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Twin edge colorings of certain square graphs and product graphs

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Abstract

A twin edge k-coloring of a graph G is a proper edge k-coloring of G with the elements of $\mathbb{Z}_k$ so that the induced vertex k-coloring, in which the color of a vertex v in G is the sum in $\mathbb{Z}_k$ of the colors of the edges incident with v, is a proper vertex k-coloring. The minimum k for which G has a twin edge k-coloring is called the twin chromatic index of G. Twin chromatic index of the square P_n^2, $n \geq 4$, and the square C_n^2, $n \geq 6$, are determined. In fact, the twin chromatic index of the square C_7^2 is $\Delta + 2$, where Δ is the maximum degree. Twin chromatic index of $C_m \,\square\, P_n$ is determined, where $\square$ denotes the Cartesian product. C_r and P_r are, respectively, the cycle, and the path on r vertices each.

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1. Introduction

Let G be a simple graph. A *proper vertex coloring* of G is an assignment from a given set of colors to the set of vertices of G, where adjacent vertices are colored differently. The minimum number of colors needed in a proper vertex coloring of G is the *chromatic number* of G and it is denoted by $\chi(G)$. A *proper edge coloring* of G is an assignment from a given set of colors to the

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set of edges of G, where adjacent edges are colored differently. The minimum number of colors needed in a proper edge coloring of G is the *chromatic index* of G and it is denoted by $\chi'(G)$.

Recently, a related coloring was introduced by Chartrand and studied in [1] and [2]. For a connected graph G of order at least 3, let $c : E(G) \to \mathbb{Z}_k$ be a proper edge k-coloring of G for some integer $k \geq 2$. A vertex k-coloring $\sigma_c : V(G) \to \mathbb{Z}_k$ is then defined by

$$\sigma_c(v) = \sum_{e \in E_v} c(e)$$

in $\mathbb{Z}_k$, where E_v is the set of edges of G incident with a vertex v and the indicated sum is computed in $\mathbb{Z}_k$. If the induced vertex k-coloring σ_c is proper, then c is called a *twin edge k-coloring* of G. The minimum k for which G has a twin edge k-coloring is called the *twin chromatic index* of G and it is denoted by $\chi_t'(G)$. Since a twin edge coloring is not only a proper edge coloring of G but induces a proper vertex coloring of G, it follows that

$$\chi_t'(G) \geq \max\{\chi(G), \chi'(G)\}.$$

For every connected graph G that is neither an odd cycle nor a complete graph, $\chi(G) \leq \Delta(G) \leq \chi'(G)$; for an odd cycle $\chi(C_{2n+1}) = 3 = \chi'(C_{2n+1})$; for the complete graph of odd order $\chi(K_{2n+1}) = \chi'(K_{2n+1})$; and for the complete graph of even order $\chi(K_{2n}) = 1 + \chi'(K_{2n})$. Hence $\chi_t'(G) \geq \max\{\chi(G), \chi'(G)\} = \chi'(G)$ except when G is a complete graph of even order. $\chi_t'(G)$ does not exist if G is the connected graph of order 2, and it was observed in [1] that every connected graph of order at least 3 has a twin edge coloring.

In [1], Andrews *et al.* obtained the twin chromatic indexes of paths, complete graphs and complete bipartite graphs. If n, a, b are integers with $n \geq 3$, $1 \leq a \leq b$ and $b \geq 2$, then $\chi_t'(P_n) = 3$, $\chi_t'(C_n) = 3$ if $n \equiv 0 \pmod 3$, $\chi_t'(C_n) = 4$ if $n \not\equiv 0 \pmod 3$ and $n \neq 5$, $\chi_t'(C_5) = 5$, $\chi_t'(K_n) = n$ if n is odd, $\chi_t'(K_n) = n+1$ if n is even, $\chi_t'(K_{1,b}) = b+1$ if $b \not\equiv 1 \pmod 4$, $\chi_t'(K_{1,b}) = b+2$ if $b \equiv 1 \pmod 4$, $\chi_t'(K_{a,a}) = a+2 = \chi_t'(K_{a,a+1})$ if $a \geq 2$, and $\chi_t'(K_{a,b}) = b$ if $b \geq a+2$ and $a \geq 2$.

The *Cartesian product* $G \,\square\, H$ of two simple graphs G and H is the simple graph with vertex set $V(G) \times V(H)$, and two vertices (u_1, u_2) and (v_1, v_2) of $G \,\square\, H$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.

In [2], Andrews *et al.* obtained the twin chromatic indexes for grids, prisms and trees with small maximum degree. If $n \geq 3$ is an integer with $n \neq 5$, then $\chi_t'(C_n \,\square\, K_2) = 4$. For $n = 5$, $\chi_t'(C_5 \,\square\, K_2) = 5$. If $n \geq 2$ is an integer, then $\chi_t'(P_n \,\square\, K_2) = 4$. If n and q are integers with $n, q \geq 3$, then $\chi_t'(P_n \,\square\, P_q) = 5$. Every tree T having maximum degree at most 6 has $\chi_t'(T) \leq 2 + \Delta(T)$. Finally, in [2], Andrews et. al conjectured the following:

Conjecture 1.1. *If G is a connected graph of order at least 3 that is not a 5-cycle, then $\chi_t'(G) \leq 2 + \Delta(G)$.*

Observation 1.1. *If a connected graph G contains two adjacent vertices of degree $\Delta(G)$, then $\chi_t'(G) \geq 1 + \Delta(G)$.*

The k-th *power* of a simple graph G is the simple graph G^k with vertex set $V(G)$ and edge set $\{uv \mid d_G(u, v) \leq k\}$. Notation and terminalogy not mentioned here can be found in [3].

2. $\chi_t'(P_n^2)$

Let $P_n := x_1x_2x_3\ldots x_n$. Consider P_n^2. For $i \in \{1,2,\ldots,n-1\}$, let $e_i = x_ix_{i+1}$ and for $i \in \{1,2,\ldots,n-2\}$, let $f_i = x_ix_{i+2}$.

Define $c : E(P_4^2) \to \mathbb{Z}_4$ as follows: $c(e_1) = 3$, $c(e_2) = 2$, $c(e_3) = 3$, $c(f_1) = 0$, $c(f_2) = 1$. The induced vertex coloring is: $\sigma_c(x_1) = 3$, $\sigma_c(x_2) = 2$, $\sigma_c(x_3) = 1$, $\sigma_c(x_4) = 0$, and it is proper. Hence $\chi_t'(P_4^2) \le 4$. By Observation 1.1, $\chi_t'(P_4^2) \ge 4$ and so $\chi_t'(P_4^2) = 4$.

Define $c : E(P_5^2) \to \mathbb{Z}_4$ as follows: $c(e_1) = 1$, $c(e_2) = 2$, $c(e_3) = 3$, $c(e_4) = 2$, $c(f_1) = c(f_2) = 0$, $c(f_3) = 1$. The induced vertex coloring is: $\sigma_c(x_1) = 1$, $\sigma_c(x_2) = 3$, $\sigma_c(x_3) = 2$, $\sigma_c(x_4) = 1$, $\sigma_c(x_5) = 3$, and it is proper. Hence $\chi_t'(P_5^2) \le 4$. As $\chi_t'(P_5^2) \ge \Delta(P_5^2) = 4$, we have $\chi_t'(P_5^2) = 4$.

For $n \ge 6$, define $c : E(P_n^2) \to \mathbb{Z}_5$ as follows:

$c(e_1) = 3$,

$c(e_i) = i \pmod 5$ if $2 \le i \le n-2$,

$$c(e_{n-1}) = \begin{cases} 2 \text{ if } n \equiv 0 \pmod 5, \\ 3 \text{ if } n \equiv 1,\ 4 \pmod 5, \\ 4 \text{ if } n \equiv 2 \pmod 5, \\ 0 \text{ if } n \equiv 3 \pmod 5, \end{cases}$$

$c(f_1) = 0$,

$c(f_i) = (i-2) \pmod 5$ if $2 \le i \le n-3$,

$$c(f_{n-2}) = \begin{cases} (n-4) \pmod 5 \text{ if } n \equiv 1,\ 2,\ 4 \pmod 5, \\ n \pmod 5 \text{ if } n \equiv 0,\ 3 \pmod 5. \end{cases}$$

The induced vertex coloring is:

$\sigma_c(x_1) = c(e_1) + c(f_1) = 3 + 0 \equiv 3$;

$\sigma_c(x_2) = c(e_1) + c(e_2) + c(f_2) = 3 + 2 + 0 \equiv 0$;

$\sigma_c(x_3) = c(e_2) + c(e_3) + c(f_1) + c(f_3) = 2 + 3 + 0 + 1 \equiv 1$;

for $4 \le i \le n-3$,

$$\sigma_c(x_i) = c(e_{i-1}) + c(e_i) + c(f_{i-2}) + c(f_i) = (i-1) + i + (i-4) + (i-2) = 4i - 7$$
$$\equiv \begin{cases} 4 \text{ if } i \equiv 4 \pmod 5, \\ 3 \text{ if } i \equiv 0 \pmod 5, \\ 2 \text{ if } i \equiv 1 \pmod 5, \\ 1 \text{ if } i \equiv 2 \pmod 5, \\ 0 \text{ if } i \equiv 3 \pmod 5; \end{cases}$$

$$\sigma_c(x_{n-2}) = c(e_{n-3}) + c(e_{n-2}) + c(f_{n-4}) + c(f_{n-2}) = (n-3) + (n-2) + (n-6) + c(f_{n-2})$$
$$= 3n - 11 + c(f_{n-2}) \equiv \begin{cases} 4 \text{ if } n \equiv 0,\ 1 \pmod 5, \\ 3 \text{ if } n \equiv 2 \pmod 5, \\ 1 \text{ if } n \equiv 3,\ 4 \pmod 5; \end{cases}$$

$$\sigma_c(x_{n-1}) = c(e_{n-2}) + c(e_{n-1}) + c(f_{n-3}) = (n-2) + c(e_{n-1}) + (n-5) = 2n - 7 + c(e_{n-1})$$

$$\equiv \begin{cases} 0 \text{ if } n \equiv 0 \pmod{5}, \\ 3 \text{ if } n \equiv 1 \pmod{5}, \\ 1 \text{ if } n \equiv 2 \pmod{5}, \\ 4 \text{ if } n \equiv 3,\ 4 \pmod{5}; \end{cases}$$

$$\sigma_c(x_n) = c(e_{n-1}) + c(f_{n-2}) \equiv \begin{cases} 2 \text{ if } n \equiv 0,\ 2 \pmod{5}, \\ 0 \text{ if } n \equiv 1 \pmod{5}, \\ 3 \text{ if } n \equiv 3,\ 4 \pmod{5}. \end{cases}$$

The sequence $\{\sigma_c(x_1), \sigma_c(x_2), \sigma_c(x_3), \dots\}$ is of the form $\{3, 0, 1, 4, 3, 2, 1, 0, 4, 3, 2, 1, 0, \dots\}$ and its end $\{\dots, \sigma_c(x_{n-2}), \sigma_c(x_{n-1}), \sigma_c(x_n)\}$ is of the form $\{\dots, 4, 3, 2, 1, 0,\ 4, 3, 2, 1, 4, 0, 2\}$ if $n \equiv 0 \pmod{5}$, $\{\dots, 4, 3, 2, 1, 0,\ 4, 3, 0\}$ if $n \equiv 1 \pmod{5}$, $\{\dots, 4, 3, 2, 1, 0,\ 4, 3, 1, 2\}$ if $n \equiv 2 \pmod{5}$, $\{\dots, 4, 3, 2, 1, 0,\ 4, 3, 1, 4, 3\}$ if $n \equiv 3 \pmod{5}$, and $\{\dots, 4, 3, 2, 1, 0,\ 4, 3, 2, 1, 4, 3\}$ if $n \equiv 4 \pmod{5}$. Hence c is a twin edge 5-coloring of P_n^2 and therefore $\chi_t'(P_n^2) \le 5$. By Observation 1.1, $\chi_t'(P_n^2) \ge 5$, and so $\chi_t'(P_n^2) = 5$.

Thus, we have the following theorem.

Theorem 2.1. $\chi_t'(P_4^2) = 4$, $\chi_t'(P_5^2) = 4$, *and for* $n \ge 6$, $\chi_t'(P_n^2) = 5$.

3. $\chi_t'(C_n^2)$

Let $C_n := x_1 x_2 x_3 \dots x_n x_1$. For $n \ge 6$, consider C_n^2. For $i \in \{1, 2, \dots, n\}$, $e_i = x_i x_{i+1}$ and $f_i = x_i x_{i+2}$, where $x_{n+1} = x_1$ and $x_{n+2} = x_2$. By Observation 1.1, $\chi_t'(C_n^2) \ge 5$.

• For $n \equiv 0 \pmod{6}$, define $c : E(C_n^2) \to \mathbb{Z}_5$ as follows:

$$c(e_i) = \begin{cases} 0 \text{ if } i \text{ is odd}, \\ 1 \text{ if } i \text{ is even}, \end{cases}$$

$$c(f_i) = \begin{cases} 2 \text{ if } i \equiv 1 \pmod{3}, \\ 3 \text{ if } i \equiv 2 \pmod{3}, \\ 4 \text{ if } i \equiv 0 \pmod{3}. \end{cases}$$

The induced vertex coloring is:

$$\sigma_c(x_i) = c(e_{i-1}) + c(e_i) + c(f_{i-2}) + c(f_i) = \begin{cases} 1 \text{ if } i \equiv 1 \pmod{3}, \\ 3 \text{ if } i \equiv 2 \pmod{3}, \\ 2 \text{ if } i \equiv 0 \pmod{3}. \end{cases}$$

Hence c is a twin edge 5-coloring of C_n^2 and therefore $\chi_t'(C_n^2) \le 5$.

• For $n \equiv 5 \pmod{6}$ and $n \ge 11$, define $c : E(C_n^2) \to \mathbb{Z}_5$ as follows:

For $i \in \{1, 2, \dots, n-3\}$, $c(e_i) = \begin{cases} 0 \text{ if } i \text{ is odd}, \\ 1 \text{ if } i \text{ is even}, \end{cases}$

$c(e_{n-2}) = 2$,
$c(e_{n-1}) = 3$,

$c(e_n) = 4.$

For $i \in \{1, 2, \ldots, n-3\}$, $c(f_i) = \begin{cases} 3 \text{ if } i \equiv 1 \pmod 3, \\ 4 \text{ if } i \equiv 2 \pmod 3, \\ 2 \text{ if } i \equiv 0 \pmod 3, \end{cases}$

$c(f_{n-2}) = 0,$
$c(f_{n-1}) = 1,$
$c(f_n) = 2.$

The induced vertex coloring is:
For $i \in \{2, 3, \ldots, n-3\}$, $\sigma_c(x_i) = c(e_{i-1}) + c(e_i) + c(f_{i-2}) + c(f_i) = 1 + c(f_{i-2}) + c(f_i)$

$$= \begin{cases} 1+4+3 = 3 \text{ if } i \equiv 1 \pmod 3, \\ 1+2+4 = 2 \text{ if } i \equiv 2 \pmod 3, \\ 1+3+2 = 1 \text{ if } i \equiv 0 \pmod 3. \end{cases}$$

$\sigma_c(x_{n-2}) = c(e_{n-3}) + c(e_{n-2}) + c(f_{n-4}) + c(f_{n-2}) = 1+2+3+0 = 1.$
$\sigma_c(x_{n-1}) = c(e_{n-2}) + c(e_{n-1}) + c(f_{n-3}) + c(f_{n-1}) = 2+3+4+1 = 0.$
$\sigma_c(x_n) = c(e_{n-1}) + c(e_n) + c(f_{n-2}) + c(f_n) = 3+4+0+2 = 4.$
$\sigma_c(x_1) = c(e_n) + c(e_1) + c(f_{n-1}) + c(f_1) = 4+0+1+3 = 3.$
Hence c is a twin edge 5-coloring of C_n^2 and therefore $\chi_t^{'}(C_n^2) \leq 5$.

• For $n \equiv 4 \pmod 6$ and $n \geq 10$, define $c : E(C_n^2) \to \mathbb{Z}_5$ as follows:

For $i \in \{1, 2, \ldots, n-8\}$, $c(e_i) = \begin{cases} 0 \text{ if } i \text{ is odd}, \\ 1 \text{ if } i \text{ is even}, \end{cases}$

$c(e_{n-7}) = c(e_{n-2}) = 2,$
$c(e_{n-6}) = c(e_{n-1}) = 3,$
$c(e_{n-5}) = c(e_n) = 4,$
$c(e_{n-4}) = 0,$
$c(e_{n-3}) = 1.$

For $i \in \{1, 2, \ldots, n-8\}$, $c(f_i) = \begin{cases} 3 \text{ if } i \equiv 1 \pmod 3, \\ 4 \text{ if } i \equiv 2 \pmod 3, \\ 2 \text{ if } i \equiv 0 \pmod 3, \end{cases}$

$c(f_{n-7}) = c(f_{n-2}) = 0,$
$c(f_{n-6}) = c(f_{n-1}) = 1,$
$c(f_{n-5}) = c(f_n) = 2,$
$c(f_{n-4}) = 3,$
$c(f_{n-3}) = 4.$

The induced vertex coloring is:
For $i \in \{2, 3, \ldots, n-8\}$, $\sigma_c(x_i) = c(e_{i-1}) + c(e_i) + c(f_{i-2}) + c(f_i) = 1 + c(f_{i-2}) + c(f_i)$

$$= \begin{cases} 1+4+3=3 \text{ if } i \equiv 1 \pmod 3, \\ 1+2+4=2 \text{ if } i \equiv 2 \pmod 3, \\ 1+3+2=1 \text{ if } i \equiv 0 \pmod 3). \end{cases}$$

$\sigma_c(x_{n-7}) = c(e_{n-8}) + c(e_{n-7}) + c(f_{n-9}) + c(f_{n-7}) = 1+2+3+0 = 1.$
$\sigma_c(x_{n-6}) = c(e_{n-7}) + c(e_{n-6}) + c(f_{n-8}) + c(f_{n-6}) = 2+3+4+1 = 0.$
$\sigma_c(x_{n-5}) = c(e_{n-6}) + c(e_{n-5}) + c(f_{n-7}) + c(f_{n-5}) = 3+4+0+2 = 4.$
$\sigma_c(x_{n-4}) = c(e_{n-5}) + c(e_{n-4}) + c(f_{n-6}) + c(f_{n-4}) = 4+0+1+3 = 3.$
$\sigma_c(x_{n-3}) = c(e_{n-4}) + c(e_{n-3}) + c(f_{n-5}) + c(f_{n-3}) = 0+1+2+4 = 2.$
$\sigma_c(x_{n-2}) = c(e_{n-3}) + c(e_{n-2}) + c(f_{n-4}) + c(f_{n-2}) = 1+2+3+0 = 1.$
$\sigma_c(x_{n-1}) = c(e_{n-2}) + c(e_{n-1}) + c(f_{n-3}) + c(f_{n-1}) = 2+3+4+1 = 0.$
$\sigma_c(x_n) = c(e_{n-1}) + c(e_n) + c(f_{n-2}) + c(f_n) = 3+4+0+2 = 4.$
$\sigma_c(x_1) = c(e_n) + c(e_1) + c(f_{n-1}) + c(f_1) = 4+0+1+3 = 3.$
Hence c is a twin edge 5-coloring of C_n^2 and therefore $\chi_t'(C_n^2) \leq 5$.

• For $n \equiv 3 \pmod 6$ and $n \geq 15$, define $c : E(C_n^2) \rightarrow \mathbb{Z}_5$ as follows:

For $i \in \{1, 2, \ldots, n-13\}$, $c(e_i) = \begin{cases} 0 \text{ if } i \text{ is odd}, \\ 1 \text{ if } i \text{ is even}, \end{cases}$

$c(e_{n-12}) = c(e_{n-7}) = c(e_{n-2}) = 2,$
$c(e_{n-11}) = c(e_{n-6}) = c(e_{n-1}) = 3,$
$c(e_{n-10}) = c(e_{n-5}) = c(e_n) = 4,$
$c(e_{n-9}) = c(e_{n-4}) = 0,$
$c(e_{n-8}) = c(e_{n-3}) = 1.$

For $i \in \{1, 2, \ldots, n-13\}$, $c(f_i) = \begin{cases} 3 \text{ if } i \equiv 1 \pmod 3, \\ 4 \text{ if } i \equiv 2 \pmod 3, \\ 2 \text{ if } i \equiv 0 \pmod 3, \end{cases}$

$c(f_{n-12}) = c(f_{n-7}) = c(f_{n-2}) = 0,$
$c(f_{n-11}) = c(f_{n-6}) = c(f_{n-1}) = 1,$
$c(f_{n-10}) = c(f_{n-5}) = c(f_n) = 2,$
$c(f_{n-9}) = c(f_{n-4}) = 3,$
$c(f_{n-8}) = c(f_{n-3}) = 4.$

The induced vertex coloring is:

For $i \in \{2, 3, \ldots, n-13\}$, $\sigma_c(x_i) = \begin{cases} 3 \text{ if } i \equiv 1 \pmod 3, \\ 2 \text{ if } i \equiv 2 \pmod 3, \\ 1 \text{ if } i \equiv 0 \pmod 3). \end{cases}$

$\sigma_c(x_{n-12}) = \sigma_c(x_{n-7}) = \sigma_c(x_{n-2}) = 1.$
$\sigma_c(x_{n-11}) = \sigma_c(x_{n-6}) = \sigma_c(x_{n-1}) = 0.$
$\sigma_c(x_{n-10}) = \sigma_c(x_{n-5}) = \sigma_c(x_n) = 4.$
$\sigma_c(x_{n-9}) = \sigma_c(x_{n-4}) = \sigma_c(x_1) = 3.$
$\sigma_c(x_{n-8}) = \sigma_c(x_{n-3}) = 2.$
Hence c is a twin edge 5-coloring of C_n^2 and therefore $\chi_t'(C_n^2) \leq 5$.

• For $n = 9$, define $c : E(C_9^2) \to \mathbb{Z}_5$ as follows:
$c(e_1) = c(e_3) = c(e_5) = c(f_7) = 0$,
$c(f_3) = c(f_6) = c(f_9) = 1$,
$c(f_2) = c(f_5) = c(f_8) = 2$,
$c(e_4) = c(e_6) = c(e_8) = c(f_1) = 3$,
$c(e_2) = c(e_7) = c(e_9) = c(f_4) = 4$.

The induced vertex coloring is:
$\sigma_c(x_8) = 0$, $\sigma_c(x_5) = 1$, $\sigma_c(x_2) = 2$, $\sigma_c(x_3) = \sigma_c(x_6) = \sigma_c(x_9) = 3$,
$\sigma_c(x_1) = \sigma_c(x_4) = \sigma_c(x_7) = 4$.
Hence c is a twin edge 5-coloring of C_9^2 and therefore $\chi_t^{'}(C_9^2) \leq 5$.

• For $n \equiv 2 \pmod 6$ and $n \geq 20$, define $c : E(C_n^2) \to \mathbb{Z}_5$ as follows:
For $i \in \{1, 2, \ldots, n-18\}$, $c(e_i) = \begin{cases} 0 \text{ if } i \text{ is odd,} \\ 1 \text{ if } i \text{ is even,} \end{cases}$
$c(e_{n-17}) = c(e_{n-12}) = c(e_{n-7}) = c(e_{n-2}) = 2$,
$c(e_{n-16}) = c(e_{n-11}) = c(e_{n-6}) = c(e_{n-1}) = 3$,
$c(e_{n-15}) = c(e_{n-10}) = c(e_{n-5}) = c(e_n) = 4$,
$c(e_{n-14}) = c(e_{n-9}) = c(e_{n-4}) = 0$,
$c(e_{n-13}) = c(e_{n-8}) = c(e_{n-3}) = 1$.
For $i \in \{1, 2, \ldots, n-18\}$, $c(f_i) = \begin{cases} 3 \text{ if } i \equiv 1 \pmod 3, \\ 4 \text{ if } i \equiv 2 \pmod 3, \\ 2 \text{ if } i \equiv 0 \pmod 3, \end{cases}$
$c(f_{n-17}) = c(f_{n-12}) = c(f_{n-7}) = c(f_{n-2}) = 0$,
$c(f_{n-16}) = c(f_{n-11}) = c(f_{n-6}) = c(f_{n-1}) = 1$,
$c(f_{n-15}) = c(f_{n-10}) = c(f_{n-5}) = c(f_n) = 2$,
$c(f_{n-14}) = c(f_{n-9}) = c(f_{n-4}) = 3$,
$c(f_{n-13}) = c(f_{n-8}) = c(f_{n-3}) = 4$.

The induced vertex coloring is:
For $i \in \{2, 3, \ldots, n-18\}$, $\sigma_c(x_i) = \begin{cases} 3 \text{ if } i \equiv 1 \pmod 3, \\ 2 \text{ if } i \equiv 2 \pmod 3, \\ 1 \text{ if } i \equiv 0 \pmod 3. \end{cases}$
$\sigma_c(x_{n-17}) = \sigma_c(x_{n-12}) = \sigma_c(x_{n-7}) = \sigma_c(x_{n-2}) = 1$.
$\sigma_c(x_{n-16}) = \sigma_c(x_{n-11}) = \sigma_c(x_{n-6}) = \sigma_c(x_{n-1}) = 0$.
$\sigma_c(x_{n-15}) = \sigma_c(x_{n-10}) = \sigma_c(x_{n-5}) = \sigma_c(x_n) = 4$.
$\sigma_c(x_{n-14}) = \sigma_c(x_{n-9}) = \sigma_c(x_{n-4}) = \sigma_c(x_1) = 3$.
$\sigma_c(x_{n-13}) = \sigma_c(x_{n-8}) = \sigma_c(x_{n-3}) = 2$.
Hence c is a twin edge 5-coloring of C_n^2 and therefore $\chi_t^{'}(C_n^2) \leq 5$.

• For $n = 8$, define $c : E(C_8^2) \to \mathbb{Z}_5$ as follows:

$c(e_5) = c(f_2) = c(f_7) = 0$, $c(e_7) = c(f_1) = c(f_4) = 1$, $c(e_1) = c(f_3) = c(f_6) = 2$,
$c(e_3) = c(f_5) = c(f_8) = 3$, $c(e_2) = c(e_4) = c(e_6) = c(e_8) = 4$.
The induced vertex coloring is:
$\sigma_c(x_3) = \sigma_c(x_8) = 0$, $\sigma_c(x_1) = \sigma_c(x_6) = 2$, $\sigma_c(x_4) = \sigma_c(x_7) = 3$, $\sigma_c(x_2) = \sigma_c(x_5) = 4$.
Hence c is a twin edge 5-coloring of C_8^2 and therefore $\chi_t^{'}(C_8^2) \leq 5$.

• For $n = 14$, define $c : E(C_{14}^2) \to \mathbb{Z}_5$ as follows:
$c(e_5) = c(e_7) = c(e_9) = c(e_{11}) = c(f_2) = c(f_{13}) = 0$,
$c(e_{13}) = c(f_1) = c(f_4) = c(f_7) = c(f_{10}) = 1$, $c(e_1) = c(f_3) = c(f_6) = c(f_9) = c(f_{12}) = 2$,
$c(e_3) = c(f_5) = c(f_8) = c(f_{11}) = c(f_{14}) = 3$,
$c(e_2) = c(e_4) = c(e_6) = c(e_8) = c(e_{10}) = c(e_{12}) = c(e_{14}) = 4$.
The induced vertex coloring is:
$\sigma_c(x_3) = \sigma_c(x_{14}) = 0$, $\sigma_c(x_1) = \sigma_c(x_6) = \sigma_c(x_9) = \sigma_c(x_{12}) = 2$,
$\sigma_c(x_4) = \sigma_c(x_7) = \sigma_c(x_{10}) = \sigma_c(x_{13}) = 3$, $\sigma_c(x_2) = \sigma_c(x_5) = \sigma_c(x_8) = \sigma_c(x_{11}) = 4$.
Hence c is a twin edge 5-coloring of C_{14}^2 and therefore $\chi_t^{'}(C_{14}^2) \leq 5$.

• For $n \equiv 1 \pmod 6$ and $n \geq 25$, define $c : E(C_n^2) \to \mathbb{Z}_5$ as follows:
For $i \in \{1, 2, \ldots, n-23\}$, $c(e_i) = \begin{cases} 0 \text{ if } i \text{ is odd}, \\ 1 \text{ if } i \text{ is even}, \end{cases}$
$c(e_{n-22}) = c(e_{n-17}) = c(e_{n-12}) = c(e_{n-7}) = c(e_{n-2}) = 2$,
$c(e_{n-21}) = c(e_{n-16}) = c(e_{n-11}) = c(e_{n-6}) = c(e_{n-1}) = 3$,
$c(e_{n-20}) = c(e_{n-15}) = c(e_{n-10}) = c(e_{n-5}) = c(e_n) = 4$,
$c(e_{n-19}) = c(e_{n-14}) = c(e_{n-9}) = c(e_{n-4}) = 0$,
$c(e_{n-18}) = c(e_{n-13}) = c(e_{n-8}) = c(e_{n-3}) = 1$.
For $i \in \{1, 2, \ldots, n-23\}$, $c(f_i) = \begin{cases} 3 \text{ if } i \equiv 1 \pmod 3, \\ 4 \text{ if } i \equiv 2 \pmod 3, \\ 2 \text{ if } i \equiv 0 \pmod 3, \end{cases}$
$c(f_{n-22}) = c(f_{n-17}) = c(f_{n-12}) = c(f_{n-7}) = c(f_{n-2}) = 0$,
$c(f_{n-21}) = c(f_{n-16}) = c(f_{n-11}) = c(f_{n-6}) = c(f_{n-1}) = 1$,
$c(f_{n-20}) = c(f_{n-15}) = c(f_{n-10}) = c(f_{n-5}) = c(f_n) = 2$,
$c(f_{n-19}) = c(f_{n-14}) = c(f_{n-9}) = c(f_{n-4}) = 3$,
$c(f_{n-18}) = c(f_{n-13}) = c(f_{n-8}) = c(f_{n-3}) = 4$.

The induced vertex coloring is:
For $i \in \{2, 3, \ldots, n-23\}$, $\sigma_c(x_i) = \begin{cases} 3 \text{ if } i \equiv 1 \pmod 3, \\ 2 \text{ if } i \equiv 2 \pmod 3, \\ 1 \text{ if } i \equiv 0 \pmod 3). \end{cases}$
$\sigma_c(x_{n-22}) = \sigma_c(x_{n-17}) = \sigma_c(x_{n-12}) = \sigma_c(x_{n-7}) = \sigma_c(x_{n-2}) = 1$.
$\sigma_c(x_{n-21}) = \sigma_c(x_{n-16}) = \sigma_c(x_{n-11}) = \sigma_c(x_{n-6}) = \sigma_c(x_{n-1}) = 0$.
$\sigma_c(x_{n-20}) = \sigma_c(x_{n-15}) = \sigma_c(x_{n-10}) = \sigma_c(x_{n-5}) = \sigma_c(x_n) = 4$.
$\sigma_c(x_{n-19}) = \sigma_c(x_{n-14}) = \sigma_c(x_{n-9}) = \sigma_c(x_{n-4}) = \sigma_c(x_1) = 3$.

$\sigma_c(x_{n-18}) = \sigma_c(x_{n-13}) = \sigma_c(x_{n-8}) = \sigma_c(x_{n-3}) = 2.$
Hence c is a twin edge 5-coloring of C_n^2 and therefore $\chi_t^{'}(C_n^2) \leq 5.$

• For $n = 19$, define $c : E(C_{19}^2) \to \mathbb{Z}_5$ as follows:
$c(e_1) = c(e_3) = c(e_5) = c(e_7) = c(e_9) = c(e_{11}) = c(e_{13}) = c(e_{15}) = c(f_{17}) = 0,$
$c(e_{16}) = c(e_{18}) = c(f_1) = c(f_4) = c(f_7) = c(f_{10}) = c(f_{13}) = 1,$
$c(e_{12}) = c(e_{17}) = c(f_3) = c(f_6) = c(f_9) = c(f_{14}) = c(f_{19}) = 2,$
$c(e_2) = c(e_4) = c(e_6) = c(e_8) = c(e_{10}) = c(f_{12}) = c(f_{15}) = c(f_{18}) = 3,$
$c(e_{14}) = c(e_{19}) = c(f_2) = c(f_5) = c(f_8) = c(f_{11}) = c(f_{16}) = 4.$

The induced vertex coloring is:
$\sigma_c(x_{18}) = 0$, $\sigma_c(x_3) = \sigma_c(x_6) = \sigma_c(x_9) = \sigma_c(x_{12}) = \sigma_c(x_{17}) = 1,$
$\sigma_c(x_{13}) = \sigma_c(x_{16}) = \sigma_c(x_{19}) = 2$, $\sigma_c(x_1) = \sigma_c(x_4) = \sigma_c(x_7) = \sigma_c(x_{10}) = \sigma_c(x_{15}) = 3,$
$\sigma_c(x_2) = \sigma_c(x_5) = \sigma_c(x_8) = \sigma_c(x_{11}) = \sigma_c(x_{14}) = 4.$
Hence c is a twin edge 5-coloring of C_{19}^2 and therefore $\chi_t^{'}(C_{19}^2) \leq 5.$

• For $n = 13$, define $c : E(C_{13}^2) \to \mathbb{Z}_5$ as follows:
$c(e_1) = c(e_3) = c(e_5) = c(e_7) = c(e_9) = c(f_{11}) = 0,$
$c(e_{10}) = c(e_{12}) = c(f_1) = c(f_4) = c(f_7) = 1,$
$c(e_6) = c(e_{11}) = c(f_3) = c(f_8) = c(f_{13}) = 2$, $c(e_2) = c(e_4) = c(f_6) = c(f_9) = c(f_{12}) = 3,$
$c(e_8) = c(e_{13}) = c(f_2) = c(f_5) = c(f_{10}) = 4.$

The induced vertex coloring is:
$\sigma_c(x_{12}) = 0$, $\sigma_c(x_3) = \sigma_c(x_6) = \sigma_c(x_{11}) = 1,$
$\sigma_c(x_7) = \sigma_c(x_{10}) = \sigma_c(x_{13}) = 2$, $\sigma_c(x_1) = \sigma_c(x_4) = \sigma_c(x_9) = 3,$
$\sigma_c(x_2) = \sigma_c(x_5) = \sigma_c(x_8) = 4.$
Hence c is a twin edge 5-coloring of C_{13}^2 and therefore $\chi_t^{'}(C_{13}^2) \leq 5.$

• For $n = 7$, define $c : E(C_7^2) \to \mathbb{Z}_6$ as follows:
$c(e_2) = c(e_4) = c(e_6) = 0,$
$c(e_1) = 1$, $c(e_3) = c(e_5) = c(e_7) = 2,$
$c(f_1) = c(f_4) = c(f_7) = 3$, $c(f_2) = c(f_5) = 4$, $c(f_3) = c(f_6) = 5.$

The induced vertex coloring is:
$\sigma_c(x_2) = 2$, $\sigma_c(x_4) = \sigma_c(x_7) = 3$, $\sigma_c(x_3) = \sigma_c(x_6) = 4$, $\sigma_c(x_1) = \sigma_c(x_5) = 5.$
Hence c is a twin edge 6-coloring of C_7^2 and therefore $\chi_t^{'}(C_7^2) \leq 6.$

Lemma 3.1. *Let G be a k-regular graph of odd order at least $k + 2$. If for any two nonadjacent vertices u and v, $N_G(u) \cup N_G(v) = V(G) \setminus \{u, v\}$, then $\chi_t^{'}(G) \geq k + 2$.*

Proof. Suppose $\chi_t^{'}(G) = k+1$. Then there exists a twin edge $(k+1)$-coloring $c : E(G) \to \mathbb{Z}_{k+1}$. As G is k-regular, for any vertex x, some color c_i is not represented for the edges incident at x. Since there are $k + 1$ colors and $|V(G)| > k + 1$, by pigeonhole principle, some color, say, i is not represented at two vertices. Since σ_c is equal for these two vertices, they are nonadjacent. Let the two nonadjacent vertices be u and v. By hypothesis, i must be represented at all the vertices of $V(G) \setminus \{u, v\}$. This is clearly impossible, since $|V(G) \setminus \{u, v\}|$ is odd. □

By Lemma 3.1, $\chi_t'(C_7^2) \geq 6$.
Thus we have:

Theorem 3.1. *Let $n \geq 6$. If $n \neq 7$, then $\chi_t'(C_n^2) = 5$. Also, $\chi_t'(C_7^2) = \Delta(C_7^2) + 2 = 6$.*

4. $\chi_t'(C_m \square P_n)$

Let $m \geq 3$, $n \geq 3$, $C_m := x_1x_2x_3 \ldots x_mx_1$ and $P_n := y_1y_2y_3 \ldots y_n$. For convenience, assume $x_{m+1} = x_1$.
By Observation 1.1, $\chi_t'(C_m \square P_n) \geq 5$.

Theorem 4.1. *For $m \geq 3$ and $n \geq 3$, $\chi_t'(C_m \square P_n) = 5$.*

Proof. We consider three cases and in each case, we first define $c : V(C_m \square P_n) \to \mathbb{Z}_5$.

Case 1. m is even.

$$c((x_i, y_j)(x_{i+1}, y_j)) = \begin{cases} 0 \text{ if } i \text{ is even,} \\ 1 \text{ if } i \text{ is odd;} \end{cases}$$

for $j \equiv 1 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 3 \text{ if } i \text{ is odd,} \\ 2 \text{ if } i \text{ is even;} \end{cases}$

for $j \equiv 2 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 4 \text{ if } i \text{ is odd,} \\ 3 \text{ if } i \text{ is even;} \end{cases}$

for $j \equiv 0 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 2 \text{ if } i \text{ is odd,} \\ 4 \text{ if } i \text{ is even.} \end{cases}$

Then the induced vertex coloring is:

$$\sigma_c((x_i, y_1)) = \begin{cases} 4 \text{ if } i \text{ is odd,} \\ 3 \text{ if } i \text{ is even.} \end{cases}$$

For $n \equiv 2 \pmod 3$, $\sigma_c((x_i, y_n)) = \begin{cases} 4 \text{ if } i \text{ is odd,} \\ 3 \text{ if } i \text{ is even.} \end{cases}$

For $n \equiv 0 \pmod 3$, $\sigma_c((x_i, y_n)) = \begin{cases} 0 \text{ if } i \text{ is odd,} \\ 4 \text{ if } i \text{ is even.} \end{cases}$

For $n \equiv 1 \pmod 3$, $\sigma_c((x_i, y_n)) = \begin{cases} 3 \text{ if } i \text{ is odd,} \\ 0 \text{ if } i \text{ is even.} \end{cases}$

For $j \equiv 2 \pmod 3$ and for $j \neq n$, $\sigma_c((x_i, y_j)) = \begin{cases} 3 \text{ if } i \text{ is odd,} \\ 1 \text{ if } i \text{ is even.} \end{cases}$

For $j \equiv 0 \pmod 3$ and for $j \neq n$, $\sigma_c((x_i, y_j)) = \begin{cases} 2 \text{ if } i \text{ is odd,} \\ 3 \text{ if } i \text{ is even.} \end{cases}$

For $j \equiv 1 \pmod 3$ and for $j \notin \{1, n\}$, $\sigma_c((x_i, y_j)) = \begin{cases} 1 \text{ if } i \text{ is odd,} \\ 2 \text{ if } i \text{ is even.} \end{cases}$

It can be verified that c is a twin edge 5-coloring of $C_m \square P_n$.

Case 2. m is odd and $n \not\equiv 2 \pmod 3$.

For $i \in \{1, 2, \ldots, m-2\}$, $c((x_i, y_j)(x_{i+1}, y_j)) = \begin{cases} 1 \text{ if } i \text{ is odd,} \\ 0 \text{ if } i \text{ is even;} \end{cases}$

$c((x_{m-1}, y_j)(x_m, y_j)) = 2;$

$c((x_m, y_j)(x_1, y_j)) = 0;$

for $i \in \{1, 2, \ldots, m-2\}$ and $j \equiv 1 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 2 \text{ if } i \text{ is odd,} \\ 3 \text{ if } i \text{ is even;} \end{cases}$

for $j \equiv 1 \pmod 3$, $c((x_{m-1}, y_j)(x_{m-1}, y_{j+1})) = 4$, $c((x_m, y_j)(x_m, y_{j+1})) = 3;$

for $i \in \{1, 2, \ldots, m-2\}$ and $j \equiv 2 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 4 \text{ if } i \text{ is odd,} \\ 2 \text{ if } i \text{ is even;} \end{cases}$

for $j \equiv 2 \pmod 3$, $c((x_{m-1}, y_j)(x_{m-1}, y_{j+1})) = 3$, $c((x_m, y_j)(x_m, y_{j+1})) = 1;$

for $i \in \{1, 2, \ldots, m-2\}$ and $j \equiv 0 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 3 \text{ if } i \text{ is odd,} \\ 4 \text{ if } i \text{ is even;} \end{cases}$

for $j \equiv 0 \pmod 3$, $c((x_{m-1}, y_j)(x_{m-1}, y_{j+1})) = 0$, $c((x_m, y_j)(x_m, y_{j+1})) = 4.$

Then the induced vertex coloring is:

For $i \in \{1, 2, \ldots, m-2\}$,

$$\begin{aligned}\sigma_c((x_i, y_1)) &= c((x_{i-1}, y_1)(x_i, y_1)) + c((x_i, y_1)(x_{i+1}, y_1)) + c((x_i, y_1)(x_i, y_2)) \\ &= \begin{cases} 1 + 2 = 3 \text{ if } i \text{ is odd,} \\ 1 + 3 = 4 \text{ if } i \text{ is even.} \end{cases}\end{aligned}$$

$$\begin{aligned}\sigma_c((x_{m-1}, y_1)) &= c((x_{m-2}, y_1)(x_{m-1}, y_1)) + c((x_{m-1}, y_1)(x_m, y_1)) + c((x_{m-1}, y_1)(x_{m-1}, y_2)) \\ &= 1 + 2 + 4 = 2.\end{aligned}$$

$$\begin{aligned}\sigma_c((x_m, y_1)) &= c((x_{m-1}, y_1)(x_m, y_1)) + c((x_m, y_1)(x_1, y_1)) + c((x_m, y_1)(x_m, y_2)) \\ &= 2 + 0 + 3 = 0.\end{aligned}$$

For $i \in \{1, 2, \ldots, m-2\}$,

$$\begin{aligned}\sigma_c((x_i, y_n)) &= c((x_{i-1}, y_n)(x_i, y_n)) + c((x_i, y_n)(x_{i+1}, y_n)) + c((x_i, y_{n-1})(x_i, y_n)) \\ &= 1 + c((x_i, y_{n-1})(x_i, y_n));\end{aligned}$$

for $n \equiv 0 \pmod 3$, $\sigma_c((x_i, y_n)) = \begin{cases} 1 + 4 = 0 \text{ if } i \text{ is odd,} \\ 1 + 2 = 3 \text{ if } i \text{ is even;} \end{cases}$

for $n \equiv 1 \pmod 3$, $\sigma_c((x_i, y_n)) = \begin{cases} 1 + 3 = 4 \text{ if } i \text{ is odd,} \\ 1 + 4 = 0 \text{ if } i \text{ is even;} \end{cases}$

for $n \equiv 2 \pmod 3$, $\sigma_c((x_i, y_n)) = \begin{cases} 1 + 2 = 3 \text{ if } i \text{ is odd,} \\ 1 + 3 = 4 \text{ if } i \text{ is even.} \end{cases}$

$\sigma_c((x_{m-1}, y_n))$

$= c((x_{m-2}, y_n)(x_{m-1}, y_n)) + c((x_{m-1}, y_n)(x_m, y_n)) + c((x_{m-1}, y_{n-1})(x_{m-1}, y_n))$
$= 1 + 2 + c((x_{m-1}, y_{n-1})(x_{m-1}, y_n));$
for $n \equiv 0 \pmod 3$, $\sigma_c((x_{m-1}, y_n)) = 1 + 2 + 3 = 1;$
for $n \equiv 1 \pmod 3$, $\sigma_c((x_{m-1}, y_n)) = 1 + 2 + 0 = 3;$
for $n \equiv 2 \pmod 3$, $\sigma_c((x_{m-1}, y_n)) = 1 + 2 + 4 = 2.$
$\sigma_c((x_m, y_n)) = c((x_{m-1}, y_n)(x_m, y_n)) + c((x_m, y_n)(x_1, y_n)) + c((x_{m-1}, y_n)(x_m, y_n))$
$= 2 + 0 + c((x_m, y_{n-1})(x_m, y_n));$
for $n \equiv 0 \pmod 3$, $\sigma_c((x_m, y_n)) = 2 + 0 + 1 = 3;$
for $n \equiv 1 \pmod 3$, $\sigma_c((x_m, y_n)) = 2 + 0 + 4 = 1;$
for $n \equiv 2 \pmod 3$, $\sigma_c((x_m, y_n)) = 2 + 0 + 3 = 0.$
For $i \in \{1, 2, \ldots, m-2\}$,
$\sigma_c((x_i, y_j))$
$= c((x_{i-1}, y_j)(x_i, y_j)) + c((x_i, y_j)(x_{i+1}, y_j)) + c((x_i, y_{j-1})(x_i, y_j)) + c((x_i, y_j)(x_i, y_{j+1}));$
for $j \equiv 0 \pmod 3$ and $j \neq n$, $\sigma_c((x_i, y_j)) = \begin{cases} 4+3+0+1 = 3 \text{ if } i \text{ is odd,} \\ 2+4+1+0 = 2 \text{ if } i \text{ is even;} \end{cases}$
for $j \equiv 1 \pmod 3$ and $j \notin \{1, n\}$, $\sigma_c((x_i, y_j)) = \begin{cases} 3+2+0+1 = 1 \text{ if } i \text{ is odd,} \\ 4+3+1+0 = 3 \text{ if } i \text{ is even;} \end{cases}$
for $j \equiv 2 \pmod 3$ and $j \neq n$, $\sigma_c((x_i, y_j)) = \begin{cases} 2+4+0+1 = 2 \text{ if } i \text{ is odd,} \\ 3+2+1+0 = 1 \text{ if } i \text{ is even.} \end{cases}$
$\sigma_c((x_{m-1}, y_j)) = c((x_{m-2}, y_j)(x_{m-1}, y_j)) + c((x_{m-1}, y_j)(x_m, y_j)) + c((x_{m-1}, y_{j-1})(x_{m-1}, y_j)) + c((x_{m-1}, y_j)(x_{m-1}, y_{j+1}));$
for $j \equiv 0 \pmod 3$ and $j \neq n$, $\sigma_c((x_{m-1}, y_j)) = 3 + 0 + 1 + 2 = 1;$
for $j \equiv 1 \pmod 3$ and $j \notin \{1, n\}$, $\sigma_c((x_{m-1}, y_j)) = 0 + 4 + 1 + 2 = 2;$
for $j \equiv 2 \pmod 3$ and $j \neq n$, $\sigma_c((x_{m-1}, y_j)) = 4 + 3 + 1 + 2 = 0.$
$\sigma_c((x_m, y_j))$
$= c((x_{m-1}, y_j)(x_m, y_j)) + c((x_m, y_j)(x_1, y_j)) + c((x_m, y_{j-1})(x_m, y_j)) + c((x_m, y_j)(x_m, y_{j+1}));$
for $j \equiv 0 \pmod 3$ and $j \neq n$, $\sigma_c((x_m, y_j)) = 1 + 4 + 2 + 0 = 2;$
for $j \equiv 1 \pmod 3$ and $j \notin \{1, n\}$, $\sigma_c((x_m, y_j)) = 4 + 3 + 2 + 0 = 4;$
for $j \equiv 2 \pmod 3$ and $j \neq n$, $\sigma_c((x_m, y_j)) = 3 + 1 + 2 + 0 = 1.$

It can be verified that c is a twin edge 5-coloring of $C_m \,\square\, P_n$.

Case 3. m is odd and $n \equiv 2 \pmod 3$.

For $i \in \{1, 2, \ldots, m-1\}$, $c((x_i, y_j)(x_{i+1}, y_j)) = \begin{cases} 0 \text{ if } i \text{ is odd,} \\ 1 \text{ if } i \text{ is even;} \end{cases}$
$c((x_m, y_j)(x_1, y_j)) = 2;$
for $i \in \{1, 2, \ldots, m-2\}$ and $j \equiv 1 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 3 \text{ if } i \text{ is odd,} \\ 2 \text{ if } i \text{ is even;} \end{cases}$
for $j \equiv 1 \pmod 3$, $c((x_{m-1}, y_j)(x_{m-1}, y_{j+1})) = 4$, $c((x_m, y_j)(x_m, y_{j+1})) = 0;$
for $i \in \{1, 2, \ldots, m-2\}$ and $j \equiv 2 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 4 \text{ if } i \text{ is odd,} \\ 3 \text{ if } i \text{ is even;} \end{cases}$

for $j \equiv 2 \pmod 3$, $c((x_{m-1}, y_j)(x_{m-1}, y_{j+1})) = 2$, $c((x_m, y_j)(x_m, y_{j+1})) = 3$;
for $j \equiv 0 \pmod 3$, $c((x_1, y_j)(x_1, y_{j+1})) = 1$;
for $i \in \{2, 3, \ldots, m-2\}$ and $j \equiv 0 \pmod 3$, $c((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 2 \text{ if } i \text{ is odd,} \\ 4 \text{ if } i \text{ is even;} \end{cases}$
for $j \equiv 0 \pmod 3$, $c((x_{m-1}, y_j)(x_{m-1}, y_{j+1})) = 3$, $c((x_m, y_j)(x_m, y_{j+1})) = 4$.

Then the induced vertex coloring is:
$\sigma_c((x_1, y_1)) = c((x_m, y_1)(x_1, y_1)) + c((x_1, y_1)(x_2, y_1)) + c((x_1, y_1)(x_1, y_2)) = 2 + 0 + 3 = 0$.
For $i \in \{2, 3, \ldots, m-2\}$,
$\sigma_c((x_i, y_1)) = c((x_{i-1}, y_1)(x_i, y_1)) + c((x_i, y_1)(x_{i+1}, y_1)) + c((x_i, y_1)(x_i, y_2))$
$= \begin{cases} 1 + 3 = 4 \text{ if } i \text{ is odd,} \\ 1 + 2 = 3 \text{ if } i \text{ is even.} \end{cases}$
$\sigma_c((x_{m-1}, y_1)) = c((x_{m-2}, y_1)(x_{m-1}, y_1)) + c((x_{m-1}, y_1)(x_m, y_1)) + c((x_{m-1}, y_1)(x_{m-1}, y_2))$
$= 0 + 1 + 4 = 0$.
$\sigma_c((x_m, y_1)) = c((x_{m-1}, y_1)(x_m, y_1)) + c((x_m, y_1)(x_1, y_1)) + c((x_m, y_1)(x_m, y_2))$
$= 1 + 2 + 0 = 3$.
$\sigma_c((x_1, y_n)) = c((x_m, y_n)(x_1, y_n)) + c((x_1, y_n)(x_2, y_n)) + c((x_1, y_{n-1})(x_1, y_n))$
$= 2 + 0 + 3 = 0$.
For $i \in \{2, 3, \ldots, m-2\}$,
$\sigma_c((x_i, y_n)) = c((x_{i-1}, y_n)(x_i, y_n)) + c((x_i, y_n)(x_{i+1}, y_n)) + c((x_i, y_{n-1})(x_i, y_n))$
$= \begin{cases} 1 + 3 = 4 \text{ if } i \text{ is odd,} \\ 1 + 2 = 3 \text{ if } i \text{ is even.} \end{cases}$
$\sigma_c((x_{m-1}, y_n))$
$= c((x_{m-2}, y_n)(x_{m-1}, y_n)) + c((x_{m-1}, y_n)(x_m, y_n)) + c((x_{m-1}, y_{n-1})(x_{m-1}, y_n))$
$= 0 + 1 + 4 = 0$.
$\sigma_c((x_m, y_n)) = c((x_{m-1}, y_n)(x_m, y_n)) + c((x_m, y_n)(x_1, y_n)) + c((x_m, y_{n-1})(x_m, y_n))$
$= 1 + 2 + 0 = 3$.
For $j \notin \{1, n\}$,
$\sigma_c((x_1, y_j))$
$= c((x_m, y_j)(x_1, y_j)) + c((x_1, y_j)(x_2, y_j)) + c((x_1, y_{j-1})(x_1, y_j)) + c((x_1, y_j)(x_1, y_{j+1}))$;
for $j \equiv 0 \pmod 3$, $\sigma_c((x_m, y_j)) = 2 + 0 + 4 + 1 = 2$;
for $j \equiv 1 \pmod 3$, $\sigma_c((x_m, y_j)) = 2 + 0 + 1 + 3 = 1$;
for $j \equiv 2 \pmod 3$, $\sigma_c((x_m, y_j)) = 2 + 0 + 3 + 4 = 4$.
For $i \in \{2, 3, \ldots, m-2\}$ and $j \notin \{1, n\}$,
$\sigma_c((x_i, y_j))$
$= c((x_{i-1}, y_j)(x_i, y_j)) + c((x_i, y_j)(x_{i+1}, y_j)) + c((x_i, y_{j-1})(x_i, y_j)) + c((x_i, y_j)(x_i, y_{j+1}))$;
for $j \equiv 0 \pmod 3$, $\sigma_c((x_i, y_j)) = \begin{cases} 1 + 0 + 4 + 2 = 2 \text{ if } i \text{ is odd,} \\ 0 + 1 + 3 + 4 = 3 \text{ if } i \text{ is even;} \end{cases}$
for $j \equiv 1 \pmod 3$, $\sigma_c((x_i, y_j)) = \begin{cases} 1 + 0 + 2 + 3 = 1 \text{ if } i \text{ is odd,} \\ 0 + 1 + 4 + 2 = 2 \text{ if } i \text{ is even;} \end{cases}$

for $j \equiv 2 \pmod 3$, $\sigma_c((x_i, y_j)) = \begin{cases} 1+0+3+4 = 3 \text{ if } i \text{ is odd,} \\ 0+1+2+3 = 1 \text{ if } i \text{ is even.} \end{cases}$

For $j \notin \{1, n\}$,

$\sigma_c((x_{m-1}, y_j)) = c((x_{m-2}, y_j)(x_{m-1}, y_j)) + c((x_{m-1}, y_j)(x_m, y_j)) + c((x_{m-1}, y_{j-1})(x_{m-1}, y_j)) + c((x_{m-1}, y_j)(x_{m-1}, y_{j+1}))$;

for $j \equiv 0 \pmod 3$, $\sigma_c((x_{m-1}, y_j)) = 0+1+2+3 = 1$;

for $j \equiv 1 \pmod 3$, $\sigma_c((x_{m-1}, y_j)) = 0+1+3+4 = 3$;

for $j \equiv 2 \pmod 3$, $\sigma_c((x_{m-1}, y_j)) = 0+1+4+2 = 2$.

For $j \notin \{1, n\}$,

$\sigma_c((x_m, y_j)) = c((x_{m-1}, y_j)(x_m, y_j)) + c((x_m, y_j)(x_1, y_j)) + c((x_m, y_{j-1})(x_m, y_j)) + c((x_m, y_j)(x_m, y_{j+1}))$;

for $j \equiv 0 \pmod 3$, $\sigma_c((x_m, y_j)) = 1+2+3+4 = 0$;

for $j \equiv 1 \pmod 3$, $\sigma_c((x_m, y_j)) = 1+2+4+0 = 2$;

for $j \equiv 2 \pmod 3$, $\sigma_c((x_m, y_j)) = 1+2+0+3 = 1$.

It can be verified that c is a twin edge 5-coloring of $C_m \square P_n$.

This completes the proof. □

5. $\chi_t'(G) = 2 + \Delta(G)$

By Section 1, $\chi_t'(G) = 2 + \Delta(G)$ for $G \in \{C_n : n \geq 3, n \neq 5 \text{ and } n \not\equiv 0 \pmod 3\} \cup \{K_n : n \text{ is even and } n \geq 4\} \cup \{K_{1,b} : b \equiv 1 \pmod 4 \text{ and } b \geq 2\} \cup \{K_{a,a} : a \geq 2\} \cup \{C_5 \square K_2\}$.

By Theorem 3.1, $\chi_t'(G) = 2 + \Delta(G)$ for $G = C_7^2$.

Consider $G = K_{2n+1} - E(H)$, where H is a triangle-free r-regular spanning subgraph of K_{2n+1}. It is a $(2n-r)$-regular graph on $2n+1$ vertices. For any two nonadjacent vertices u and v of G, $N_G(u) \cup N_G(v) = V(G) \setminus \{u, v\}$. Otherwise, there exist $w \notin N_G(u) \cup N_G(v)$. But then $\{u, v, w\}$ is an independent set in G, and therefore it is a triangle in H, a contradiction. So, by Lemma 3.1, $\chi_t'(G) \geq 2 + \Delta(G)$.

In particular, consider $K_9 - E(C_9)$. Let $V(K_9) = \{v_0, v_1, \ldots, v_8\}$ and $C_9 = v_0 v_1 \ldots v_8 v_0$. The table below yields a twin edge 8-coloring of it. Consequently,

$$\chi_t'(K_9 - E(C_9)) \geq 2 + \Delta(K_9 - E(C_9)).$$

Finally, we propose the following problem:

Problem 1. *If possible, find a twin edge $(2+\Delta)$-coloring of $K_{2n+1} - E(H)$, where H is a triangle-free 2-factor of K_{2n+1}.*

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_0	–	–	5	4	3	2	1	0	–
v_1	–	–	–	7	1	0	2	6	5
v_2	5	–	–	–	0	7	3	2	4
v_3	4	7	–	–	–	5	6	1	3
v_4	3	1	0	–	–	–	5	7	2
v_5	2	0	7	5	–	–	–	4	1
v_6	1	2	3	6	5	–	–	–	0
v_7	0	6	2	1	7	4	–	–	–
v_8	–	5	4	3	2	1	0	–	–

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