* if  $x(yz) = (xy)z$ , for all  $x, y, z \in R$ . The following fact is very easy and in the case of rings it is a part of folklore knowledge. Here we include a complete proof for semirings, since this proposition permeates the study of incidence semirings of digraphs.

**Proposition 3.1.** *The multiplication in  $I_D(R)$  is associative if and only if  $D$  is balanced.*

*Proof.* The ‘if’ part. Suppose that  $D$  is balanced. The distributive law shows that it suffices to verify the associative law for elements  $x, y, z \in I_D(R)$  of the form  $x = (g_1, g_2), y = (g_2, g_3), z = (g_3, g_4)$ , where  $g_1, g_2, g_3, g_4 \in V$ , because an arbitrary element of  $I_D(R)$  is a linear combination of elements of this form.

If  $(g_1, g_4) \notin E$ , then (1) implies that  $x(yz) = 0 = (xy)z$ , and so the associative law holds in this case. Further, we assume that  $(g_1, g_4) \in E$ , and so the balanced property of  $D$  applies. If  $(g_1, g_3) \in E$ , then  $(g_2, g_4) \in E$ , because  $D$  is balanced. Hence it follows from (1) that  $x(yz) = (g_1, g_4) = (xy)z$ . On the other hand, if  $(g_1, g_3) \notin E$ , then  $(g_2, g_4) \notin E$ , and so (1) implies that  $x(yz) = 0 = (xy)z$ . Thus, the associative law holds in all cases.

The ‘only if’ part. Suppose to the contrary that the associative law is satisfied in  $I_D(R)$ , but  $D$  is not balanced. Without loss of generality we may assume that there exist  $g_1, g_2, g_3, g_4 \in V$  such that  $(g_1, g_2), (g_2, g_3), (g_3, g_4), (g_1, g_4) \in E$  and  $(g_1, g_3) \in E$ , but  $(g_2, g_4) \notin E$ . Then it follows from (1) that  $x(yz) = (g_1, g_4)$  and  $(xy)z = 0$ . This contradicts the associative law and shows that  $D$  must be balanced. This completes the proof.  $\square$

It follows immediately from (1) and Proposition 3.1 that many natural and well known classes of graphs have incidence semirings.

*Example 3.1.* The incidence construction is a semiring for each of the following digraphs: (a) complete digraph, (b) null digraph, (c) directed cycle, (d) Petersen digraph.

A semiring  $R$  is said to be *nilpotent* if there exists a positive integer  $n$  such that  $x_1 \cdots x_n = 0$  for all  $x_1, \dots, x_n \in R$ .

**Proposition 3.2.** *Let  $D = (V, E)$  be a balanced finite graph, and let  $R$  be a semiring with identity element. Then the incidence semiring  $I_D(R)$  is nilpotent if and only if  $E$  does not contain loops.*

*Proof.* The ‘if’ part. Assuming that  $E$  does not contain any loops, suppose to the contrary that  $I_D(R)$  is not nilpotent. Put  $n = |V|$ . Then there exists a nonzero product  $(u_1, u_2) \cdots (u_n, u_{n+1}) = (u_1, u_{n+1}) \neq 0$  in  $I_D(R)$ , where  $(u_1, u_2), (u_2, u_3), \dots, (u_n, u_{n+1})$  in  $E$ . However, since  $|V| = n$ , it follows that there exist  $1 \leq i < j \leq n + 1$  such that  $u_i = u_j$ . Since  $(u_i, u_{i+1}) \cdots (u_{j-1}, u_j) =$

$(u_i, u_j) \neq 0$ , we get  $(u_i, u_j) = (u_i, u_i) \in E$ . This contradiction shows that  $I_D(R)$  has to be nilpotent.

The ‘only if’ part. Suppose to the contrary that  $I_D(R)$  is nilpotent, but  $E$  contains a loop  $(u, u)$ . It follows from (1) that for any  $n \in \mathbb{N}$  the product of  $n$  copies of  $(u, u)$  is equal to  $(u, u)$ , and so it is nonzero. This contradicts the nilpotency of  $I_D(R)$  and completes the proof.  $\square$

Examples and theoretical results of this sort confirm that the general problem of investigating the relations of properties of digraphs and properties of their incidence semirings is worth considering. Furthermore, here we include a few open questions concerning a few more advanced properties well known in graph theory. Let us start with the Moore digraphs. A *Moore digraph* is a digraph that meets the directed Moore bound. Let us refer to the survey [21] and articles [15, 16, 17, 18, 19] for more information and previous results on the Moore digraphs.

**Problem 1.** Find a semiring property such that the incidence semiring  $I_D(R)$  satisfies it if and only if  $D$  is a Moore graph.

Another important and well known class of digraphs is that of digraphs with edge antimagic labelling (cf. the survey [10], book [14] and papers [11, 12]).

**Problem 2.** Find a semiring property such that the incidence semiring  $I_D(R)$  satisfies it if and only if  $D$  has an edge antimagic labelling.

The readers are referred to [25, 26, 41, 42, 43, 44] for more information on power graphs.

**Problem 3.** Find a semiring property such that the incidence semiring  $I_D(R)$  satisfies it if and only if  $D$  is isomorphic to a power graph of a group or a semigroup.

Let us refer to [24, 32, 35, 36, 40, 45, 51, 55] for more complete bibliography and background information on the Cayley graphs.

**Problem 4.** Find a semiring property such that the incidence semiring  $I_D(R)$  satisfies it if and only if  $D$  is isomorphic to a Cayley graph of a group or a semigroup.

#### 4. Main Theorem

The following definition was introduced in [29] by analogy to a similar concept of ring theory considered, for example, in [49].

*Definition 2.* A subset  $S$  of a semiring  $R$  is called a *visible basis* for ideals, or a *visible ideal basis* if, for every subset  $T$  of  $S$ , the weight of the ideal  $\text{id}(T)$  generated by  $T$  in  $R$  is equal to the minimum of the weights of all elements of  $T$ .

Visible ideal bases are convenient for determining the weights of ideals. Let  $D = (V, E)$  be a digraph, and let  $g$  be a vertex in  $V$ . We use the following notation for two sets of vertices

$$\text{In}(g) = \{h \in V \mid (h, g) \in E\}, \tag{3}$$

$$\text{Out}(g) = \{h \in V \mid (g, h) \in E\}. \tag{4}$$

Define two sets of edges of the digraph  $D = (V, E)$

$$E_r = \{(g, h) \in E \mid \text{Out}(g) \cap \text{Out}(h) = \emptyset\}, \quad (5)$$

$$E_\ell = \{(g, h) \in E \mid \text{In}(g) \cap \text{In}(h) = \emptyset\}. \quad (6)$$

Denote the subgraph of  $D = (V, E)$  with the same set  $V$  of vertices and the set  $E_\ell$  of edges by

$$D_\ell = (V, E_\ell). \quad (7)$$

Likewise, denote the subgraph of  $D = (V, E)$  with the same set  $V$  of vertices and the set  $E_r$  of edges by

$$D_r = (V, E_r). \quad (8)$$

For any positive integer  $k$ , let  $\mathcal{P}_k$  stand for the set of all pairs  $(S, v)$ , where  $v \in V$  and  $S \subseteq \text{In}(v)$  are such that  $|S| = k$ ,  $(u, v) \in E_\ell$  for all  $u \in S$ , and the intersection  $\text{Out}(v) \cap \text{Out}(u)$  is equal to the same set for all vertices  $u$  in  $S$  (so that  $\text{Out}(v) \cap \text{Out}(u_1) = \text{Out}(v) \cap \text{Out}(u_2)$ , for all  $u_1, u_2 \in S$ ). Thus,

$$\mathcal{P}_k = \{(S, v) \mid v \in V, S \subseteq \text{In}(v), |S| = k, (u, v) \in E_\ell \text{ for all } u \in S, \text{Out}(v) \cap \text{Out}(u_1) = \text{Out}(v) \cap \text{Out}(u_2), \text{ for all } u_1, u_2 \in S\}. \quad (9)$$

Let  $\mathcal{H}_{L,k}$  be the set of all elements  $x = \sum_{u \in S} r(u, v) \in I_{D_\ell}(R)$ , for all pairs  $(S, v) \in \mathcal{P}_k$  and all  $0 \neq r \in R$ , i.e.,

$$\mathcal{H}_{L,k} = \left\{ \sum_{u \in S} r(u, v) \in I_{D_\ell}(R) \mid (S, v) \in \mathcal{P}_k, 0 \neq r \in R \right\}. \quad (10)$$

Denote by  $M_L$  the largest positive integer such that the set  $\mathcal{P}_{M_L}$  is not empty, or zero if such integers do not exist, and put

$$\mathcal{H}_L = \mathcal{H}_{L, M_L}. \quad (11)$$

Similarly, for any positive integer  $k$ , let us introduce the sets

$$\mathcal{Q}_k = \{(v, S) \mid v \in V, S \subseteq \text{Out}(v), |S| = k, (v, u) \in E_r \text{ for all } u \in S, \text{In}(v) \cap \text{In}(u_1) = \text{In}(v) \cap \text{In}(u_2), \text{ for all } u_1, u_2 \in S\}, \quad (12)$$

$$\mathcal{H}_{R,k} = \left\{ \sum_{u \in S} r(v, u) \in I_{D_r}(R) \mid (v, S) \in \mathcal{Q}_k, 0 \neq r \in R \right\}. \quad (13)$$

Denote by  $M_R$  the largest positive integer such that the set  $\mathcal{Q}_{M_R}$  is not empty, or zero if such integers do not exist, and put

$$\mathcal{H}_R = \mathcal{H}_{R, M_R}. \quad (14)$$

Let  $\mathcal{H}_Z$  be the set of all elements  $x = \sum_{(u,v) \in E_\ell \cap E_r} r(u, v) \in I_D(R)$  for all  $(u, v) \in E_\ell \cap E_r$  and  $0 \neq r \in R$ . Finally, put

$$B_D(R) = \mathcal{H}_Z \cup \mathcal{H}_R \cup \mathcal{H}_L. \tag{15}$$

Our main theorem shows that all incidence semirings of digraphs have convenient visible bases, which can be used to generate two-sided ideals of the largest weight.

**Theorem 4.1.** *Let  $D$  be a balanced finite digraph, and let  $R$  be an idempotent semiring with identity element. Then the following conditions hold:*

- (i) *The set  $B_D(R)$  is a visible ideal basis in  $I_D(R)$ .*
- (ii) *If  $I_D(R)$  has any ideal of weight greater than one, then the set  $B_D(R)$  contains an element  $x \in B_D(R)$  such that the ideal  $\text{id}(x)$  has the largest possible weight among the weights of all two-sided ideals in  $I_D(R)$ .*

It is nice that our main theorem eliminates algorithmic questions from the problem of generating ideals of the largest weight, because it gives us visible bases which can be used directly to generate ideals of the largest weight.

## 5. Proofs

A semiring  $F$  is said to be *zerosumfree* if, for all  $x_1, \dots, x_n \in F$ ,

$$x_1 + \dots + x_n = 0 \iff x_1 = \dots = x_n = 0. \tag{16}$$

The following lemma is easy and well known.

**Lemma 5.1.** ([30]) *All idempotent semirings are zerosumfree.*

For any semiring  $R$ , the *left annihilator* of  $R$  is the set

$$\text{Ann}_\ell(R) = \{x \in R \mid xR = 0\}, \tag{17}$$

and the *right annihilator* of  $R$  is the set

$$\text{Ann}_r(R) = \{x \in R \mid Rx = 0\}. \tag{18}$$

**Lemma 5.2.** *If  $D$  is a balanced digraph and  $R$  is a semiring with identity element, then the following equalities hold:*

$$\text{Ann}_r(I_D(R)) = I_{D_\ell}(R), \tag{19}$$

$$\text{Ann}_\ell(I_D(R)) = I_{D_r}(R). \tag{20}$$

*Proof.* Here we include only the proof of equality (19), because the proof of equality (20) is dual.

Let us first prove the inclusion  $\text{Ann}_r(I_D(R)) \supseteq I_{D_\ell}(R)$ . Pick any nonzero element  $x$  in  $I_{D_\ell}(R)$ . By (7), the element  $x$  can be written down in the form  $x = \sum_{i=1}^n x_i(g_i, h_i)$ , where  $x_i \in R$ ,  $(g_i, h_i) \in E_\ell$ . We claim that  $I_D(R)x = 0$ .

To verify this, suppose to the contrary that there exists an edge  $(u, v) \in E$  such that  $(u, v)x \neq 0$ . Then  $(u, v)(g_i, h_i) \neq 0$ , for some  $i$ . It follows from (1) that  $v = g_i$  and  $(u, h_i) \in E$ . Hence  $u \in \text{In}(g_i) \cap \text{In}(h_i)$ , and so  $(g_i, h_i) \notin E_\ell$ . This contradiction shows that the assumption made in the beginning of this paragraph was wrong, and in fact  $I_D(R)x = 0$ . This means that  $x \in \text{Ann}_r(I_D(R))$ . Thus,  $\text{Ann}_r(I_D(R)) \supseteq I_{D_\ell}(R)$ .

To prove the reversed inclusion, take any element  $x$  in  $\text{Ann}_r(I_D(R))$ . We can write it down as  $x = \sum_{i=1}^n x_i(g_i, h_i)$ , where  $0 \neq x_i \in R$ ,  $(g_i, h_i) \in E$ . Suppose to the contrary that  $x$  does not belong to  $I_{D_\ell}(R)$ . It follows from (7) that there exists  $i$  such that  $(g_i, h_i) \notin E_\ell$ . Hence (6) yields that there exists  $w \in V$  such that  $(w, g_i), (w, h_i) \in E$ . Therefore  $1(w, g_i) \cdot 1(g_i, h_i) = 1(w, h_i) \neq 0$  in  $I_D(R)$ . Since  $(w, g_i)(g_i, h_i)$  is a summand of  $(w, g_i)x$  with coefficient  $x_i$  and since  $x$  was chosen in the right annihilator  $\text{Ann}_r(I_D(R))$  we get  $(w, g_i)x = 0$ ; whence  $x_i = 0$ . This contradiction shows that our assumption made in the beginning of this paragraph was wrong, and in fact  $x \in I_{D_\ell}(R)$ . Therefore  $\text{Ann}_r(I_D(R)) \subseteq I_{D_\ell}(R)$ . This completes the proof of equality (19).  $\square$

**Lemma 5.3.** *If  $D$  is a balanced digraph,  $R$  is an idempotent semiring with identity element, and  $x \in \mathcal{H}_Z$ , then  $\{x\}$  is a visible ideal basis in  $I_D(R)$ .*

*Proof.* Since  $x$  belongs to  $\mathcal{H}_Z$ , we can write it down as a sum

$$x = \sum_{(i,j) \in E_\ell \cap E_r} r(i, j) \in I_D(R), \tag{21}$$

where  $0 \neq r \in R$ . Evidently,  $\text{wt}(x) = |E_\ell \cap E_r|$ . Consider an arbitrary nonzero element  $y$  in  $\text{id}(x)$ . It follows from (2) that

$$y = \sum_{j=1}^k \ell_j x r_j, \tag{22}$$

for some  $\ell_j, r_j \in I_D(R) \cup \{1\}$ . Lemma 5.2 shows that all summands  $\ell_j x r_j$  in (22) are equal to zero whenever  $\ell_j \in I_D(R)$  or  $r_j \in I_D(R)$ . We may assume that only nonzero summands are included in (22). This means that  $\ell_j = r_j = 1$  for all  $j = 1, \dots, k$ . Since  $R$  is an idempotent semiring, it follows from Lemma 5.1 that

$$\text{wt}(y) = \text{wt} \left( \sum_{j=1}^k x \right) = \text{wt}(x). \tag{23}$$

Therefore  $\text{wt}(\text{id}(x)) = \text{wt}(x)$ , which means that  $\{x\}$  is a visible ideal basis in  $I_D(R)$ .  $\square$

**Lemma 5.4.** *If  $D$  is a balanced digraph,  $R$  is an idempotent semiring with identity element,  $k$  is a positive integer and  $x \in \mathcal{H}_{L,k}$ , then  $\{x\}$  is a visible ideal basis in  $I_D(R)$ . In particular,  $\{y\}$  is a visible ideal basis in  $I_D(R)$ , for each  $y \in \mathcal{H}_L$ .*



*Proof.* Since  $x$  belongs to  $\mathcal{H}_{L,k}$ , it can be recorded as a sum

$$x = \sum_{s \in S} r(s, v) \in I_D(R), \tag{24}$$

where  $(S, v) \in \mathcal{P}_k$  and  $0 \neq r \in R$ . Hence  $\text{wt}(x) = |S| = k$ . We need to prove that  $\text{wt}(\text{id}(x)) = \text{wt}(x)$ .

Pick any nonzero element  $y$  in  $\text{id}(x)$ . We have to verify that  $\text{wt}(y) \geq \text{wt}(x)$ . It follows from (2) that  $y$  can be written down as

$$y = \sum_{j=1}^k \ell_j x r_j, \tag{25}$$

for some  $\ell_j, r_j \in I_D(R) \cup \{1\}$ . We may assume that only nonzero summands  $\ell_j x r_j$  are included in (25). It follows from  $x \in I_{D_\ell}(R)$  and Lemma 5.2 that  $\ell_j x = 0$  for all  $\ell_j \in I_D(R)$ . Therefore further we may assume that all the  $\ell_j$  are equal to 1 in the expression (25).

Since  $I_D(R) = \bigoplus_{(u,w) \in E} R(u, w)$ , in view of the distributive law we may assume that every element  $r_j \neq 1$  in (22) is homogeneous, which means that it belongs to the union  $\cup_{(u,w) \in E} R(u, w)$ . Furthermore, since  $x r_j \neq 0$ , it follows from (1) that all the  $r_j$  in (25) belong to  $\cup_{(v,w) \in E} R(v, w)$ , where the element  $v$  is associated to  $x$  in (24). By (9), the intersection  $\text{Out}(v) \cap \text{Out}(u)$  is equal to one and the same set  $T$  for all  $u$  in  $S$ . Since  $x r_j \neq 0$ , we see that all the  $r_j$  belong to  $\cup_{w \in T} R(v, w)$ . Therefore  $r_j = t_j(v, w_j)$ , for some  $0 \neq t_j \in R$  and  $w_j \in V$ . Since  $x r_j \neq 0$ , we get  $x r_j = \sum_{s \in S} r t_j(s, w_j)$ . Hence  $\text{wt}(x r_j) = \text{wt}(x)$ . Given that  $R$  is an idempotent semiring, it follows from Lemma 5.1 that  $\text{wt}(y) \geq \text{wt}(x)$ , as required.  $\square$

**Lemma 5.5.** *If  $D$  is a balanced digraph,  $R$  is an idempotent semiring with identity element,  $k$  is a positive integer, and  $x \in \mathcal{H}_{R,k}$ , then  $\{x\}$  is a visible ideal basis in  $I_D(R)$ . In particular,  $\{y\}$  is a visible ideal basis in  $I_D(R)$ , for each  $y \in \mathcal{H}_R$ .*

*Proof.* We omit the proof, because Lemma 5.5 is dual to Lemma 5.4.  $\square$

**Lemma 5.6.** *Let  $D$  be a balanced digraph,  $R$  an idempotent semiring with identity element, and let  $B$  be a subset of  $I_D(R)$ . Then  $B$  is a visible ideal basis in  $I_D(R)$  if and only if every set  $\{x\}$  is a visible ideal basis in  $I_D(R)$  for each  $x \in B$ .*

*Proof.* The ‘only if’ part follows immediately, because the definition of a visible ideal basis implies that every subset  $S$  of each visible ideal basis  $T$  is a visible ideal basis too.

To prove the ‘if’ part, suppose that all sets  $\{x\}$  are visible ideal bases in  $I_D(R)$ , for all  $x \in B$ . Choose an arbitrary subset  $T$  of  $B_D(R)$  and consider a nonzero element  $y$  in  $\text{id}(T)$  such that the weight of  $y$  is minimal, so that  $\text{wt}(y) = \text{wt}(\text{id}(T))$ . By (2), we can write  $y$  down as a sum

$$y = \sum_{i=1}^k \ell_i t_i r_i \tag{26}$$

for some  $t_i \in T$ ,  $k \in \mathbb{N}_0$ ,  $\ell_i, r_i \in I_D(R) \cup \mathbb{N}$ . Combining similar terms in (26) by collecting all summands  $\ell_i t_i r_i$  with equal values of the  $t_i \in T$  into separate subsums, we can rewrite  $y$  as

$$y = \sum_{i=1}^a \left( \sum_{j=1}^{b_i} \ell_{i,j} t_i r_{i,j} \right) \tag{27}$$

where  $t_i \in T$ ,  $\ell_{i,j}, r_{i,j} \in I_D(R) \cup \mathbb{N}$ , and where  $t_i \neq t_j$  for all  $0 \leq i < j \leq a$ . We may remove all sums equal to zero from (27) and assume that  $\sum_{j=1}^{b_i} \ell_{i,j} t_i r_{i,j} \neq 0$  for all  $i$ . Lemmas 5.3, 5.4 and 5.5 show that  $\text{wt}(\sum_{j=1}^{b_i} \ell_{i,j} t_i r_{i,j}) = \text{wt}(t_i)$ , because  $\sum_{j=1}^{b_i} \ell_{i,j} t_i r_{i,j} \in \text{id}(t_i)$ . Finally, Lemma 5.1 implies that

$$\text{wt}(y) \geq \text{wt}(t_1), \dots, \text{wt}(t_a).$$

This means that the set  $T$  is a visible ideal basis of  $I_D(R)$ , which completes the proof. □

*Proof of Theorem 4.1.* Condition (i) immediately follows from (15) and Lemmas 5.3, 5.4, 5.5 and 5.6.

Next, we are going to prove condition (ii). Suppose that the incidence semiring  $I_D(R)$  has an ideal of weight greater than one. Choose an ideal  $J$  of  $I_D(R)$  such that the weight of  $J$  is the largest possible one among the weights of all ideals in  $I_D(R)$ . Choose a nonzero element  $y$  of minimal weight in  $J$ . Then  $\text{wt}(y) = \text{wt}(J) \geq 2$ . Since  $y \in I_D(R)$ , there exist  $n \geq 2$ ,  $y_1, \dots, y_n \in R$ ,  $(g_1, h_1), \dots, (g_n, h_n) \in E$ , such that

$$y = \sum_{i=1}^n y_i(g_i, h_i). \tag{28}$$

We may assume that likely terms in (28) have been combined so that all edges  $(g_1, h_1), \dots, (g_n, h_n)$  are pairwise distinct. Then  $\text{wt}(y) = n$ . Here we have to consider several cases.

**Case 1.**  $(g_1, h_1), \dots, (g_n, h_n) \in E_\ell \cap E_r$ . Then  $|E_\ell \cap E_r| \geq \text{wt}(y) \geq 2$ . Hence the set  $\mathcal{H}_Z$  is nonempty. Take any nonzero element  $x \in \mathcal{H}_Z$ . Lemma 5.3 tells us that  $\text{wt}(x) = \text{wt}(\text{id}(x)) = |E_\ell \cap E_r|$ . Hence  $\text{wt}(y) \leq \text{wt}(x)$ . It follows from the maximality of the weight of  $\text{wt}(J) = \text{wt}(y)$  that  $\text{wt}(y) = \text{wt}(x)$ . This means that condition (ii) is satisfied in this case.

**Case 2.**  $(g_1, h_1), \dots, (g_n, h_n) \in E_r$ , but there exists  $i$  such that  $(g_i, h_i) \notin E_\ell$ . Without loss of generality we may assume that  $(g_1, h_1) \notin E_\ell$ . By (6), there exists  $v \in \text{In}(g_1) \cap \text{In}(h_1)$ .

Fix any  $j$  such that  $1 \leq j \leq n$ . Consider an arbitrary element  $w \in \text{In}(g_j) \cap \text{In}(h_j)$ . We get  $(w, g_j)y \neq 0$ , because  $y_j(w, h_j) = 1(w, g_j) \cdot y_j(g_j, h_j)$  is a summand of  $(w, g_j)y$ . Since  $(w, g_1)y \in \text{id}(y) \subseteq J$ , the minimality of the weight of  $y$  in  $J$  implies that  $\text{wt}((w, g_j)y) \geq \text{wt}(y)$ . However, (28) yields us that

$$(w, g_j)y = \sum_{i=1}^n y_i(w, g_j)(g_i, h_i), \tag{29}$$

and so  $\text{wt}((w, g_j)y) \leq n = \text{wt}(y)$ . Hence  $\text{wt}((w, g_j)y) = \text{wt}(y)$ . It follows that all summands  $y_j(w, g_j)(g_j, h_i)$  are nonzero. By (1), we get  $g_j = g_i$  for all  $i = 1, \dots, n$ . Therefore  $g_1 = g_2 =$

$\dots = g_n$ , for all  $j = 1, \dots, n$ . Since all summands  $y_j(w, g_1)(g_1, h_i)$  in (29) are nonzero, we get  $w \in \text{In}(g_1) \cap \text{In}(h_1)$ . It follows that  $\text{In}(g_j) \cap \text{In}(h_j) = \text{In}(g_1) \cap \text{In}(h_1)$ , for all  $j = 1, \dots, n$ .

Putting  $S = \{h_1, \dots, h_n\}$ , we get  $S \subseteq \text{Out}(g_1)$ . The hypothesis of Case 2 tells us that  $(g_1, u) \in E_r$  for all  $u \in S$ . Besides, as we have just verified in the preceding paragraph, the intersection  $\text{In}(g_1) \cap \text{In}(u)$  is equal to one and the same set for all vertices  $u$  in  $S$ . Therefore  $(g_1, S) \in \mathcal{Q}_n$ .

It follows from (13) that the element

$$z = \sum_{i=1}^n 1(g_1, h_i). \tag{30}$$

belongs to  $\mathcal{H}_{R,n}$ . Lemma 5.5 tells us that the set  $\{z\}$  is a visible ideal basis in  $I_D(R)$ . Hence  $\text{wt}(\text{id}(z)) = \text{wt}(z) = n$ . The maximality of the weight  $\text{wt}(J)$  and Lemma 5.5 imply that  $n = M_R$ . Therefore  $z \in \mathcal{H}_R$ . This means that condition (ii) holds true in this case.

**Case 3.**  $(g_1, h_1), \dots, (g_n, h_n) \in E_\ell$ , but there exists  $1 \leq i \leq n$  such that  $(g_i, h_i) \notin E_r$ . This case is dual to Case 2, and so a dual proof shows that condition (ii) holds true in this case too.

**Case 4.** There exist  $1 \leq a, b \leq n$  such that  $(g_a, h_b) \notin E_r \cup E_\ell$ . Without loss of generality we may assume that  $(g_1, h_1) \notin E_r \cup E_\ell$ . Then (5) and (6) show that we can find elements  $u \in \text{In}(g_1) \cap \text{In}(h_1)$  and  $v \in \text{Out}(g_1) \cap \text{Out}(h_1)$ . It follows from (1) and (28) that

$$(u, g_1)y(h_1, v) = y_1(u, v) \neq 0. \tag{31}$$

Since  $(u, g_1)y(h_1, v) \in J$ , we see that  $\text{wt}(J) = \text{wt}((u, g_1)y(h_1, v)) = 1$ . This contradiction shows that Case 4 is impossible.

Cases 1 through to 4 cover all possibilities. Therefore condition (ii) always holds true. This completes the proof.  $\square$

### Acknowledgements

The authors are grateful to the referee for suggesting serious improvements to the text of this article. The first author was supported by Discovery grant DP0880501 from Australian Research Council. The second author was supported by ARC Discovery grant DP0449469.

### References

- [1] J. Abawajy and A. V. Kelarev, A multi-tier ensemble construction of classifiers for phishing email detection and filtering, *Cyberspace Safety and Security, CSS 2012, Lecture Notes in Computer Science* **7672** (2012), 48–56.
- [2] J. Abawajy and A. V. Kelarev, Classification systems based on combinatorial semigroups, *Semigroup Forum* **86** (2013), 603–612.
- [3] J. H. Abawajy, A. V. Kelarev and M. Chowdhury, Multistage approach for clustering and classification of ECG data, *Computer Methods and Programs in Biomedicine* **112** (2013), 720-730.

- [4] J. Abawajy, A. Kelarev, M. Chowdhury, Large iterative multitier ensemble classifiers for security of big data, *IEEE Transaction on Emerging Topics in Computing* **2** (2014), 352–363.
- [5] J. Abawajy, A. V. Kelarev, M. Chowdhury and H. F. Jelinek, Enhancing predictive accuracy of cardiac autonomic neuropathy using blood biochemistry features and iterative multi-tier ensembles, *IEEE Journal of Biomedical and Health Informatics*, in press.
- [6] J. Abawajy, A. Kelarev, M. Chowdhury, A. Stranieri and H. F. Jelinek, Predicting cardiac autonomic neuropathy category for diabetic data with missing values, *Computers in Biology and Medicine* **43** (2013), 1328–1333.
- [7] J. Abawajy, A. V. Kelarev, M. Miller and J. Ryan, Incidence semirings of graphs and visible bases, *Bulletin of the Australian Mathematical Society* **89** (2014), 451–459.
- [8] J. Abawajy, A. V. Kelarev and J. Zeleznikow, Optimization of classification and clustering systems based on Munn semirings, *Semigroup Forum* **87** (2013), 617–626.
- [9] B. Aziz, A semiring-based quantitative analysis of mobile systems, *Electronic Notes in Theoretical Computer Science* **157** (2006), 3–21.
- [10] M. Baca, E. T. Baskoro, M. Miller, J. Ryan, R. Simanjuntak and K. A. Sugeng, Survey of edge antimagic labelings of graphs, *Journal of Indonesian Math. Society (MIHMI)* **12** (2006), 113–130.
- [11] M. Baca and L. Brankovic, Edge-antimagicness for a class of disconnected graphs, *Ars Combinatoria* **97A** (2010), 145–152.
- [12] M. Baca, L. Brankovic, M. Lascsakova, O. Phanalasy and A. Semanicova-Fenovciová, On d-antimagic labelings of plane graphs, *Electronic Journal of Graph Theory and Applications* **1** (2013), 28–39.
- [13] M. Baca, S. Jendrol, M. Miller and J. Ryan, On irregular total labellings, *Discrete Mathematics* **307** (2007), 1378–1388.
- [14] M. Baca and M. Miller, *Super Edge-Antimagic Graphs: A Wealth of Problems and Some Solutions*, Brown Walker Press, 2008.
- [15] E. T. Baskoro, Y. M. Cholily and M. Miller, Structure of selfrepeat cycles in almost Moore digraphs with selfrepeats and diameter 3, *Bulletin of the Institute of Combinatorics and its Applications* **46** (2006), 99–109.
- [16] E. T. Baskoro, Y. M. Cholily and M. Miller, Enumeration of vertex orders of almost Moore digraphs with selfrepeats, *Discrete Mathematics* **308** (2008), 123–128.
- [17] E. T. Baskoro, M. Miller and J. Plesnik, On the structure of digraphs with order close to the Moore bound, *Graphs and Combinatorics* **14** (1998), 109–119.

- [18] E. T. Baskoro, M. Miller and J. Plesnik, Further results on almost Moore digraphs, *Ars Combinatoria* **56** (2000), 43–63.
- [19] E. T. Baskoro, M. Miller, J. Plesnik and S. Znam, Digraphs of degree 3 close to Moore bound, *Journal of Graph Theory* **20** (1995), 339–349.
- [20] G. Beliakov, J. Yearwood and A. Kelarev, Application of rank correlation, clustering and classification in information security, *Journal of Networks* **7** (2012), 935–945.
- [21] M. Miller and J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, *Electronic J. Combinatorics, Dynamic Survey* **DS20**, 2013, 92pp.
- [22] E. Dahlhaus, P. Horak, M. Miller and J. Ryan, The train marshalling problem, *Discrete Applied Mathematics* **103** (2000), 41–54.
- [23] S. Bistarelli, S. N. Foley, B. O’Sullivan and F. Santini, Semiring-based frameworks for trust propagation in small-world networks and coalition formation criteria, *Security and Communication Networks* **3** (2010), 595–610.
- [24] L. Brankovic, M. Miller, J. Plesnik, J. Ryan and J. Siran, A note on constructing large Cayley graphs of given degree and diameter by voltage assignment, *Electronic J. Combinatorics* **5** (1998), 147–158.
- [25] B. Curtin and G. R. Pourgholi, A group sum inequality and its applications to power graphs, *Bulletin of the Australian Mathematical Society* **90** (2014), 418–426.
- [26] B. Curtin and G. R. Pourgholi, Edge-maximality of power graphs of finite cyclic groups, *J. Algebraic Combinatorics* **40** (2014), 313–330.
- [27] N. Damljanović, M. Ćirić and J. Ignjatović, Bisimulations for weighted automata over an additively idempotent semiring, *Theoretical Computer Science* **534** (2014), 86–100.
- [28] V. I. Danilov, A. V. Karzanov and G. A. Koshevoy, Planar flows and quadratic relations over semirings, *Journal of Algebraic Combinatorics* **36** (2012), 441–474.
- [29] D. Y. Gao, A. V. Kelarev and J. L. Yearwood, Optimization of matrix semirings for classification systems, *Bulletin of the Australian Mathematical Society* **84** (2011), 492–503.
- [30] J. S. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.
- [31] H. F. Jelinek, J. H. Abawajy, A. V. Kelarev, M. U. Chowdhury and A. Stranieri, Decision trees and multi-level ensemble classifiers for neurological diagnostics, *AIMS Medical Science* **1** (2014), 1–12.
- [32] A. V. Kelarev, On undirected Cayley graphs, *Australasian Journal of Combinatorics* **25** (2002), 73–78.

- [33] A. V. Kelarev, *Ring Constructions and Applications*, World Scientific, River Edge, NJ, 2002.
- [34] A. V. Kelarev, *Graph Algebras and Automata*, Marcel Dekker, New York, 2003.
- [35] A. V. Kelarev, Labelled Cayley graphs and minimal automata, *Australasian J. Combinatorics* **30** (2004), 95–101.
- [36] A. V. Kelarev, On Cayley graphs of inverse semigroups, *Semigroup Forum* **72** (2006), 411–418.
- [37] A. V. Kelarev, J. Abawajy, A. Stranieri and H. F. Jelinek, Empirical investigation of decision tree ensembles for monitoring cardiac complications of diabetes, *International Journal of Data Warehousing and Mining* **9** (2013), 1–18.
- [38] A. Kelarev, S. Brown, P. Watters, X.-W. Wu and R. Dazeley, Establishing reasoning communities of security experts for internet commerce security, *Technologies for Supporting Reasoning Communities and Collaborative Decision Making: Cooperative Approaches*, IGI Global, 2011, pp. 380–396.
- [39] A. V. Kelarev and D. S. Passman, A description of incidence rings of group automata, *Contemporary Mathematics* **456** (2008), 27–33.
- [40] A. V. Kelarev and C. E. Praeger, On transitive Cayley graphs of groups and semigroups, *European J. Combinatorics* **24** (2003), 59–72.
- [41] A. V. Kelarev and S. J. Quinn, A combinatorial property and power graphs of groups, *Contrib. General Algebra* **12** (2000), 229–235.
- [42] A. V. Kelarev and S. J. Quinn, Directed graphs and combinatorial properties of semigroups, *J. Algebra* **251** (2002), 16–26.
- [43] A. V. Kelarev and S. J. Quinn, A combinatorial property and power graphs of semigroups, *Comment. Math. Univ. Carolinae* **45** (2004), 1–7.
- [44] A. V. Kelarev, S. J. Quinn and R. Smolikova, Power graphs and semigroups of matrices, *Bulletin of the Australian Mathematical Society* **63** (2001), 341–344.
- [45] A. Kelarev, J. Ryan and J. Yearwood, Cayley graphs as classifiers for data mining: The influence of asymmetries, *Discrete Mathematics* **309** (2009), 5360–5369.
- [46] A. Kelarev, J. Ryan and J. Yearwood, An algorithm for the optimization of multiple classifiers in data mining based on graphs, *J. Comb. Math. Comb. Computing* **71** (2009), 65–86.
- [47] A. V. Kelarev and O. V. Sokratova, On congruences of automata defined by directed graphs, *Theoretical Computer Science* **301** (2003), 31–43.
- [48] A. V. Kelarev, J. L. Yearwood and M. A. Mammadov, A formula for multiple classifiers in data mining based on Brandt semigroups, *Semigroup Forum* **78** (2009), 293–309.

- [49] A. V. Kelarev, J. L. Yearwood and P. W. Vamplew, A polynomial ring construction for classification of data, *Bulletin of the Australian Mathematical Society* **79** (2009), 213–225.
- [50] A. V. Kelarev, J. L. Yearwood, P. Watters, X. W. Wu, J. H. Abawajy and L. Pan, Internet security applications of the Munn rings, *Semigroup Forum* **81** (2010), 162–171.
- [51] B. Khosravi and M. Mahmoudi, On Cayley graphs of rectangular groups, *Discrete Mathematics* **310** (2010), 804–811.
- [52] D. Kirsten, Decidability, undecidability, and PSPACE-completeness of the twins property in the tropical semiring, *Theoretical Computer Science* **420** (2012), 56–63.
- [53] U. Knauer, *Algebraic Graph Theory*, De Gruyter, Berlin, 2011.
- [54] G. L. Litvinov, A. Ya. Rodionov, S. N. Sergeev and A. N. Sobolevski, Universal algorithms for solving the matrix Bellman equations over semirings, *Soft Computing* **17** (2013), 1767–1785.
- [55] E. Loz, M. Macaj, M. Miller, Šiagiová, J. Širán and J. Tomanová, Small vertex-transitive and Cayley graphs of girth six and given degree: an algebraic approach, *J. Graph Theory* **68** (2011), 265–284.
- [56] A. Lugowski, S. Kamil, A. Buluc, S. Williams, E. Duriakova, L. Oliker, A. Fox, J. R. Gilbert, Parallel processing of filtered queries in attributed semantic graphs, *J. Parallel Distributed Computing* (2014), <http://dx.doi.org/10.1016/j.jpdc.2014.08.010>, in press.
- [57] F. Otto and O. Sokratova, Reduction relations for monoid semirings, *Journal of Symbolic Computation* **37** (2004), 343–376.
- [58] P. Sotin, D. Cacher and T. Jensen, Quantitative static analysis over semirings: analysing cache behaviour for Java card, *Electronic Notes in Theoretical Computer Science* **164** (2006), 153–167.
- [59] E. Spiegel and C. J. O’Donnell, *Incidence Algebras*, Marcel Dekker, New York, 1997.
- [60] A. Stranieri, J. Abawajy, A. V. Kelarev, S. Huda, M. Chowdhury and H. F. Jelinek, An approach for Ewing test selection to support the clinical assessment of cardiac autonomic neuropathy, *Artificial Intelligence in Medicine* **58** (2013), 185–193.
- [61] I. H. Witten and E. Frank, *Data Mining: Practical Machine Learning Tools and Techniques*, Elsevier/Morgan Kaufman, Amsterdam, 2005.
- [62] Y. Yu and S. Ren, Coordinating asynchronous and open distributed systems under semiring-based timing constraints, *Electronic Notes in Theoretical Computer Science* **229** (2009), 133–153.