Electronic Journal of Graph Theory and Applications **3** (1) (2015), 22–26

Electronic Journal of
Graph Theory and Applications

A new characterization of trivially perfect graphs

Christian Rubio-Montiel

Instituto de Matemáticas, Universidad Nacional Autónoma de México,
Ciudad Universitaria, 04510, D.F. - Mexico

christian@matem.unam.mx

Abstract

A graph G is *trivially perfect* if for every induced subgraph the cardinality of the largest set of pairwise nonadjacent vertices (the stability number) $\alpha(G)$ equals the number of (maximal) cliques $m(G)$. We characterize the trivially perfect graphs in terms of vertex-coloring and we extend some definitions to infinite graphs.

Keywords: Perfect graphs, complete coloring, Grundy number, forbidden graph characterization
Mathematics Subject Classification : 05C17
DOI: 10.5614/ejgta.2015.3.1.3

1. Introduction

Let G be a finite graph. A *coloring (vertex-coloring)* of G with k colors is a surjective function that assigns to each vertex of G a number from the set $\{1,\dots,k\}$. A coloring of G is called *pseudo-Grundy* if each vertex is adjacent to some vertex of each smaller color. The *pseudo-Grundy number* $\gamma(G)$ is the maximum k for which a pseudo-Grundy coloring of G exists (see [5, 6]).

A coloring of G is called *proper* if any two adjacent vertices have different color. A proper pseudo-Grundy coloring of G is called *Grundy*. The *Grundy number* $\Gamma(G)$ (also known as the first-fit chromatic number) is the maximum k for which a Grundy coloring of G exists (see [6, 11]).

Since there must be $\alpha(G)$ distinct cliques containing the members of a maximum stable set, clearly,

$$\alpha(G)\leq\theta(G)\leq m(G) \text{ and } \omega(G)\leq\chi(G)\leq\Gamma(G)\leq\gamma(G) \quad (1)$$

Received: 03 February 2014, Revised: 07 September 2014, Accepted: 01 January 2015.

where θ denotes the *clique cover* (the least number of cliques of G whose union covers $V(G)$), ω denotes the clique number and χ denotes the chromatic number. Let $a, b \in \{\alpha, \theta, m, \omega, \chi, \Gamma, \gamma\}$

α	θ	m	ω	χ	Γ	γ
2	2	3	2	2	3	3

α	θ	m	ω	χ	Γ	γ
2	2	4	2	2	2	3

Figure 1. Left; a Grundy coloring of P_4 with 3 colors. Right; a pseudo-Grundy coloring of C_4 with 3 colors.

such that $a \neq b$. A graph G is called *ab-perfect* if for every induced subgraph H of G, $a(H) = b(H)$. This definition extends the usual notion of *perfect graph* introduced by Berge [3], with this notation a perfect graph is denoted by $\omega\chi$-perfect. The concept of the ab-perfect graphs was introduced earlier by Christen and Selkow in [7] and extended in [17] and [1, 2]. A graph G without an induced subgraph H is called *H-free*. A graph H_1-free and H_2-free is called *(H_1, H_2)-free*.

Some important known results are the following: Lóvasz proved in [13] that a graph G is $\omega\chi$-perfect if and only its complement is $\omega\chi$-perfect. Consequently, a graph G is $\omega\chi$-perfect if and only if G is $\alpha\theta$-perfect, see also [4, 5, 12]. By Equation (1), a graph αm-perfect is "trivially" perfect (see [9, 10]). Chudnovsky, Robertson, Seymour and Thomas proved in [8] that a graph G is $\omega\chi$-perfect if and only if G and its complement are C_{2k+1}-free for all $k \geq 2$. Christen and Selkow proved in [7] that for any graph G the following are equivalent: G is $\omega\Gamma$-perfect, G is $\chi\Gamma$-perfect, and G is P_4-free.

The remainder of this paper is organized as follows: In Section 2: Characterizations are given of the families of finite graphs: (i) θm-perfect graphs, (ii) αm-perfect graphs (trivially perfect graphs), (iii) $\omega\gamma$-perfect graphs and (iv) $\chi\gamma$-perfect graphs. In Section 3: We further extend some definitions to locally finite graphs and denumerable graphs.

2. Characterizations for finite graphs

There exist several trivially perfect graph characterizations, e.g. [2, 9, 14, 15, 16]. We will use the following equivalence to prove Theorem 2.2:

Theorem 2.1 (Golumbic [9])**.** *A graph G is trivially perfect if and only if G is (C_4, P_4)-free.*

A consequence of Theorem 2.1 is the following characterization of θm-perfect and trivially perfect graphs.

Corollary 2.1. *A graph G is θm-perfect graph if and only if G is αm-perfect.*

Proof. Since $\theta(C_4) = \theta(P_4) = 2$, $m(C_4) = 4$ and $m(P_4) = 3$ then G is (C_4, P_4)-free, so the implication follows. For the converse, the implication is immediate from Equation (1). □

We now characterize the $\omega\gamma$-perfect and $\chi\gamma$-perfect graphs. In the following result, one should note that the finiteness of G is not necessary for the proof, the finiteness of $\omega(G)$ is sufficient.

Theorem 2.2. *For any graph G the following are equivalent: $\langle 1\rangle$ G is (C_4, P_4)-free, $\langle 2\rangle$ G is $\omega\gamma$-perfect, and $\langle 3\rangle$ G is $\chi\gamma$-perfect.*

Proof. To prove $\langle 1\rangle \Rightarrow \langle 2\rangle$ assume that G is (C_4, P_4)-free. Let ς be a pseudo-Grundy coloring of G with $\gamma(G)$ colors. We will prove by induction on n that for $n \leq \gamma(G)$, G contains a complete subgraph of n vertices with the n highest colors of ς. This proves (for $n = \gamma(G)$) that G is $\omega\gamma$-perfect since every induced subgraph of G is (C_4, P_4)-free.

For $n = 1$, there exists a vertex with color $\gamma(G)$, then the assertion is trivial. Let us now suppose that we have $n-1$ vertices $v_1, \dots, v_{n-1}$ in the $n-1$ highest colors such that they are the vertices of a complete subgraph, and define V_i as the set of vertices colored $\gamma(G)-(n-1)$ by ς adjacent to v_i ($1 \leq i < n$). Since ς is a pseudo-Grundy coloring, none V_i is empty. Any two such sets are comparable with respect to inclusion, otherwise there must be vertices p in $V_i \setminus V_j$ and q in $V_j \setminus V_i$ and the subgraph induced by $\{p, v_i, v_j, q\}$ would be isomorphic to C_4 or P_4. Therefore the $n-1$ sets V_i are linearly ordered with respect to inclusion, and there is a k ($1 \leq k < n$) with

$$V_k = \bigcap_{1 \leq i < n} V_i.$$

Thus there is a vertex v_n in V_k which is colored with $\gamma(G)-n+1$ by ς and is adjacent to each of the v_i ($1 \leq i < n$).

The proof of $\langle 2\rangle \Rightarrow \langle 3\rangle$ is immediate from Equation (1).

To prove $\langle 3\rangle \Rightarrow \langle 1\rangle$ note that if $H \in \{C_4, P_4\}$ then $\chi(H) = 2$ and $\gamma(H) = 3$ hence the implication is true (see Fig 1). □

Corollary 2.2. *Every $\chi\gamma$-perfect graph is $\omega\chi$-perfect.*

3. Extensions for infinite graphs

We presuppose here the axiom of choice. The definitions of pseudo-Grundy coloring with n colors and of proper coloring with n colors of a finite graph are generalizable to any cardinal number. It is defined the *chromatic number* χ of a graph as the smallest cardinal κ such that the graph has a proper coloring with κ colors. The *clique number* ω of a graph as the supremum of the cardinalities of the complete subgraphs of the graph (see [7]). Similarly, for any ordinal number β (such that $|\beta| = \kappa$), a *pseudo-Grundy* coloring of a graph with κ colors is a coloring of the vertices of the graph with the elements of β such that for any $\beta'' < \beta'$ and any vertex v colored β' there is a vertex colored β'' adjacent to v. The *pseudo-Grundy number* γ of a graph is the supremum of the cardinalities κ for which there is a pseudo-Grundy coloring of the graph with β such that $|\beta| = \kappa$.

Next we prove a generalization of Theorem 2.2 for some classes of infinite graphs. Afterwards we show that there exists a graph, not belonging to these classes, for which the theorem does not hold.

Theorem 3.1. *The statements $\langle 1\rangle$, $\langle 2\rangle$ and $\langle 3\rangle$ of Theorem 2.2 are equivalent for each locally finite graph and for each denumerable graph.*

Proof. To prove $\langle 1\rangle \Rightarrow \langle 2\rangle$, let H be an induced subgraph of G. If $\omega(H)$ is finite, we can use the proof of Theorem 2.2 to show that $\gamma(H) = \omega(H)$. In otherwise $\omega(H)$ is infinite, then $\gamma(H) = \omega(H)$, because $\gamma(H)$ is at most the supremum of the degrees of the vertices of H, which is at most $\aleph_0$, if G is locally finite or denumerable.

The implications $\langle 2\rangle \Rightarrow \langle 3\rangle$ and $\langle 3\rangle \Rightarrow \langle 1\rangle$ hold for any graph, finite or not. □

The following example can be found in [7]. Let G be a non-denumerable, locally denumerable graph formed by the disjoint union of $|\beta_1| = \aleph_1$ complete denumerable subgraphs of $|\beta| = \aleph_0$ vertices. Clearly $\omega(G) = \chi(G) = |\beta| = \aleph_0$, and G is (C_4, P_4)-free. But let $f\colon \beta_1 \times \beta \to \beta_1$ be such that for each $\beta' \in \beta_1$ the function $\lambda x \cdot f(\beta', x)$ is a bijection of β onto β'. Index the components of G with the denumerable ordinals, and their vertices with natural numbers. Color the n-th vertex of the β'-th component with $f(\beta', n)$. Each $\beta' < \beta_1$ is used as a color in the $(\beta'+1)$-th component. Since for each $\beta' < \beta_1$, $\lambda x \cdot f(\beta', x)$ is injective, this function defines a coloring with β_1 colors. Since $\lambda x \cdot f(\beta', x)$ is surjective for each $\beta' < \beta_1$, this function is a pseudo-Grundy coloring with $\aleph_1$ colors.

Acknowledgement

The author wishes to thank M. D. Safe and L. N. Grippo for useful discussions, D. Lara for her help and G. Araujo-Pardo for her assistance in carrying out this research.

The work was partially supported by CONACyT of Mexico, grant 166306 and 178395.

References

[1] G. Araujo-Pardo and C. Rubio-Montiel. The $\omega\psi$-perfection of graphs. *Electron. Notes Discrete Math.*, 44:163–168, 2013.

[2] G. Araujo-Pardo and C. Rubio-Montiel. On $\omega\psi$-perfect graphs. *(In review)*, 2013.

[3] C. Berge. Färbung von Graphen, deren sämtliche bzw. ungerade Kreise starr sind. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg, Math.-Natur. Reihe*, 10:114, 1961.

[4] C. Berge. *Graphs and hypergraphs*. North-Holland Publishing Co., Amsterdam, 1973.

[5] C. Berge. Perfect graphs. In *Studies in graph theory, Part I*, pages 1–22. Studies in Math., Vol. 11. Math. Assoc. Amer., Washington, D. C., 1975.

[6] G. Chartrand and P. Zhang. *Chromatic graph theory*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2009.

[7] C.A. Christen and S.M. Selkow. Some perfect coloring properties of graphs. *J. Combin. Theory Ser. B*, 27(1):49–59, 1979.

[8] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[9] M.C. Golumbic. Trivially perfect graphs. *Discrete Math.*, 24(1):105–107, 1978.

[10] M.C. Golumbic. *Algorithmic graph theory and perfect graphs*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1980.

[11] P.M. Grundy. Mathematics and games. *Eureka*, 2:6–8, 1939.

[12] L. Lovász. A characterization of perfect graphs. *J. Combinatorial Theory Ser. B*, 13:95–98, 1972.

[13] L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Math.*, 2(3):253–267, 1972.

[14] E.S. Wolk. The comparability graph of a tree. *Proc. Amer. Math. Soc.*, 13:789–795, 1962.

[15] E.S. Wolk. A note on "The comparability graph of a tree". *Proc. Amer. Math. Soc.*, 16:17–20, 1965.

[16] J. Yan, J. Chen, and G.J. Chang. Quasi-threshold graphs. *Discrete Appl. Math.*, 69(3):247–255, 1996.

[17] V. Yegnanarayanan. Graph colourings and partitions. *Theoret. Comput. Sci.*, 263(1-2):59–74, 2001.