



The signed Roman domatic number of a digraph

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Abstract

A *signed Roman dominating function* on the digraph D is a function $f : V(D) \rightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N^-[v]} f(u) \geq 1$ for every $v \in V(D)$, where $N^-[v]$ consists of v and all inner neighbors of v , and every vertex $u \in V(D)$ for which $f(u) = -1$ has an inner neighbor v for which $f(v) = 2$. A set $\{f_1, f_2, \dots, f_d\}$ of distinct signed Roman dominating functions on D with the property that $\sum_{i=1}^d f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed Roman dominating family* (of functions) on D . The maximum number of functions in a signed Roman dominating family on D is the *signed Roman domatic number* of D , denoted by $d_{sR}(D)$. In this paper we initiate the study of signed Roman domatic number in digraphs and we present some sharp bounds for $d_{sR}(D)$. In addition, we determine the signed Roman domatic number of some digraphs. Some of our results are extensions of well-known properties of the signed Roman domatic number of graphs.

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1. Introduction

In this paper we continue the study of Roman dominating functions in graphs and digraphs. Let G be a finite and simple graph with vertex set $V(G)$, and let $N_G[v] = N[v]$ be the closed neighborhood of the vertex v . A *signed Roman dominating function* (SRDF) on a graph G is defined in [1] as a function $f : V(G) \rightarrow \{-1, 1, 2\}$ such that $\sum_{x \in N[v]} f(x) \geq 1$ for each $v \in V(G)$, and every vertex $u \in V(D)$ for which $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$. The *weight* of an SRDF f is the value $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed Roman domination number* $\gamma_{sR}(G)$ of G is the minimum weight of an SRDF on G . A set $\{f_1, f_2, \dots, f_d\}$ of distinct signed Roman dominating functions on G with the property that $\sum_{i=1}^d f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed Roman dominating family* (of functions) on G . The maximum number of functions in a signed Roman dominating family (SRD family) on G is the *signed Roman domatic number* of G , denoted by $d_{sR}(G)$. This parameter was introduced and investigated in [4].

Let D be a finite and simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The *order* $|V|$ of D is denoted by $n = n(D)$, and the *size* $|A|$ is denoted by $m = m(D)$. For an arc $(x, y) \in A(D)$, the vertex y is an *out-neighbor* of x and x is an *in-neighbor* of y , we also say that x *dominates* y and y is *dominated* by x . We write $d_D^+(v) = d^+(v)$ for the *out-degree* of a vertex v and $d_D^-(v) = d^-(v)$ for its *in-degree*. The *minimum* and *maximum in-degree* are $\delta^-(D) = \delta^-$ and $\Delta^-(D) = \Delta^-$ and the *minimum* and *maximum out-degree* are $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$. The sets $N_D^-(v) = N^-(v) = \{x | (x, v) \in A(D)\}$ and $N_D^+(v) = N^+(v) = \{x | (v, x) \in A(D)\}$ are called the *in-neighborhood* and *out-neighborhood* of the vertex v . Likewise, $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X . A digraph D is *r-in-regular* when $\delta^-(D) = \Delta^-(D) = r$ and *r-out-regular* when $\delta^+(D) = \Delta^+(D) = r$. If D is *r-in-regular* and *r-out-regular*, then D is called *r-regular*. The *associated digraph* G^* of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same end as e . For a real-valued function $f : V(D) \rightarrow \mathbb{R}$, the *weight* of f is $\omega(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V(D))$. Consult Haynes, Hedetniemi and Slater [2, 3] for notation and terminology which are not defined here.

A *signed Roman dominating function* (SRDF) on a digraph D is defined in [5] as a function $f : V(D) \rightarrow \{-1, 1, 2\}$ such that $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 1$ for each $v \in V(D)$, and such that every vertex $u \in V(D)$ for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The *weight* of an SRDF f is the value $\omega(f) = \sum_{v \in V(D)} f(v)$. The *signed Roman domination number* of a digraph D , denoted by $\gamma_{sR}(D)$, equals the minimum weight of an SRDF on D . A $\gamma_{sR}(D)$ -*function* is a signed Roman dominating function of D with weight $\gamma_{sR}(D)$. A signed Roman dominating function $f : V(D) \rightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) (or (V_{-1}^f, V_1^f, V_2^f) to refer f) of $V(D)$, where $V_i = \{v \in V(D) | f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

A set $\{f_1, f_2, \dots, f_d\}$ of distinct signed Roman dominating functions on D with the property that $\sum_{i=1}^d f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed Roman dominating family* (of functions)

on D . The maximum number of functions in a signed Roman dominating family (SRD family) on D is the *signed Roman domatic number* of D , denoted by $d_{sR}(D)$. The signed Roman domatic number is well-defined and

$$d_{sR}(D) \geq 1 \tag{1}$$

for all digraphs D , since the set consisting of the SRDF with constant value 1 forms an SRD family on D .

Our purpose in this paper is to initiate the study of signed Roman domatic number in digraphs. We study basic properties and bounds for the signed Roman domatic number of a digraph. In addition, we determine the signed Roman domatic number of some classes of digraphs. Some of our results are extensions of well-known properties of the signed Roman domatic number $d_{sR}(G)$ of graphs G .

We make use of the following results in this paper.

Proposition A. ([4]) If K_n is the complete graph of order $n \geq 1$, then $d_{sR}(K_n) = n$, unless $n = 3$ in which case $d_{sR}(K_n) = 1$.

Proposition B. ([5]) If K_n^* is the complete digraph of order $n \geq 1$, then $\gamma_{sR}(K_n^*) = 1$, unless $n = 3$ in which case $\gamma_{sR}(K_n^*) = 2$.

Proposition C. ([5]) Let D be a digraph of order $n \geq 1$. Then $\gamma_{sR}(D) \leq n$, with equality if and only if D is the disjoint union of isolated vertices and oriented triangles C_3 .

Proposition D. ([5]) If D is an r -out-regular digraph of order n with $r \geq 1$, then $\gamma_{sR}(D) \geq n/(r + 1)$.

Proposition E. ([5]) Let C_n be an oriented cycle of order $n \geq 2$. Then $\gamma_{sR}(C_n) = n/2$ when n is even and $\gamma_{sR}(C_n) = (n + 3)/2$ when n is odd.

Proposition F. ([5]) Let $K_{p,p}^*$ be the complete bipartite digraph of order $n = 2p \geq 2$. Then $\gamma_{sR}(K_{1,1}^*) = 1$, $\gamma_{sR}(K_{2,2}^*) = 3$ and $\gamma_{sR}(K_{p,p}^*) = 4$ when $p \geq 3$.

Since $N_{G^*}^-[v] = N_G[v]$ for each $v \in V(G) = V(G^*)$, the following useful observation is valid.

Observation 1.1. If G^* is the associated digraph of a graph G , then $\gamma_{sR}(G^*) = \gamma_{sR}(G)$ and $d_{sR}(G^*) = d_{sR}(G)$.

Using Observation 1.1 and Proposition A, we obtain the signed Roman domatic number of complete digraphs.

Corollary 1.1. If K_n^* is the complete digraph of order $n \geq 1$, then $d_{sR}(K_n^*) = n$, unless $n = 3$ in which case $d_{sR}(K_n^*) = 1$.

2. Properties of the signed Roman domatic number

In this section we present basic properties of $d_{sR}(D)$ and sharp bounds on the signed Roman domatic number of a digraph.

Theorem 2.1. For every digraph D ,

$$d_{sR}(D) \leq \delta^-(D) + 1.$$

Moreover, if $d_{sR}(D) = \delta^-(D) + 1$, then for each SRD family $\{f_1, f_2, \dots, f_d\}$ on D with $d = d_{sR}(D)$ and each vertex v of minimum in-degree, $\sum_{u \in N^-[v]} f_i(u) = 1$ for each function f_i and $\sum_{i=1}^d f_i(u) = 1$ for all $u \in N^-[v]$.

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be an SRD family on D such that $d = d_{sR}(D)$. Assume that v is a vertex of minimum in-degree $\delta^-(D)$. It is easy to see that

$$d \leq \sum_{i=1}^d \sum_{u \in N^-[v]} f_i(u) = \sum_{u \in N^-[v]} \sum_{i=1}^d f_i(u) \leq \sum_{u \in N^-[v]} 1 = \delta^-(D) + 1.$$

Thus $d_{sR}(D) \leq \delta^-(D) + 1$.

If $d_{sR}(D) = \delta^-(D) + 1$, then the two inequalities occurring in the proof become equalities. Hence for the SRD family $\{f_1, f_2, \dots, f_d\}$ on D and for each vertex v of minimum in-degree, $\sum_{u \in N^-[v]} f_i(u) = 1$ for each function f_i and $\sum_{i=1}^d f_i(u) = 1$ for all $u \in N^-[v]$. \square

Inequality (1) and Theorem 2.1 imply the next result immediately.

Corollary 2.1. If D consists of isolated vertices or D is an oriented path, then $d_{sR}(D) = 1$.

A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. The set of leaves incident to a support vertex v is denoted by L_v .

Proposition 2.1. If G has a support vertex v of degree at least two with $|L_v| \geq (2 \deg(v) + 2)/3$, then $d_{sR}(G^*) = 1$.

Proof. It follows from Theorem 2.1 that $d_{sR}(G^*) \leq 2$. Suppose to the contrary that $d_{sR}(G^*) = 2$ and assume that $\{f_1, f_2\}$ is an SRD family on G^* . Let $L_v = \{u_1, \dots, u_k\}$. Theorem 2.1 implies that $f_1(v) + f_2(v) = 1$. Since $f_j(x) \in \{-1, 1, 2\}$ for each j and each vertex x , we deduce that $f_1(v) = -1$ and $f_2(v) = 2$ or $f_1(v) = 2$ and $f_2(v) = -1$. Assume, without loss of generality, that $f_1(v) = -1$ and $f_2(v) = 2$. By Theorem 2.1, we must have $f_2(u_i) + f_2(v) = 1$ for each $1 \leq i \leq k$ and therefore $f_2(u_i) = -1$ for each $1 \leq i \leq k$. Since $|L_v| \geq (2 \deg(v) + 2)/3$, we obtain the contradiction $1 \leq \sum_{x \in N^-[v]} f_2(x) \leq 0$. Thus $d_{sR}(G^*) = 1$. \square

Corollary 2.2. For $n \geq 2$, $d_{sR}(K_{1,n}^*) = 1$.

Theorem 2.2. If D is a digraph of order n , then

$$\gamma_{sR}(D) \cdot d_{sR}(D) \leq n.$$

Moreover, if $\gamma_{sR}(D) \cdot d_{sR}(D) = n$, then for each SRD family $\{f_1, f_2, \dots, f_d\}$ on D with $d = d_{sR}(D)$, each function f_i is a $\gamma_{sR}(D)$ -function and $\sum_{i=1}^d f_i(v) = 1$ for all $v \in V(D)$.

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be an SRD family on D such that $d = d_{sR}(D)$ and let $v \in V(D)$. Then

$$d \cdot \gamma_{sR}(D) = \sum_{i=1}^d \gamma_{sR}(D) \leq \sum_{i=1}^d \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^d f_i(v) \leq \sum_{v \in V(D)} 1 = n.$$

If $\gamma_{sR}(D) \cdot d_{sR}(D) = n$, then the two inequalities occurring in the proof become equalities. Hence for the SRD family $\{f_1, f_2, \dots, f_d\}$ on D and for each i , $\sum_{v \in V(D)} f_i(v) = \gamma_{sR}(D)$. Thus each function f_i is a $\gamma_{sR}(D)$ -function, and $\sum_{i=1}^d f_i(v) = 1$ for all $v \in V(D)$. \square

The next result follows immediately from Theorem 2.2 and Proposition C, and it demonstrates that Theorem 2.2 is sharp.

Corollary 2.3. Let D be a digraph of order $n \geq 1$. Then $\gamma_{sR}(D) = n$ and $d_{sR}(D) = 1$ if and only if D consists of the disjoint union of isolated vertices and oriented triangles C_3 .

Applying Proposition E and Theorems 2.1 and 2.2, we obtain the signed Roman domatic number for oriented cycles.

Corollary 2.4. Let C_n be an oriented cycle of length $n \geq 2$. Then $d_{sR}(C_n) = 1$ when n is odd and $d_{sR}(C_n) = 2$ when n is even.

Proof. First let n be odd. Using Proposition E and Theorem 2.2, we deduce that

$$d_{sR}(C_n) \leq \frac{n}{\gamma_{sR}(C_n)} = \frac{2n}{n+3} < 2$$

and thus $d_{sR}(C_n) = 1$.

Now let $n = 2p$ be even, and let $C_n = u_1v_1u_2v_2 \dots u_pv_pu_1$. Define the functions $f_i : V(C_n) \rightarrow \{-1, 1, 2\}$ by $f_1(u_i) = -1$ and $f_1(v_i) = 2$ and $f_2(u_i) = 2$ and $f_2(v_i) = -1$ for $1 \leq i \leq p$. Then f_1 and f_2 are SRDF such that $f_1(x) + f_2(x) = 1$ for each vertex $x \in V(C_n)$. Therefore $d_{sR}(C_n) \geq 2$. It follows from Theorem 2.1 that $d_{sR}(C_n) \leq 2$, and so $d_{sR}(C_n) = 2$ when n is even. \square

Theorem 2.3. Let $p \geq 4$ be an even integer. Then $d_{sR}(K_{p,p}^*) = \frac{p}{2}$ when $p \neq 6$.

Proof. According to Theorem 2.2 and Proposition F, we have

$$d_{sR}(K_{p,p}^*) \leq \frac{2p}{\gamma_{sR}(K_{p,p}^*)} = \frac{p}{2}$$

for $p \geq 3$.

Assume first that $p = 6t + 4$ for an integer $t \geq 0$. Let $\{u_1, v_1, u_2, v_2, \dots, u_{3t+2}, v_{3t+2}\}$ and $\{a_1, b_1, a_2, b_2, \dots, a_{3t+2}, b_{3t+2}\}$ be the partite sets of $D = K_{p,p}^*$. For $1 \leq i \leq 3t + 2$ define the function $g_i : V(D) \rightarrow \{-1, 1, 2\}$ by

$$g_i(u_i) = g_i(v_i) = g_i(u_{i+1}) = g_i(v_{i+1}) = \dots = g_i(u_{2t+i}) = g_i(v_{2t+i}) = -1,$$

$$g_i(a_i) = g_i(b_i) = g_i(a_{i+1}) = g_i(b_{i+1}) = \dots = g_i(a_{2t+i}) = g_i(b_{2t+i}) = -1$$

and $g_i(x) = 2$ otherwise, where the indices are taken modulo $p/2 = 3t + 2$. Then g_i is an SRDF on D for $1 \leq i \leq 3t + 2$ such that $\sum_{i=1}^{3t+2} g_i(x) = 1$ for each vertex $x \in V(D)$. Therefore $\{g_1, g_2, \dots, g_{3t+2}\}$ is a signed Roman dominating family on $K_{p,p}^*$. It follows that $d_{sR}(K_{p,p}^*) \geq 3t + 2 = \frac{p}{2}$ and thus $d_{sR}(K_{p,p}^*) = \frac{p}{2}$.

Assume second that $p = 6t$ for an integer $t \geq 2$. Now let $\{u_1, v_1, u_2, v_2, \dots, u_{3t}, v_{3t}\}$ and $\{a_1, b_1, a_2, b_2, \dots, a_{3t}, b_{3t}\}$ be the partite sets of $D = K_{p,p}^*$. For $1 \leq i \leq 3t$ define the function $g_i : V(D) \rightarrow \{-1, 1, 2\}$ by

$$g_i(u_i) = g_i(v_i) = g_i(u_{i+1}) = g_i(v_{i+1}) = \dots = g_i(u_{2t-i}) = g_i(v_{2t-i}) = -1,$$

$$g_i(a_i) = g_i(b_i) = g_i(a_{i+1}) = g_i(b_{i+1}) = \dots = g_i(a_{2t-i}) = g_i(b_{2t-i}) = -1,$$

$$g_i(u_{2t+1-i}) = g_i(v_{2t+1-i}) = g_i(u_{2t+2-i}) = g_i(v_{2t+2-i}) = 1,$$

$$g_i(a_{2t+1-i}) = g_i(b_{2t+1-i}) = g_i(a_{2t+2-i}) = g_i(b_{2t+2-i}) = 1$$

and $g_i(x) = 2$ otherwise, where the indices are taken modulo $p/2 = 3t$. Then g_i is an SRDF on D for $1 \leq i \leq 3t$ such $\sum_{i=1}^{3t} g_i(x) = 1$ for each vertex $x \in V(D)$. Therefore $\{g_1, g_2, \dots, g_{3t}\}$ is a signed Roman dominating family on $K_{p,p}^*$. It follows that $d_{sR}(K_{p,p}^*) \geq 3t = \frac{p}{2}$ and thus $d_{sR}(K_{p,p}^*) = \frac{p}{2}$.

Assume third that $p = 6t + 2$ for an integer $t \geq 1$. Let $\{u_1, v_1, u_2, v_2, \dots, u_{3t+1}, v_{3t+1}\}$ and $\{a_1, b_1, a_2, b_2, \dots, a_{3t+1}, b_{3t+1}\}$ be the partite sets of $D = K_{p,p}^*$. For $1 \leq i \leq 3t + 1$ define the function $g_i : V(D) \rightarrow \{-1, 1, 2\}$ by

$$g_i(u_i) = g_i(v_i) = g_i(u_{i+1}) = g_i(v_{i+1}) = \dots = g_i(u_{2t+1-i}) = g_i(v_{2t+1-i}) = -1,$$

$$g_i(a_i) = g_i(b_i) = g_i(a_{i+1}) = g_i(b_{i+1}) = \dots = g_i(a_{2t+1-i}) = g_i(b_{2t+1-i}) = -1,$$

$$g_i(u_{2t+2-i}) = g_i(v_{2t+2-i}) = g_i(a_{2t+2-i}) = g_i(b_{2t+2-i}) = 1$$

and $g_i(x) = 2$ otherwise, where the indices are taken modulo $p/2 = 3t + 1$. Then g_i is an SRDF on D for $1 \leq i \leq 3t + 1$ such $\sum_{i=1}^{3t+1} g_i(x) = 1$ for each vertex $x \in V(D)$. Therefore $\{g_1, g_2, \dots, g_{3t+1}\}$ is a signed Roman dominating family on $K_{p,p}^*$. It follows that $d_{sR}(K_{p,p}^*) \geq 3t + 1 = \frac{p}{2}$ and thus $d_{sR}(K_{p,p}^*) = \frac{p}{2}$. \square

Theorem 2.3 is a further example which shows that Theorem 2.2 is sharp.

Theorem 2.4. If D is a digraph of order $n \geq 1$, then

$$\gamma_{sR}(D) + d_{sR}(D) \leq n + 1 \tag{2}$$

with equality if and only if $D = K_n^*$ ($n \neq 3$) or D consists of the disjoint union of isolated vertices and oriented triangles.

Proof. It follows from Theorem 2.2 that

$$\gamma_{sR}(D) + d_{sR}(D) \leq \frac{n}{d_{sR}(D)} + d_{sR}(D). \tag{3}$$

According to inequality (1) and Theorem 2.1, we have $1 \leq d_{sR}(D) \leq n$. Using these bounds, and the fact that the function $g(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$, the last inequality leads to the desired bound immediately.

If $D = K_n^*$ ($n \neq 3$) then we deduce from Proposition B and Corollary 1.1 that $\gamma_{sR}(D) + d_{sR}(D) = n + 1$. If D consists of the disjoint union of isolated vertices and oriented triangles, then it follows from Proposition C and (1) that $\gamma_{sR}(D) + d_{sR}(D) \geq n + 1$ and thus $\gamma_{sR}(D) + d_{sR}(D) = n + 1$ by (2).

Conversely, let equality hold in (2). It follows from (3) that

$$n + 1 = \gamma_{sR}(D) + d_{sR}(D) \leq \frac{n}{d_{sR}(D)} + d_{sR}(D) \leq n + 1,$$

which implies that $\gamma_{sR}(D) = n$ and $d_{sR}(D) = 1$ or $d_{sR}(D) = n$ and $\gamma_{sR}(D) = 1$. If $\gamma_{sR}(D) = n$, then Proposition C shows that D consists of the disjoint union of isolated vertices and oriented triangles. If $d_{sR}(D) = n$ and $\gamma_{sR}(D) = 1$, then Theorem 2.1 implies that $\delta^-(D) = n - 1$ and hence D is a complete digraph K_n^* . Since also $\gamma_{sR}(D) = 1$, we conclude from Proposition B that $n \neq 3$ and hence $D = K_n^*$ ($n \neq 3$). \square

The *complement* \overline{D} of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u, v the arc (u, v) belongs to \overline{D} if and only if (u, v) does not belong to D . As an application of Theorems 2.1, we will prove the following Nordhaus-Gaddum type result.

Theorem 2.5. For every digraph D of order n ,

$$d_{sR}(D) + d_{sR}(\overline{D}) \leq n + 1. \tag{4}$$

Furthermore, if $d_{sR}(D) + d_{sR}(\overline{D}) = n + 1$, then D is in-regular.

Proof. Since $\delta^-(\overline{D}) = n - 1 - \Delta^-(D)$, it follows from Theorem 2.1 that

$$\begin{aligned} d_{sR}(D) + d_{sR}(\overline{D}) &\leq (\delta^-(D) + 1) + (\delta^-(\overline{D}) + 1) \\ &= (\delta^-(D) + 1) + (n - 1 - \Delta^-(D) + 1) \leq n + 1. \end{aligned}$$

If D is not in-regular, then $\Delta^-(D) - \delta^-(D) \geq 1$, and hence the above inequality chain implies the better bound $d_{sR}(D) + d_{sR}(\overline{D}) \leq n$. \square

Using Observation 1.1, Theorems 2.1, 2.2, 2.4 or 2.5, we obtain the next known results.

Corollary 2.5. ([4]) Let G be a graph of order n . Then $d_{sR}(G) \leq \delta(G) + 1$, $\gamma_{sR}(G) \cdot d_{sR}(G) \leq n$, $\gamma_{sR}(G) + d_{sR}(G) \leq n + 1$ and $d_{sR}(G) + d_{sR}(\overline{G}) \leq n + 1$.

For some out-regular graphs we will improve the upper bound given in Theorem 2.1.

Theorem 2.6. Let D be an r -out-regular digraph of order n such that $r \geq 1$. If $n \not\equiv 0 \pmod{r + 1}$, then $d_{sR}(D) \leq r$.

Proof. Since $n \not\equiv 0 \pmod{r + 1}$, we deduce that $n = p(r + 1) + t$ with integers $p \geq 1$ and $1 \leq t \leq r$. Let $\{f_1, f_2, \dots, f_d\}$ be an SRD family on D such that $d = d_{sR}(D)$. It follows that

$$\sum_{i=1}^d \omega(f_i) = \sum_{i=1}^d \sum_{v \in V} f_i(v) = \sum_{v \in V} \sum_{i=1}^d f_i(v) \leq \sum_{v \in V} 1 = n.$$

Proposition D implies $\omega(f_i) \geq \gamma_{sR}(D) \geq p + 1$ for each $i \in \{1, 2, \dots, d\}$. If we suppose to the contrary that $d \geq r + 1$, then the above inequality chain leads to the contradiction

$$n \geq \sum_{i=1}^d \omega(f_i) \geq d(p + 1) \geq (r + 1)(p + 1) = p(r + 1) + r + 1 > n.$$

Thus $d \leq r$, and the proof is complete. □

Corollary 1.1 demonstrates that Theorem 2.6 is not valid in general when $n \equiv 0 \pmod{r + 1}$. As an application of Theorem 2.6, we improve Theorem 2.5 for r -regular digraphs.

Theorem 2.7. Let D be an r -regular digraph of order n . Then $d_{sR}(D) + d_{sR}(\overline{D}) = n + 1$ if and only if $D = K_n^*$ or $\overline{D} = K_n^*$ and $n \neq 3$.

Proof. If $n \neq 3$ and $D = K_n^*$ or $\overline{D} = K_n^*$, then Corollaries 1.1 and 2.1 lead to $d_{sR}(D) + d_{sR}(\overline{D}) = n + 1$.

Conversely, assume that $d_{sR}(D) + d_{sR}(\overline{D}) = n + 1$. Since D is r -regular, \overline{D} is $(n - 1 - r)$ -regular. If $r = 0$ or $r = n - 1$, then $D = K_n^*$ or $\overline{D} = K_n^*$, and we obtain the desired result.

Next assume that $1 \leq r \leq n - 2$ and $1 \leq \delta(\overline{D}) = n - 1 - r \leq n - 2$. We assume, without loss of generality, that $r \leq (n - 1)/2$. If $n \not\equiv 0 \pmod{r + 1}$, then it follows from Theorems 2.1 and 2.6 that

$$n + 1 = d_{sR}(D) + d_{sR}(\overline{D}) \leq \delta^-(D) + (\delta^-(\overline{D}) + 1) = r + (n - 1 - r + 1) = n,$$

a contradiction. Next assume that $n \equiv 0 \pmod{r + 1}$. Then $n = p(r + 1)$ with an integer $p \geq 2$. If $n \not\equiv 0 \pmod{n - r}$, then it follows from Theorems 2.1 and 2.6 that

$$n + 1 = d_{sR}(D) + d_{sR}(\overline{D}) \leq (r + 1) + (n - 1 - r) = n,$$

a contradiction. Therefore assume that $n \equiv 0 \pmod{n - r}$. Then $n = q(n - r)$ with an integer $q \geq 2$. Since $r \leq (n - 1)/2$, this leads to the contradiction

$$n = q(n - r) \geq q \left(n - \frac{n - 1}{2} \right) = \frac{q(n + 1)}{2} \geq n + 1,$$

and the proof is complete. □

Corollary 2.6. If T is a tournament of odd order $n \geq 3$, then $d_{sR}(T) + d_{sR}(\overline{T}) \leq n - 1$.

Proof. If T is an r -regular tournament, then \overline{T} is also an r -regular tournament such that $n = 2r + 1$. Therefore it follows from Theorem 2.6 that $d_{sR}(T) + d_{sR}(\overline{T}) \leq r + r = n - 1$.

Assume now that T is not regular. Then $\delta^-(T) \leq (n - 3)/2$ and $\delta^-(\overline{T}) \leq (n - 3)/2$, and we deduce from Theorem 2.1 that

$$d_{sR}(T) + d_{sR}(\overline{T}) \leq (\delta^-(T) + 1) + (\delta^-(\overline{T}) + 1) \leq \left(\frac{n-1}{2}\right) + \left(\frac{n-1}{2}\right) = n - 1.$$

□

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