



A remark on the second neighborhood problem

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Abstract

Seymour's second neighborhood conjecture states that every simple digraph (without digons) has a vertex whose first out-neighborhood is at most as large as its second out-neighborhood. Such a vertex is said to have the second neighborhood property (SNP). We define "good" digraphs and prove a statement that implies that every feed vertex of a tournament has the SNP. In the case of digraphs missing a matching, we exhibit a feed vertex with the SNP by refining a proof due to Fidler and Yuster and using good digraphs. Moreover, in some cases we exhibit two vertices with SNP.

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1. Introduction

In this paper, a digraph D is a pair of two sets (V, D) , where $E \subseteq V \times V$. V and E are the vertex set and edge set of D and denoted by $V(D)$ and $E(D)$ respectively. An oriented graph is a digraph that contains neither loops nor digons. If $K \subseteq V(D)$ then the induced restriction of D to K is

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denoted by $D[K]$. As usual, $N_D^+(v)$ (resp. $N_D^-(v)$) denotes the (first) out-neighborhood (resp. in-neighborhood) of a vertex $v \in V$. $N_D^{++}(v)$ (resp. $N_D^{--}(v)$) denotes the second *out-neighborhood* (*in-neighborhood*) of v , which is the set of vertices that are at distance 2 from v (resp. to v). We also denote $d_D^+(v) = |N_D^+(v)|$, $d_D^{++}(v) = |N_D^{++}(v)|$, $d_D^-(v) = |N_D^-(v)|$ and $d_D^{--}(v) = |N_D^{--}(v)|$. We omit the subscript if the digraph is clear from the context. For short, we write $x \rightarrow y$ if the arc $(x, y) \in E$. A vertex $v \in V(D)$ is called *whole* if $d(v) := d^+(v) + d^-(v) = |V(D)| - 1$, otherwise v is non whole. A *sink* v is a vertex with $d^+(v) = 0$. For $x, y \in V(D)$, we say xy is a *missing edge* of D if neither (x, y) nor (y, x) are in $E(D)$. The *missing graph* G of D is the graph whose edges are the missing edges of D and whose vertices are the non whole vertices of D . In this case, we say that D is *missing* G . So, a tournament does not have missing edges.

A vertex v of D is said to have the *second neighborhood property* (SNP) if $d_D^+(v) \leq d_D^{++}(v)$. In 1990, Seymour conjectured the following:

Conjecture 1. (Seymour’s Second Neighborhood Conjecture (SNC))[1] *Every oriented graph has a vertex with the SNP.*

In 1996, Fisher [2] solved the SNC for tournaments by using a certain probability distribution on the vertices. Another proof of Dean’s conjecture was established in 2000 by Havet and Thomassé [3]. Their short proof uses a tool called median orders. Furthermore, they have proved that if a tournament has no sink vertex then there are at least two vertices with the SNP.

Let $D = (V, E)$ be a digraph (vertex) weighted by a positive real valued function $\omega : V \rightarrow \mathcal{R}_+$. The couple (D, ω) (or simply D) is called a *weighted digraph*. The weight of an arc $e = (x, y)$ is $\omega(e) := \omega(x) \cdot \omega(y)$. The weight of a set of vertices (resp. edges) is the sum of the weights of its members. We say that a vertex v has the *weighted SNP* if $\omega(N^+(v)) \leq \omega(N^{++}(v))$. It is known that the SNC is equivalent to its weighted version: *Every weighted oriented graph has a vertex with the weighted SNP.*

Let $L = v_1 v_2 \dots v_n$ be an ordering of the vertices of a weighted digraph (D, ω) . An arc $e = (v_i, v_j)$ is *forward* with respect to L if $i < j$. Otherwise e is a *backward* arc. A *weighted median order* $L = v_1 v_2 \dots v_n$ of D is an order of the vertices of D that maximizes the weight of the set of forward arcs of D , i.e., the set $\{(v_i, v_j) \in E(D); i < j\}$. In other words, $L = v_1 v_2 \dots v_n$ is a weighted median order of D if $\omega(L) = \max\{\omega(L'); L' \text{ is an ordering of their vertices of } D\}$. In fact, the weighted median order L satisfies the *feedback property*: For all $1 \leq i \leq j \leq n$:

$$\omega(N_{D[i,j]}^+(v_i)) \geq \omega(N_{D[i,j]}^-(v_i))$$

and

$$\omega(N_{D[i,j]}^-(v_j)) \geq \omega(N_{D[i,j]}^+(v_j))$$

where $[i, j] := \{v_i, v_{i+1}, \dots, v_j\}$.

Indeed, suppose to the contrary that $\omega(N_{D[i,j]}^+(v_i)) < \omega(N_{D[i,j]}^-(v_i))$. Consider the order $L' = v_1 \dots v_{i-1} v_{i+1} \dots v_j v_i v_{j+1} \dots v_n$ obtained from L by inserting v_i just after v_j . Then we have:

$$\begin{aligned} \omega(L') &= \omega(L) + \omega(\{(v_k, v_i) \in E(D); i \leq k \leq j\}) - \omega(\{(v_i, v_k) \in E(D); i \leq k \leq j\}) \\ &= \omega(L) + \omega(v_i) \cdot \omega(N_{D[i,j]}^-(v_i)) - \omega(v_i) \cdot \omega(N_{D[i,j]}^+(v_j)) \\ &= \omega(L) + \omega(v_i) \cdot (\omega(N_{D[i,j]}^-(v_i)) - \omega(N_{D[i,j]}^+(v_j))) > \omega(L), \end{aligned}$$

which contradicts the maximality of $\omega(L)$.

It is also known that if we reverse the orientation of a backward arc $e = (v_i, v_j)$ of D with respect to L , then L is again a weighted median order of the new weighted digraph $D' = D - (v_i, v_j) + (v_j, v_i)$.

When $\omega = 1$, we obtain the definition of median orders of a digraph ([3, 4]).

Let $L = v_1 v_2 \dots v_n$ be a weighted median order. Among the vertices not in $N^+(v_n)$ two types are distinguished: A vertex v_j is *good* if there is $i \leq j$ such that $v_n \rightarrow v_i \rightarrow v_j$, otherwise v_j is a *bad vertex*. The set of good vertices of L is denoted by G_L^D [3] (or G_L if there is no confusion). Clearly, $G_L \subseteq N^{++}(v_n)$. The last vertex v_n is called a feed vertex of (D, ω) .

In 2007, Fidler and Yuster [4] proved that SNC holds for oriented graphs missing a matching. They have used median orders and another tool called the dependency digraph. However, there proof does not guarantee that the vertex found to have the SNP is a feed vertex.

In 2012, Ghazal also used the notion of weighted median order to prove the weighted SNC for digraphs missing a generalized star. As a corollary, the weighted version holds for digraphs missing a star, complete graph or a sun [5]. He also used the dependency digraph to prove SNC for other classes of oriented graphs [6].

We say that a missing edge $x_1 y_1$ *loses to* a missing edge $x_2 y_2$ if: $x_1 \rightarrow x_2, y_2 \notin N^+(x_1) \cup N^{++}(x_1), y_1 \rightarrow y_2$ and $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$. The *dependency digraph* Δ of D is defined as follows: Its vertex set consists of all the missing edges and $(ab, cd) \in E(\Delta)$ if ab loses to cd [4, 6]. Note that Δ may contain digons.

Definition 1. [5] In a digraph D , a missing edge ab is called a *good missing edge* if:

- (i) $(\forall v \in V \setminus \{a, b\})[(v \rightarrow a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ or
- (ii) $(\forall v \in V \setminus \{a, b\})[(v \rightarrow b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$.

If ab satisfies (i) we say that (a, b) is a convenient orientation of ab .

If ab satisfies (ii) we say that (b, a) is a convenient orientation of ab .

We will need the following observation:

Lemma 1.1. [4] Let D be an oriented graph and let Δ denote its dependency digraph. A missing edge ab is good if and only if its in-degree in Δ is zero.

In the next section, we will define good median orders and good digraphs and prove a statement which implies that every feed vertex of a weighted tournament has the weighted SNP. In the last section, we refine the proof of Fidler and Yuster and use good median orders to exhibit a feed vertex with the SNP in the case of oriented graphs missing a matching.

2. Good median orders

Let D be a (weighted) digraph and let Δ denote its dependency digraph. Let C be a connected component of Δ . Set $K(C) = \{u \in V(D); \text{there is a vertex } v \text{ of } D \text{ such that } uv \text{ is a missing edge and belongs to } C\}$. The *interval graph of D* , denoted by \mathcal{I}_D is defined as follows. Its vertex set consists of the connected components of Δ and two vertices C_1 and C_2 are adjacent if $K(C_1) \cap K(C_2) \neq \emptyset$. So \mathcal{I}_D is the intersection graph of the family $\{K(C); C \text{ is a connected component of } \Delta\}$. Let ξ be a connected component of \mathcal{I}_D . We set $K(\xi) = \cup_{C \in \xi} K(C)$. Clearly, if uv is a missing edge in D then there is a unique connected component ξ of \mathcal{I}_D such that u and v belong to $K(\xi)$. For $f \in V(D)$, we set $J(f) = \{f\}$ if f is a whole vertex, otherwise $J(f) = K(\xi)$, where ξ is the unique connected component of \mathcal{I}_D such that $f \in K(\xi)$. Clearly, if $x \in J(f)$ then $J(f) = J(x)$ and if $x \notin J(f)$ then x is adjacent to every vertex in $J(f)$.

Let $L = x_1 \cdots x_n$ be a (weighted) median order of a digraph D . For $i < j$, the sets $[i, j] := [x_i, x_j] := \{x_i, x_{i+1}, \dots, x_j\}$ and $]i, j[= [i, j] \setminus \{x_i, x_j\}$ are called *intervals* of L . We recall that $K \subseteq V(D)$ is an *interval of D* if for every $u, v \in K$ we have: $N^+(u) \setminus K = N^+(v) \setminus K$ and $N^-(u) \setminus K = N^-(v) \setminus K$. The following shows a relation between the intervals of D and the intervals of L .

Proposition 2.1. *Let $\mathcal{I} = \{I_1, \dots, I_r\}$ be a set of pairwise disjoint intervals of D . Then for every weighted median order L of D , there is a weighted median order L' of D such that: L and L' have the same feed vertex and every interval in \mathcal{I} is an interval of L' .*

Proof. Let $L = x_1 x_2 \dots x_n$ be a weighted median order of a weighted digraph (D, ω) and let $\mathcal{I} = \{I_1, \dots, I_r\}$ be a set of pairwise disjoint intervals of D . We will use the feedback property to prove it. Suppose $a, b \in I_1$ with $a = x_i, b = x_j, i < j$ and $[x_i, x_j] \cap I_1 = \{x_i, x_j\}$. Since I_1 is an interval of D , we have $N^+_{]i,j[}(x_i) = N^+_{]i,j[}(x_j)$ and $N^-_{]i,j[}(x_i) = N^-_{]i,j[}(x_j)$. So, $\omega(N^-_{]i,j[}(x_i)) \leq \omega(N^+_{]i,j[}(x_i)) = N^+_{]i,j[}(x_j) \leq \omega(N^-_{]i,j[}(x_j)) = \omega(N^-_{]i,j[}(x_i))$, where the two inequalities are by the feedback property. Whence, all the quantities in the previous statement are equal. In particular, $\omega(N^+_{]i,j[}(x_i)) = \omega(N^-_{]i,j[}(x_i))$. Let L_1 be the enumeration $x_1 \dots x_{i-1} x_{i+1} \dots x_{j-1} x_i x_j x_{j+1} \dots x_n$. Then $\omega(L_1) = \omega(L) + \omega(N^-_{]i,j[}(x_i)) - \omega(N^+_{]i,j[}(x_i)) = \omega(L)$. Thus, L_1 is a weighted median order of D . By successively repeating this argument, we obtain a weighted median order in which I_1 is an interval of L . Again, by successively repeating the argument for each $I \in \mathcal{I}$, we obtain the desired order. \square

We say that D is *good digraph* if the sets $K(\xi)$'s are intervals of D . By the previous proposition, every good digraph has a (weighted) median order L such that the $K(\xi)$'s form intervals of L . Such an enumeration is called a *good (weighted) median order* of the good digraph D .

Theorem 2.1. *Let (D, ω) be a good weighted oriented graph and let L be a good weighted median order of (D, ω) , with feed vertex f . Then for every $x \in J(f)$, we have $\omega(N^+(x) \setminus J(f)) \leq \omega(G_L \setminus J(f))$. So if x has the weighted SNP in $(D[J(f)], \omega)$, then it has the weighted SNP in D .*

Proof. The proof is by induction n , the number of vertices of D . It is trivial for $n = 1$. Let $L = x_1 \dots x_n$ be a good weighted median order of (D, ω) . Set $f = x_n$, $J(f) = [x_t, x_n]$, $L_1 = x_1 \dots x_t$ and $D_1 = D[x_1, x_t]$. Then (D_1, ω) is a good weighted oriented graph and L_1 is a good weighted median order of (D_1, ω) in which $J(x_t) = \{x_t\}$. Suppose that $t < n$. Then by the induction hypothesis, $\omega(N_{D_1}^+(x_t)) \leq \omega(G_{L_1})$. However, $J(f)$ is an interval of D , then for every $x \in J(f)$, we have $\omega(N^+(x) \setminus J(f)) = \omega(N^+(x_t) \setminus J(f)) = \omega(N_{D_1}^+(x_t)) \leq \omega(G_{L_1}) = \omega(G_L \setminus J(f))$. Now suppose that $t = n$. If L does not have any bad vertex then $N^-(x_n) = G_L$. Whence, $\omega(N^+(x_n)) \leq \omega(N^-(x_n)) = \omega(G_L)$ where the inequality is by the feedback property. Now suppose that L has a bad vertex and let i be the smallest such that x_i is bad. Since $J(x_i)$ is an interval of D and L , then every vertex in $J(x_i)$ is bad and thus $J(x_i) = [x_i, x_p]$ for some $p < n$. For $j < i$, x_j is either an out-neighbor of x_n or a good vertex, by definition of i . Moreover, if $x_j \in N^+(x_n)$ then $x_j \in N^+(x_i)$. So $N^+(x_n) \cap [1, i] \subseteq N^+(x_i) \cap [1, i]$. Equivalently, $N^-(x_i) \cap [1, i] \subseteq G_L \cap [1, i]$. Therefore, $\omega(N^+(x_n) \cap [1, i]) \leq \omega(N^+(x_i) \cap [1, i]) \leq \omega(N^-(x_i) \cap [1, i]) \leq \omega(G_L \cap [1, i])$, where the second inequality is by the feedback property. Now $L' = x_{p+1} \dots x_n$ is good also. By induction, $\omega(N^+(x_n) \cap [p+1, n]) \leq \omega(G_{L'})$. Note that $G_{L'} \subseteq G_L \cap [p+1, n]$. Whence $\omega(N^+(x_n)) = \omega(N^+(x_n) \cap [1, i]) + \omega(N^+(x_n) \cap [p+1, n]) \leq \omega(G_L \cap [1, i]) + \omega(G_L \cap [p+1, n]) = \omega(G_L)$. The second part of the statement is obvious. \square

Since every (weighted) tournament is a good (weighted) oriented graph, we automatically obtain the second neighborhood conjecture for weighted and non-weighted tournaments respectively.

Corollary 2.1. ([4]) *Let L be a weighted median order of a weighted tournament (T, ω) with feed vertex f . Then $\omega(N^+(f)) \leq \omega(G_L)$.*

Corollary 2.2. ([3]) *Let L be a median order of a tournament with feed vertex f . Then $|N^+(f)| \leq |G_L|$.*

Let L be a good weighted median order of a good oriented graph D and let f denote its feed vertex. By theorem 2.1, for every $x \in J(f)$, $\omega(N^+(x) \setminus J(f)) \leq \omega(G_L \setminus J(f))$. Let b_1, \dots, b_r denote the bad vertices of L not in $J(f)$ and v_1, \dots, v_s denote the non bad vertices of L not in $J(f)$, both enumerated in increasing order with respect to their index in L . If $\omega(N^+(f) \setminus J(f)) < \omega(G_L \setminus J(f))$, we set $Sed(L) = L$. If $\omega(N^+(f) \setminus J(f)) = \omega(G_L \setminus J(f))$, we set $sed(L) = b_1 \dots b_r J(f) v_1 \dots v_s$. This new order is called the *sedimentation* of L .

Lemma 2.1. *Let L be a good weighted median order of a good weighted oriented graph (D, ω) . Then $Sed(L)$ is a good weighted median order of (D, ω) .*

Proof. Let $L = x_1 \dots x_n$ be a good weighted local median order of (D, ω) . If $Sed(L) = L$, there is nothing to prove. Otherwise, we may assume that $\omega(N^+(x_n) \setminus J(x_n)) = \omega(G_L \setminus J(x_n))$. The proof is by induction on r the number of bad vertices not in $J(x_n)$. Set $J(x_n) = [x_t, x_n]$. If $r = 0$ then we have $N^-(x_n) \setminus J(x_n) = G_L \setminus J(x_n)$. Whence, $\omega(N^+(x_n) \setminus J(x_n)) = \omega(G_L \setminus J(x_n)) =$

$\omega(N^-(x_n) \setminus J(x_n))$. Thus, $Sed(L) = J(x_n)x_1 \dots x_{t-1}$ is a good weighted median order. Now suppose that $r > 0$ and let i be the smallest such that $x_i \notin J(x_n)$ and is bad. As in the proof of theorem 2.1, $J(x_i) = [x_i, x_p]$ for some $p < n$, $\omega(N^+(x_n) \cap [1, i]) \leq \omega(N^+(x_i) \cap [1, i]) \leq \omega(N^-(x_i) \cap [1, i]) \leq \omega(G_L \cap [1, i])$ and $\omega(N^+(x_n) \cap [p+1, t-1]) \leq \omega(G_L \cap [p+1, t-1])$. However, $\omega(N^+(x_n) \setminus J(x_n)) = \omega(G_L \setminus J(x_n))$, then the previous inequalities are equalities. In particular, $\omega(N^+(x_i) \cap [1, i]) = \omega(N^-(x_i) \cap [1, i])$. Since $J(x_i)$ is an interval of L and D , then for every $x \in J(x_i)$ we have $\omega(N^+(x) \cap [1, i]) = \omega(N^-(x) \cap [1, i])$. Thus $J(x_i)x_1 \dots x_{i-1}x_{p+1} \dots x_n$ is a good weighted median order. To conclude, apply the induction hypothesis to the good weighted median order $x_1 \dots x_{i-1}x_{p+1} \dots x_n$.

□

Define now inductively $Sed^0(L) = L$ and $Sed^{q+1}(L) = Sed(Sed^q(L))$. If the process reaches a rank q such that $Sed^q(L) = y_1 \dots y_n$ and $\omega(N^+(y_n) \setminus J(y_n)) < \omega(G_{Sed^q(L)} \setminus J(y_n))$, call the order L stable. Otherwise call L periodic. These new order are used by Havet and Thomassé to exhibit a second vertex with the SNP in tournaments that do not have any sink. We will use them for the same purpose but for other classes of oriented graphs.

3. Case of oriented graph missing a matching

In this section, D is an oriented graph missing a matching and Δ denotes its dependency digraph. We begin by the following lemma:

Lemma 3.1. [4] *The maximum out-degree of Δ is one and the maximum in-degree of Δ is one. Thus Δ is composed of vertex disjoint directed paths and directed cycles.*

Proof. Assume that a_1b_1 loses to a_2b_2 and a_1b_1 loses to $a'_2b'_2$, with $a_1 \rightarrow a_2$ and $a_1 \rightarrow a'_2$. The edge a'_2b_2 is not a missing edge of D . If $a'_2 \rightarrow b_2$ then $b_1 \rightarrow a'_2 \rightarrow b_2$, a contradiction. If $b_2 \rightarrow a'_2$ then $b_1 \rightarrow b_2 \rightarrow a'_2$, a contradiction. Thus, the maximum out-degree of Δ is one. Similarly, the maximum in-degree is one. □

In the following, $C = a_1b_1, \dots, a_kb_k$ denotes a directed cycle of Δ , namely $a_i \rightarrow a_{i+1}, b_{i+1} \notin N^{++}(a_i) \cup N^+(a_i), b_i \rightarrow b_{i+1}$ and $a_{i+1} \notin N^{++}(b_i) \cup N^+(b_i)$, for all $i < k$. In [4], it is proved that $D[K(C)]$ has a vertex with the SNP. Here we prove that every vertex of $K(C)$ has the SNP in $D[K(C)]$.

Lemma 3.2. ([4]) *If k is odd then $a_k \rightarrow a_1, b_1 \notin N^{++}(a_k) \cup N^+(a_k), b_k \rightarrow b_1$ and $a_1 \notin N^{++}(b_k) \cup N^+(b_k)$. If k is even then $a_k \rightarrow b_1, a_1 \notin N^{++}(a_k) \cup N^+(a_k), b_k \rightarrow a_1$ and $b_1 \notin N^{++}(b_k) \cup N^+(b_k)$.*

Lemma 3.3. [4] *$K(C)$ is an interval of D .*

Proof. Let $f \notin K(C)$. Then f is adjacent to every vertex in $K(C)$. If $a_1 \rightarrow f$ then $b_2 \rightarrow f$, since otherwise $b_2 \in N^{++}(a_1) \cup N^+(a_1)$ which is a contradiction. So $N^+(a_1) \setminus K(C) \subseteq N^+(b_2) \setminus K(C)$. Applying this to every losing relation of C yields $N^+(a_1) \setminus K(C) \subseteq N^+(b_2) \setminus K(C) \subseteq N^+(a_3) \setminus K(C) \dots \subseteq N^+(b_k) \setminus K(C) \subseteq N^+(b_1) \setminus K(C) \subseteq N^+(a_2) \setminus K(C) \dots \subseteq N^+(a_k) \setminus K(C) \subseteq N^+(a_1) \setminus K(C)$ if k is even. So these inclusion are equalities. An analogous argument proves the same result for odd cycles. □

Lemma 3.4. In $D[K(C)]$ we have:

If k is odd then:

$$N^+(a_1) = N^-(b_1) = \{a_2, b_3, \dots, a_{k-1}, b_k\}$$

$$N^-(a_1) = N^+(b_1) = \{b_2, a_3, \dots, b_{k-1}, a_k\},$$

If k is even then:

$$N^+(a_1) = N^-(b_1) = \{a_2, b_3, \dots, b_{k-1}, a_k\}$$

$$N^-(a_1) = N^+(b_1) = \{b_2, a_3, \dots, a_{k-1}, b_k\}.$$

Proof. Suppose that k is odd. Set $K := K(C)$. Then $b_k \in N_{D[K]}^+(a_1)$ by lemma 3.2. Since $a_{k-1}b_{k-1}$ loses to a_k, b_k and $(a_1, b_k) \in E(D)$ then $(a_1, a_{k-1}) \in E(D)$ and so $a_{k-1} \in N_{D[K]}^+(a_1)$, since otherwise $(a_{k-1}, a_1) \in E(D)$ and so $b_k \in N_{D[K]}^{++}(a_{k-1})$, which is a contradiction to the definition of the losing relation $a_{k-1}b_{k-1} \rightarrow a_k b_k$. And so on $b_{k-2}, a_{k-3}, \dots, b_3, a_2 \in N_{D[K]}^+(a_1)$. Again, since a_1b_1 loses to a_2, b_2 then $b_2 \in N_{D[K]}^-(a_1)$. Since a_2b_2 loses to a_3, b_3 and $(b_2, a_1) \in E(D)$ then $(a_3, a_1) \in E(D)$ and so $a_3 \in N_{D[K(C)]}^-(a_1)$. And so on, $b_4, a_5, \dots, b_{k-1}, a_k \in N_{D[K]}^-(a_1)$. We use the same argument for finding $N_{D[K]}^+(b_1)$ and $N_{D[K]}^-(b_1)$. Also we use the same argument when k is even. \square

Lemma 3.5. In $D[K(C)]$ we have: $N^+(a_i) = N^-(b_i)$, $N^-(a_i) = N^+(b_i)$, $N^{++}(a_i) = N^-(a_i) \cup \{b_i\} \setminus \{b_{i+1}\}$ and $N^{++}(b_i) = N^-(b_i) \cup \{a_i\} \setminus \{a_{i+1}\}$ for all $i = 1, \dots, k$ where $a_{k+1} := a_1$, $b_{k+1} := b_1$ if k is odd and $a_{k+1} := b_1$, $b_{k+1} := a_1$ if k is even.

Proof. The first part is due to the previous lemma and the symmetry in these cycles. For the second part it is enough to prove it for $i = 1$ and a_1 . Suppose first that k is odd. By definition of losing relation between a_1b_1 and a_2b_2 we have $b_2 \notin N^{++}(a_1) \cup N^+(a_1)$. Moreover $a_1 \rightarrow a_2 \rightarrow b_1$, whence $b_1 \in N^{++}(a_1)$. Note that for $i = 1, \dots, k - 1$, $a_i \rightarrow a_{i+1}$ and $b_i \rightarrow b_{i+1}$. Combining this with the previous lemma we find that $N^{++}(a_1) = N^-(a_1) \cup \{b_1\} \setminus \{b_2\}$. Similar argument is used when k is even. \square

So we have:

Lemma 3.6. $d^{++}(v) = d^+(v) = d^-(v) = k - 1$ for all $v \in K(C)$.

Let $P = a_1b_1, a_2b_2, \dots, a_k b_k$ be a connected component of Δ , which is also a maximal path in Δ , namely $a_i \rightarrow a_{i+1}, b_i \rightarrow b_{i+1}$ for $i = 1, \dots, k - 1$. Since a_1b_1 is a good edge then (a_1, b_1) or (b_1, a_1) is a convenient orientation. If (a_1, b_1) is a convenient orientation, then we orient (a_i, b_i) for $i = 1, \dots, k$. Otherwise, we orient $a_i b_i$ as (b_i, a_i) . We do this for every such a path of Δ . Denote the set of these new arcs by F . Set $D' = D + F$.

Since we have oriented all the missing edges of D that form the connected components of Δ that are paths, then they are no longer missing edges of D' and thus, the dependency digraph of D' is composed of only directed cycles. Then by lemma 3.3 we have:

Lemma 3.7. D' is a good digraph.

Now, we are ready to prove the following statement:

Theorem 3.1. Every feed vertex of D' has the SNP in D and D' .

Proof. Let L be a good median order of D' and let f denote its feed vertex. We have $|N_{D'}^+(f) \setminus J(f)| \leq |G_L^{D'} \setminus J(f)|$ by theorem 2.1.

Suppose that f is not incident to any new arc of F . Then $J(f) = \{f\}$ or $J(f) = K(C)$ (in D and D') for some cycle C of Δ , $N_{D'}^+(f) = N^+(f)$ and f has the SNP in $D[J(f)]$. Let $y \in N_{D'}^{++}(f) \setminus J(f)$. There is a vertex x such that $f \rightarrow x \rightarrow y \rightarrow f$ in D' . Note that the arcs (f, x) and (y, f) are in D . If $(x, y) \in D$ or is a convenient orientation then $y \in N^{++}(f)$. Otherwise, there is a missing edge rs that loses to xy , namely $s \rightarrow y$ and $x \notin N^{++}(s) \cup N^+(s)$. But fs is not a missing edge then we must have $(f, s) \in D$. Thus $y \in N^{++}(f)$. Hence $N_{D'}^{++}(f) \setminus J(f) \subseteq N^{++}(f) \setminus J(f)$. Thus $|N^+(f)| = |N_{D'}^+(f)| = |N_{D'}^+(f) \setminus J(f)| + |N_{D'}^+(f) \cap J(f)| \leq |G_L^{D'} \setminus J(f)| + |N_{D'}^{++}(f) \cap J(f)| \leq |N^{++}(f) \setminus J(f)| + |N_{D[J(f)]}^{++}(f)| = |N^{++}(f)|$.

Suppose that F is incident to a new arc of F . Then there is a path $P = a_1b_1, a_2b_2, \dots, a_kb_k$ in Δ , which is also a connected component Δ , namely $a_t \rightarrow a_{t+1}, b_t \rightarrow b_{t+1}$ for $t = 1, \dots, k-1$, such that $f = a_i$ or $f = b_i$. We may suppose without loss of generality that $(a_t, b_t) \in D', \forall t \in \{1, \dots, k\}$. Suppose first that $f = a_i$ and $i < k$. Then f gains only b_i as a first out-neighbor and b_{i+1} as a second out-neighbor. Indeed, let $y \in N_{D'}^{++}(f) \setminus \{b_{i+1}\}$. There is a vertex x such that $f \rightarrow x \rightarrow y \rightarrow f$ in D' . Suppose that $b_i \neq x$. Note that the arcs (f, x) and (y, f) are in D . If $(x, y) \in D$ or is a convenient orientation then $y \in N^{++}(f)$. Otherwise, there is a missing edge rs that loses to xy , namely $s \rightarrow y$ and $x \notin N^{++}(s) \cup N^+(s)$. But fs is not a missing edge then we must have $(f, s) \in D$. Thus $y \in N^{++}(f)$. Suppose that $b_i = x$. Since $b_i \rightarrow y, a_{i+1} \notin N^{++}(b_i) \cup N^+(b_i)$ and $a_{i+1}y$ is not a missing edge, then we must have $(y, a_{i+1}) \in D$. Thus $f \rightarrow a_{i+1} \rightarrow y$ in D and $y \in N^{++}(f)$. Hence $N_{D'}^{++}(f) \setminus \{b_{i+1}\} \subseteq N^{++}(f)$. Note that $J(f) = \{f\}$ in D' . Combining this with theorem 2.1, we get $|N^+(f)| = |N_{D'}^+(f)| - 1 \leq |N_{D'}^{++}(f)| - 1 \leq |N^{++}(f)|$. Now suppose that $f = a_k$. We reorient the missing edge a_kb_k as (b_k, a_k) and let D'' denote the new oriented graph. Then L is a good median order of the good oriented graph D'' , $N_{D''}^+(f) = N^+(f)$, $J(f) = \{f\}$ in D'' , and f has the SNP in D'' . Let $y \in N_{D''}^{++}(f)$. There is a vertex x such that $f \rightarrow x \rightarrow y \rightarrow f$ in D'' . Note that the arcs (f, x) and (y, f) are in D . If $(x, y) \in D$ or is a convenient orientation then $y \in N^{++}(f)$. Otherwise, there is a missing edge rs that loses to xy , namely $s \rightarrow y$ and $x \notin N^{++}(s) \cup N^+(s)$. But fs is not a missing edge then we must have $(f, s) \in D$. Thus $y \in N^{++}(f)$ and $N_{D''}^{++}(f) \subseteq N^{++}(f)$. Thus f has the SNP in D . Finally, suppose that $f = b_i$. We use the same argument of the case $f = a_k$ to prove that f has the SNP in D . □

We note that our method guarantees that the vertex f found with the SNP is a feed vertex of some digraph containing D . This is not guaranteed by the proof presented in [4]. Recall that F is the set of the new arcs added to D to obtain the good oriented graph D' . So if $F = \phi$ then D is a good oriented graph.

Theorem 3.2. *Let D be an oriented graph missing a matching and suppose that $F = \phi$. If D has no sink vertex then it has at least two vertices with the SNP.*

Proof. Consider a good median order $L = x_1 \dots x_n$ of D . If $J(x_n) = K(C)$ for some directed cycle C of Δ then by theorem 2.1 and lemma 3.6 the result holds. Otherwise, x_n is a whole vertex (i.e. $J(x_n) = \{x_n\}$). By theorem 2.1, x_n has the SNP in D . So we need to find another vertex with SNP. Consider the good median order $L' = x_1 \dots x_{n-1}$. Suppose first that L' is stable. There is q for which $Sed^q(L') = y_1 \dots y_{n-1}$ and $|N^+(y_{n-1}) \setminus J(y_{n-1})| < |G_{Sed^q(L')} \setminus J(y_{n-1})|$. Note that $y_1 \dots y_{n-1} x_n$ is also a good median order of D . By theorem 2.1 and lemma 3.6, $y := y_{n-1}$ has the SNP in $D[y_1, y_{n-1}]$. So $|N^+(y)| = |N_{D[y_1, y_{n-1}]}^+(y)| + 1 \leq |G_{Sed^q(L')}| \leq |N^{++}(y)|$. Now suppose that L' is periodic. Since D has no sink then x_n has an out-neighbor x_j . Choose j to be the greatest (so that it is the last vertex of its corresponding interval). Note that for every q , x_n is an out-neighbor of the feed vertex of $Sed^q(L')$. So x_j is not the feed vertex of any $Sed^q(L')$. Since L' is periodic, x_j must be a bad vertex of $Sed^q(L')$ for some integer q , otherwise the index of x_j would always increase during the sedimentation process. Let q be such an integer. Set $Sed^q(L') = y_1 \dots y_{n-1}$. Lemma 3.6 and theorem 2.1 guarantee that the vertex $y := y_{n-1}$ with the SNP in $D[y_1, y_{n-1}]$. Note that $y \rightarrow x_n \rightarrow x_j$ and $G_{Sed^q(L')} \cup \{x_j\} \subseteq N^{++}(y)$. So $|N^+(y)| = |N_{D[y_1, y_{n-1}]}^+(y)| + 1 = |G_{Sed^q(L')}| + 1 = |G_{Sed^q(L')} \cup \{x_j\}| \leq |N^{++}(y)|$.

□

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