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A note on edge-disjoint contractible Hamiltonian cycles in polyhedral maps

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Abstract

We present a necessary and sufficient condition for the existence of edge-disjoint contractible Hamiltonian cycles in the edge graph of polyhedral maps.

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1. Introduction and Definitions

Recall the following definitions (see Maity and Upadhyay [7]) that a *graph* $G := (V, E)$ is a simple graph with vertex set V and edge set E. A *surface* S is a connected, compact, 2-dimensional manifold without boundary. A *map* on a surface S is an embedding of a finite graph G such that the closure of components of $S \setminus G$ is p-gonal 2-disc for $p \geq 3$. The components are also called *facets*. The map M is called a *polyhedral map* if nonempty intersection of any two facets of the map is either a vertex or an edge. We call G the *edge graph* of the map and denote it by $EG(M)$. The vertices and edges of G are also called vertices and edges of the map, respectively. A *path* P in a graph G is a subgraph $P : [v_1v_2 \dots v_n]$ of G, such that the vertex set of P is $V(P) = \{v_1, v_2, \dots, v_n\} \subseteq V(G)$ and v_iv_{i+1} are edges in P for $1 \leq i \leq n-1$. A path $P : [v_1, v_2, \dots, v_n]$ in G is said to be a *cycle* if v_nv_1 is also an edge in P. A graph without any cycle is called a *tree*. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then $G_1 \bigcup G_2$ is defined to be a graph $G(V, E)$ for

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which $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2$. In this case G is called *union* of the graphs G_1 and G_2. Similarly, $G_1 \bigcap G_2$ is the graph $G(V, E)$ for which $V = V_1 \bigcap V_2$ and $E = E_1 \bigcap E_2$. In this case G is called *intersection* of G_1 and G_2. These definitions remain valid for a finite number of graphs as well. See Mohar and Thomassen [8] for details about graphs on surfaces and Bondy and Murthy [1] for terminology related to graph theory.

In this note we are interested in finding out whether edge-disjoint Hamiltonian cycles exist in the edge graph of a polyhedral map. Such cycles in graphs have been studied previously. For example, Nash-Williams [10] generalised a result of Dirac [4] about existence of Hamiltonian cycles and showed that every graph on n vertices of minimum degree at least $\frac{n}{2}$ contains at least $\lfloor \frac{5n}{224} \rfloor$ edge-disjoint Hamiltonian cycles. Christofides, Kühn and Osthus [2] improved the bound of Nash-Williams and showed there is a positive integer n_0 such that every graph on $n \geq n_0$ vertices with minimum degree $(\frac{1}{2} + \alpha)n$ (for every $\alpha > 0$) contains at least $\frac{n}{8}$ edge-disjoint Hamiltonian cycles. They also showed that if such a graph is almost regular, then it can almost be decomposed into edge-disjoint Hamiltonian cycles. In this note we present a necessary and sufficient condition for the existence of edge-disjoint contractible Hamiltonian cycles in the edge graph of a polyhedral map. To show this result we define a subgraph in the edge graph of dual of a polyhedral map K as *admissible graph* (see Definition 1.2). We use this admissible graph and enumerate the edge-disjoint contractible Hamiltonian cycles in the polyhedral map K. To show this result we use the concept of proper tree and the Proposition 1.1.

We begin with some terminology defined in Maity and Upadhyay [7] which will be needed in the course of the proof of the main Theorem 1.1. We call a cycle in the edge graph of a map to be *contractible* if it bounds a 2-disk (2-cell) (see Upadhyay [9] and Hachimori [5]). For example, the boundary cycle of a facet is contractible. If v is a vertex of a map K, then the number of edges incident with v is called the *degree* of v and it is denoted by $\deg(v)$. If the number of vertices, edges and facets of K are denoted by $f_0(K), f_1(K)$ and $f_2(K)$ respectively, then the integer $\chi(K) = f_0(K) - f_1(K) + f_2(K)$ is called the *Euler characteristic of* K. The *dual map* M of K is defined to be the map on the same surface as K, which has for its vertices the set of facets of K and two vertices u_1 and u_2 of M are ends of an edge of M if the corresponding facets in K have an edge in common. The well-known maps of type $\{3, 6\}$ and $\{6, 3\}$ on the surface of torus are examples of mutually dual maps.

Consider a polyhedral map K on a surface S that has n vertices.

Definition 1.1. *(See Maity and Upadhyay [7]) Let M denote the dual map of K. Let $T := (V, E)$ denote a tree in the edge graph $EG(M)$ of M. We say that T is a proper tree if the following conditions hold :*

1. $\sum_{i=1}^{k} \deg(v_i) = n + 2(k-1)$*, where* $V = \{v_1, v_2, ..., v_k\}$ *and* $\deg(v)$ *denotes degree of* v *in* $EG(M)$*,*
2. *whenever two vertices u_1 and u_2 of T lie on a face F in M, a path $P[u_1, u_2]$ joining u_1 and u_2 in the boundary ∂F of F is a subtree of T, and*
3. *any path P in T which lies in a face F of M is of length at most $q - 2$, where q is the length of ∂F.*

Definition 1.2. *Let M denote the dual map of a polyhedral map K on n vertices. Let $H := (V, E)$ denote a subgraph in the edge graph $EG(M)$ of M. We say that H is an admissible graph if the following conditions hold:*

1. *H has a decomposition into proper trees $T_1, T_2, \ldots, T_r$ such that $H = T_1 \bigcup T_2 \bigcup \ldots \bigcup T_r$ and $T_i \neq T_j$ for $i \neq j$, $i, j \in \{1, \ldots, r\}$,*
2. *$T_i \bigcap T_j$ is a set of paths and for $v \in V(T_i \bigcap T_j)$ we have $\deg(v)$ in $EG(M)$ is equal to $\deg(v)$ in $T_i \bigcup T_j$, for $i \neq j$ and $i, j \in \{1, \ldots, r\}$, and*
3. *the graph $T_i \bigcup T_j$ does not contain a pair of vertices u_i, u_j with $u_i \in V(T_i)$ and $u_j \in V(T_j)$ such that $u_iu_j \in E(EG(M))$ and $u_iu_j \notin E(T_i \bigcup T_j)$ for $i \neq j$ and $i, j \in \{1, \ldots, r\}$.*

Remark 1.1. : Let $v \in V(T_{i_1}), \ldots, V(T_{i_t})$, $i_1, \ldots, i_t \in \{1, \ldots, r\}$, then $\displaystyle\sum_{v \in V(H)} t \deg(v) = rn + 2\sum_{i=1}^{r}(k_i - 1)$, where $H = T_1 \bigcup T_2 \bigcup \cdots \bigcup T_r$, $n = |V(EG(K))|$ and $k_i = |V(T_i)|$.

By the Definition 1.1 we have $\displaystyle\sum_{j=1}^{k_i} \deg(v_j) = n + 2(k_i - 1)$ for the proper tree T_i and $1 \leq i \leq r$, where $k_i = |V(T_i)|$. Hence $\displaystyle\sum_{i=1}^{r}\sum_{j=1}^{k_i} \deg(v_j) = \sum_{i=1}^{r}(n + 2(k_i - 1)) = rn + 2\sum_{i=1}^{r}(k_i - 1)$.

Proposition 1.1. *[Maity, Upadhyay] [7] The edge graph $EG(K)$ of a map K on a surface has a contractible Hamiltonian cycle if and only if the edge graph of the corresponding dual map of K has a proper tree.*

The main result of this note is:

Theorem 1.1. *Let K be a map on the surface S with n vertices. Then, K contains r edge-disjoint contractible Hamiltonian cycles, if and only if the dual map M of K contains an admissible graph H that has a decomposition into r proper trees.*

In particular, we prove:

Corollary 1.1. *Let K be a map on the surface S with n vertices. Then, K contains r face-disjoint contractible Hamiltonian cycles, if and only if the dual map M of K contains an admissible graph H that has a decomposition into r disjoint proper trees.*

In the next section, we give examples of an admissible graph and the existence of edge- and face-disjoint contractible Hamiltonian cycles in polyhedral maps. Then, in the following section we present the proofs of Theorem 1.1 and Corollary 1.1.

2. Examples

Example 2.1. Figure 1 depicts a triangulation of a surface M_1 of $\chi = 0$ on 7 vertices (see Datta and Upadhyay [3]). K depicts the dual of M_1 in Figure 2. Graph $H := (V, E)$ where $V := \{w_1, w_2, w_4, w_6, w_9, w_{10}, w_{13}, w_{14}\}$ and $E := \{w_1w_2, w_1w_6, w_1w_{14}, w_{13}w_{14}, w_4w_{13}, w_9w_{14}, w_9w_{10}\}$ is an admissible graph in K. Let $T_1 := (V_1, E_1)$ where $V_1 := \{w_1, w_2, w_9, w_{10}, w_{14}\}$ and $E_1 := \{w_1w_2, w_1w_{14}, w_9w_{14}, w_9w_{10}\}$, and $T_2 := (V_2, E_2)$ where $V_2 := \{w_1, w_4, w_6, w_{13}, w_{14}\}$ and $E_1 := \{w_1w_6, w_1w_{14}, w_{13}w_{14}, w_4w_{13}\}$. Then, graph H has a decomposition into T_1 and T_2.

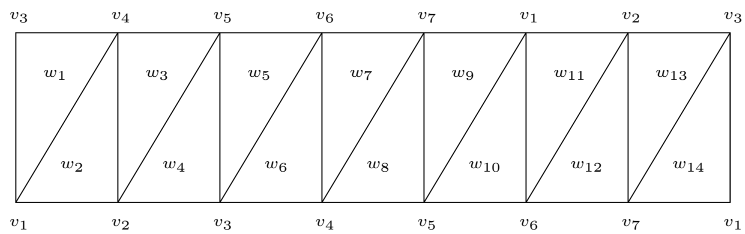

Figure 1 : M_1

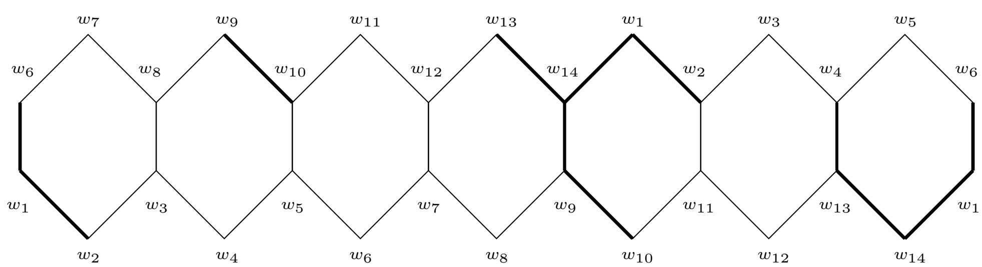

Figure 2 : K

Example 2.2. Figure 3 depicts a triangulation of a surface M_2 of $\chi = -3$ on 9 vertices taken from Lutz [6]. $\partial D_1 = C(1, 6, 4, 2, 3, 5, 7, 9, 8)$ in Figure 4 and $\partial D_2 = C(5, 2, 7, 1, 3, 8, 6, 9, 4)$ in Figure 5 depict edge-disjoint contractible Hamiltonian cycles in M_2.

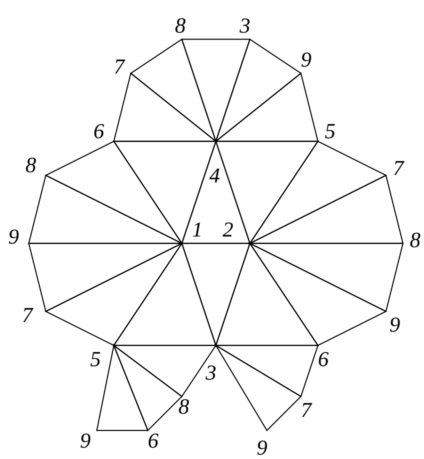

Figure 3 : M_2

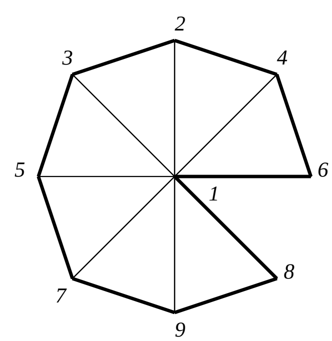

Figure 4 : D_1

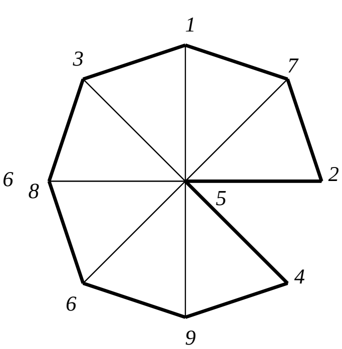

Figure 5 : D_2

Example 2.3. Figure 6 Lutz [6] depicts a triangulation of a surface M_3 of $\chi = -10$ on 12 vertices. This triangulation contains two face disjoint cycles $\partial D'_1 = C(1, 7, 8, c, 4, 5, 3, 9, 6, a, b, 2)$ in Figure 7 and $\partial D'_2 = C(1, 8, 2, 5, c, 6, 7, b, 3, a, 9, 4)$ in Figure 8 as shown below.

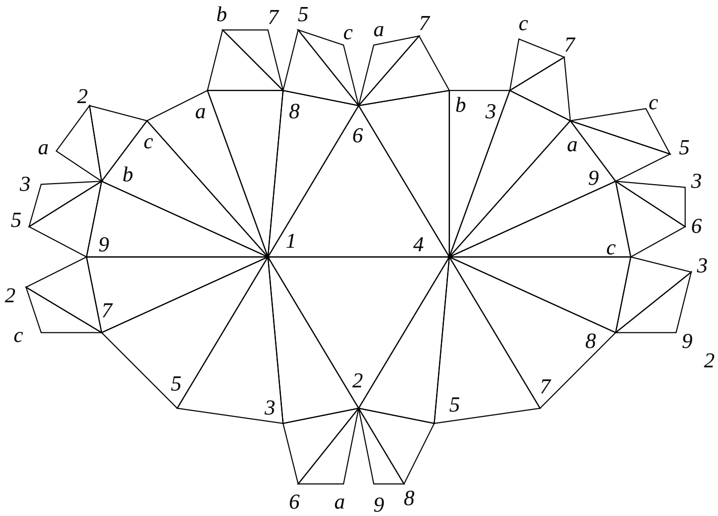

Figure 6 : M_3

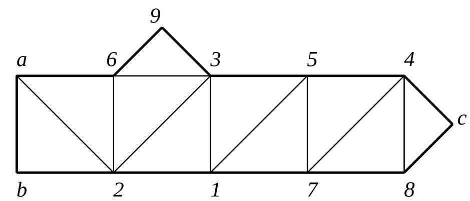

Figure 7 : D'_1

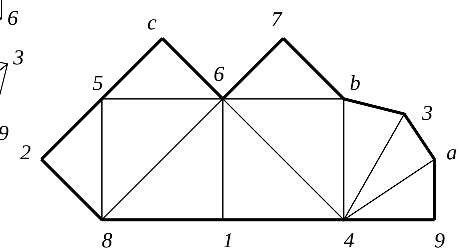

Figure 8 : D'_2

3. Proof of the Theorem 1.1

PROOF OF THEOREM 1.1 : Let M be as in the statement of Theorem 1.1 and containing an admissible graph H. By Definition 1.2, it has a decomposition $H = T_1 \bigcup T_2 \bigcup \cdots \bigcup T_r$. Then by the Proposition 1.1 see Maity and Upadhyay [7], the map K contains contractible Hamiltonian cycles C_i corresponding to T_i for $1 \leq i \leq r$. Hence the map K contains r contractible Hamiltonian cycles. We now show that these cycles are pairwise edge-disjoint.

Suppose, on the contrary, $E(C_i) \bigcap E(C_j)$ contains an edge uv. Then uv belongs to two faces, say, F_1 and F_2. Let $D(C_i)$ denote the 2-disk which is bounded by the cycle C_i and v_{F_1} denote the vertex corresponding to F_1 in the dual. Two situations may arise. In the first, if $F_1 \in DC_i$ and $F_2 \in DC_j$, then edge $v_{F_1}v_{F_2}$ does not belong to the graph $T_i \bigcup T_j$. That is, $v_{F_1}v_{F_2} \notin E(T_i \bigcup T_j)$ and $v_{F_1}v_{F_2} \in E(EG(M))$. This contradicts the condition 3 of Definition 1.2. Further, in the second situation if one of the two faces F_1 and F_2, say F_1, belongs to both disks DC_i and DC_j then F_1 lies in both disks. Hence the degree of v_{F_1} in $T_i \bigcup T_j$ is less than the degree of v_{F_1} in $EG(M)$. This contradicts the condition 2 in Definition 1.2. Therefore $E(C_i) \bigcap E(C_j) = \emptyset$ for $i \neq j$ and $i, j \in \{1, \ldots, r\}$. Hence the map K contains r edge-disjoint contractible Hamiltonian cycles.

Suppose the map M has r edge-disjoint Hamiltonian cycles $C_1, C_2, \ldots, C_r$ and let the dual of the disk DC_i be the tree T_i. We define $H := T_1 \bigcup T_2 \bigcup \cdots \bigcup T_r$. Since all the T_is are distinct proper trees, it is easy to check that H satisfies the condition 1 in Definition 1.2. Suppose there are two trees T_i and T_j such that the graph $T_i \bigcap T_j$ contains a vertex v with $\deg(v)$ in the graph $EG(M)$ that is greater than its degree in $T_i \bigcup T_j$. Thus there exists an edge vw that does not belong to the graph $T_i \bigcup T_j$. Consider the dual face F_v corresponding to vertex v. Face F_v belongs to both disks DC_i and DC_j as v belongs to $V(T_i \bigcap T_j)$. So the dual edge corresponding to vw shall lie in the boundary of the 2-disks DC_i and DC_j. Hence C_i and C_j are not edge-disjoint. This is a contradiction. Hence $\deg(v)$ in $EG(M)$ is greater than $\deg(v)$ in $T_i \bigcup T_j$ for all the vertices of $T_i \bigcap T_j$. This gives the condition 2 in the Definition 1.2. Let $u_i \in V(T_i)$ and $u_j \in V(T_j)$ be such that $u_i u_j \in E(EG(M))$ and $u_i u_j \notin E(T_i \bigcup T_j)$. Then face F_{u_i} belongs to the disk DC_i and face F_{u_j} belongs to the disk DC_j. Moreover, the dual edge corresponding to $u_i u_j$ will lie in both faces F_{u_i} and F_{u_j}. Hence edge $u_i u_j$ will be on the boundary of both the 2-disks DC_i and DC_j. Therefore both the cycles C_i and C_j contain the dual edge corresponding to $u_i u_j$. So C_i and C_j are not edge-disjoint. This is a contradiction. So we see that the condition 3 in Definition 1.2 is also satisfied. Thus H is the required admissible graph. □

PROOF OF COROLLARY 1.1 : To prove the corollary we proceed exactly same as in the previous proof of Theorem 1.1 and we use disjoint proper trees instead of proper trees. □

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