



# Modular colorings of join of two special graphs

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## Abstract

For  $k \geq 2$ , a modular  $k$ -coloring of a graph  $G$  without isolated vertices is a coloring of the vertices of  $G$  with the elements in  $\mathbb{Z}_k$  having the property that for every two adjacent vertices of  $G$ , the sums of the colors of their neighbors are different in  $\mathbb{Z}_k$ . The minimum  $k$  for which  $G$  has a modular  $k$ -coloring is the modular chromatic number of  $G$ . In this paper, we determine the modular chromatic number of join of two special graphs.

*Keywords:* modular coloring; modular chromatic number; join of two graphs

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## 1. Introduction

For graph-theoretical terminology and notation, we in general follow [1]. For a vertex  $v$  of a graph  $G$ , let  $N_G(v)$ , the *neighborhood of  $v$* , denote the set of vertices adjacent to  $v$  in  $G$ . For a graph  $G$  without isolated vertices, let  $c : V(G) \rightarrow \mathbb{Z}_k$ ,  $k \geq 2$ , be a vertex coloring of  $G$  where adjacent vertices may be colored the same. The *color sum*  $\sigma(v) = \sum_{u \in N_G(v)} c(u)$  of a vertex  $v$  of  $G$  is the sum of the colors of the vertices in  $N_G(v)$ . The coloring  $c$  is called a *modular  $k$ -coloring* of  $G$  if  $\sigma(x) \neq \sigma(y)$  in  $\mathbb{Z}_k$  for all pairs  $x, y$  of adjacent vertices in  $G$ . The *modular chromatic number*  $mc(G)$  of  $G$  is the minimum  $k$  for which  $G$  has a modular  $k$ -coloring. This concept was introduced by Okamoto, Salehi and Zhang [2].

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Okamoto, Salehi and Zhang proved, in [2], that: every nontrivial connected graph  $G$  has a modular  $k$ -coloring for some integer  $k \geq 2$  and  $mc(G) \geq \chi(G)$ , where  $\chi(G)$  denotes the chromatic number of  $G$ ; for the cycle  $C_n$  of length  $n$ ,  $mc(C_n)$  is 2 if  $n \equiv 0 \pmod 4$  and it is 3 otherwise; every nontrivial tree has modular chromatic number 2 or 3; for the complete multipartite graph  $G$ ,  $mc(G) = \chi(G)$ ; for the cartesian product  $G = K_r \square K_2$ ,  $mc(G)$  is  $r$  if  $r \equiv 2 \pmod 4$  and it is  $r + 1$  otherwise; for the wheel  $W_n = C_n \vee K_1$ ,  $n \geq 3$ ,  $mc(W_n) = \chi(W_n)$ , where  $\vee$  denotes the join of two graphs; for  $n \geq 3$ ,  $mc(C_n \vee K_2^c) = \chi(C_n \vee K_2^c)$ , where  $G^c$  denotes the complement of  $G$ ; and for  $n \geq 2$ ,  $mc(P_n \vee K_2) = \chi(P_n \vee K_2)$ , where  $P_n$  denotes the path of length  $n - 1$ ; and in [3] that: for  $m, n \geq 2$ ,  $mc(P_m \square P_n) = 2$ .

For graphs  $G_1$  and  $G_2$ , their union  $G_1 \cup G_2$  is the graph with vertex set  $V(G_1) \cup V(G_2)$  and edge set  $E(G_1) \cup E(G_2)$ . For vertex-disjoint graphs  $G_1$  and  $G_2$ , their *join*  $G_1 \vee G_2$  is the supergraph of  $G_1 \cup G_2$  in which each vertex of  $G_1$  is adjacent to every vertex of  $G_2$  and both  $G_1$  and  $G_2$  are induced subgraphs.

In this paper, we compute the modular chromatic number of join of two special graphs.

## 2. Join of two bipartite graphs

### 2.1. Sufficient condition for $mc = 4$

Let  $i, j, k \in \mathbb{Z}_4$  with  $i \neq j$ . Set  $M_4^{i,j;k} = \{G : G \text{ is a bipartite graph such that } G \text{ has a modular 4-coloring } c \text{ with the property that for every } v \in V(G), \sigma(v) \pmod 4 \in \{i, j\} \text{ and } \sum_{v \in V(G)} c(v) \equiv k \pmod 4\}$ .

**Lemma 2.1.** *Let  $G$  and  $H$  be two vertex-disjoint nonempty bipartite graphs. If any one of the following holds, then  $mc(G \vee H) = 4$ .*

- (1)  $G \in M_4^{0,1;0}$  and  $H \in M_4^{2,3;0}$ ;
- (2)  $G \in M_4^{0,1;1}$  and  $H \in M_4^{2,3;1}$ ;
- (3)  $G \in M_4^{0,1;2}$  and  $H \in M_4^{2,3;2}$ ;
- (4)  $G \in M_4^{0,1;3}$  and  $H \in M_4^{2,3;3}$ ;
- (5)  $G \in M_4^{0,2;0}$  and  $H \in M_4^{1,3;0}$ ;
- (6)  $G \in M_4^{0,2;1}$  and  $H \in M_4^{1,3;1}$ ;
- (7)  $G \in M_4^{0,2;2}$  and  $H \in M_4^{1,3;2}$ ;
- (8)  $G \in M_4^{0,2;3}$  and  $H \in M_4^{1,3;3}$ ;
- (9)  $G \in M_4^{0,3;0}$  and  $H \in M_4^{1,2;0}$ ;
- (10)  $G \in M_4^{0,3;1}$  and  $H \in M_4^{1,2;1}$ ;
- (11)  $G \in M_4^{0,3;2}$  and  $H \in M_4^{1,2;2}$ ;
- (12)  $G \in M_4^{0,3;3}$  and  $H \in M_4^{1,2;3}$ ;
- (13)  $G \in M_4^{0,1;0}$  and  $H \in M_4^{0,1;2}$ ;
- (14)  $G \in M_4^{0,1;1}$  and  $H \in M_4^{0,1;3}$ ;
- (15)  $G \in M_4^{0,3;0}$  and  $H \in M_4^{0,3;2}$ ;
- (16)  $G \in M_4^{0,3;1}$  and  $H \in M_4^{0,3;3}$ ;
- (17)  $G \in M_4^{1,2;0}$  and  $H \in M_4^{1,2;2}$ ;
- (18)  $G \in M_4^{1,2;1}$  and  $H \in M_4^{1,2;3}$ ;
- (19)  $G \in M_4^{2,3;0}$  and  $H \in M_4^{2,3;2}$ ;

- (20)  $G \in M_4^{2,3;1}$  and  $H \in M_4^{2,3;3}$ ;
- (21)  $G \in M_4^{0,2;0}$  and  $H \in M_4^{0,2;1}$ ;
- (22)  $G \in M_4^{0,2;1}$  and  $H \in M_4^{0,2;2}$ ;
- (23)  $G \in M_4^{0,2;2}$  and  $H \in M_4^{0,2;3}$ ;
- (24)  $G \in M_4^{0,2;3}$  and  $H \in M_4^{0,2;0}$ ;
- (25)  $G \in M_4^{1,3;0}$  and  $H \in M_4^{1,3;1}$ ;
- (26)  $G \in M_4^{1,3;1}$  and  $H \in M_4^{1,3;2}$ ;
- (27)  $G \in M_4^{1,3;2}$  and  $H \in M_4^{1,3;3}$ ;
- (28)  $G \in M_4^{1,3;3}$  and  $H \in M_4^{1,3;0}$ ;
- (29)  $G \in M_4^{0,1;0}$  and  $H \in M_4^{0,3;1}$ ;
- (30)  $G \in M_4^{0,1;1}$  and  $H \in M_4^{0,3;2}$ ;
- (31)  $G \in M_4^{0,1;2}$  and  $H \in M_4^{0,3;3}$ ;
- (32)  $G \in M_4^{0,1;3}$  and  $H \in M_4^{0,3;0}$ ;
- (33)  $G \in M_4^{0,1;0}$  and  $H \in M_4^{1,2;3}$ ;
- (34)  $G \in M_4^{0,1;1}$  and  $H \in M_4^{1,2;0}$ ;
- (35)  $G \in M_4^{0,1;2}$  and  $H \in M_4^{1,2;1}$ ;
- (36)  $G \in M_4^{0,1;3}$  and  $H \in M_4^{1,2;2}$ ;
- (37)  $G \in M_4^{0,2;0}$  and  $H \in M_4^{1,3;2}$ ;
- (38)  $G \in M_4^{0,2;1}$  and  $H \in M_4^{1,3;3}$ ;
- (39)  $G \in M_4^{0,2;2}$  and  $H \in M_4^{1,3;0}$ ;
- (40)  $G \in M_4^{0,2;3}$  and  $H \in M_4^{1,3;1}$ ;
- (41)  $G \in M_4^{0,3;0}$  and  $H \in M_4^{2,3;1}$ ;
- (42)  $G \in M_4^{0,3;1}$  and  $H \in M_4^{2,3;2}$ ;
- (43)  $G \in M_4^{0,3;2}$  and  $H \in M_4^{2,3;3}$ ;
- (44)  $G \in M_4^{0,3;3}$  and  $H \in M_4^{2,3;0}$ ;
- (45)  $G \in M_4^{1,2;0}$  and  $H \in M_4^{2,3;3}$ ;
- (46)  $G \in M_4^{1,2;1}$  and  $H \in M_4^{2,3;0}$ ;
- (47)  $G \in M_4^{1,2;2}$  and  $H \in M_4^{2,3;1}$ ;
- (48)  $G \in M_4^{1,2;3}$  and  $H \in M_4^{2,3;2}$ .

*Proof.* Clearly,  $mc(G \vee H) \geq \chi(G \vee H) = \chi(G) + \chi(H) = 4$ . We prove (48) and the proofs of (1) to (47) are similar.  $G \in M_4^{1,2;3}$  implies that  $G$  has a modular 4-coloring  $c'$  such that for every  $v \in V(G)$ ,  $\sigma'(v) \pmod 4 \in \{1, 2\}$  and  $\sum_{v \in V(G)} c'(v) \equiv 3 \pmod 4$  and  $H \in M_4^{2,3;2}$  implies

that  $H$  has a modular 4-coloring  $c''$  such that for every  $v \in V(H)$ ,  $\sigma''(v) \pmod 4 \in \{2, 3\}$  and  $\sum_{v \in V(H)} c''(v) \equiv 2 \pmod 4$ . Define  $c : V(G \vee H) \rightarrow \mathbb{Z}_4$  by  $c(v) = c'(v)$  for  $v \in V(G)$  and

$c(v) = c''(v)$  for  $v \in V(H)$ . Then, for every  $v \in V(G)$ ,  $\sigma(v) \pmod 4 = 3 \Leftrightarrow \sigma'(v) \pmod 4 = 1$ , and  $\sigma(v) \pmod 4 = 0 \Leftrightarrow \sigma'(v) \pmod 4 = 2$ ; and for every  $v \in V(H)$ ,  $\sigma(v) \pmod 4 = 1 \Leftrightarrow \sigma''(v) \pmod 4 = 2$ , and  $\sigma(v) \pmod 4 = 2 \Leftrightarrow \sigma''(v) \pmod 4 = 3$ . Hence,  $c$  is a modular 4-coloring of  $G \vee H$ . Consequently,  $mc(G \vee H) \leq 4$ . □

Using Lemma 2.1, we compute  $mc(G \vee H)$  for some special graphs  $G$  and  $H$ .

2.2. Join of two paths

**Theorem 2.1.** For  $m \geq 2$  and  $n \geq 2$ ,  $mc(P_m \vee P_n) = 4$ .

*Proof.* Case 1.  $n \not\equiv 1 \pmod 4$ .

First, we claim that  $P_m \in M_4^{0,2;0} \cup M_4^{0,2;2}$ . To see this, for  $m \equiv 0 \pmod 4$ , label the vertices of  $P_m$  by  $0, 0, 2, 0, 0, 0, 2, 0, \dots, 0, 0, 2, 0$  in order; for  $m \equiv 1 \pmod 4$ , label the vertices of  $P_m$  by  $0, 0, 2, 0, 0, 0, 2, 0, \dots, 0, 0, 2, 0, 0$  in order; for  $m \equiv 2 \pmod 4$ , label the vertices of  $P_m$  by  $2, 0, 0, 0, 2, 0, 0, 0, \dots, 2, 0, 0, 0, 2, 0$  in order; and for  $m \equiv 3 \pmod 4$ , label the vertices of  $P_m$  by  $0, 0, 2, 0, 0, 0, 2, 0, \dots, 0, 0, 2, 0, 0, 0, 2$  in order.

Next, we claim that  $P_n \in M_4^{1,3;0}$ . To see this, for  $n \equiv 0 \pmod 4$ , label the vertices of  $P_n$  by  $0, 1, 3, 0, 0, 1, 3, 0, \dots, 0, 1, 3, 0$  in order; for  $n \equiv 2 \pmod 4$ , label the vertices of  $P_n$  by  $1, 3, 0, 0, 1, 3, 0, 0, \dots, 1, 3, 0, 0, 1, 3$  in order; and for  $n \equiv 3 \pmod 4$ , label the vertices of  $P_n$  by  $0, 1, 3, 0, 0, 1, 3, 0, \dots, 0, 1, 3, 0, 0, 1, 3$  in order.

Finally, apply Lemma 2.1 (5) and (39).

Now, by symmetry, assume that both  $m$  and  $n$  are  $\equiv 1 \pmod 4$ . Again, by symmetry, it is enough if we consider the following cases.

Case 2.  $m \equiv 1 \pmod{16}$  and  $n \equiv 1 \pmod{16}$ .

First, we claim that  $P_m \in M_4^{0,3;0}$ . To see this, label the vertices of  $P_m$  by  $0, 0, 3, 0, 0, 0, 3, 0, \dots, 0, 0, 3, 0, 0$  in order. Next, we claim that  $P_n \in M_4^{0,1;3}$ . To see this, label the vertices of  $P_n$  by  $3, 0, 2, 0, 3, 0, 2, 0, \dots, 3, 0, 2, 0, 3$  in order. Finally, apply Lemma 2.1 (32).

Case 3.  $m \equiv 1 \pmod{16}$  and  $n \equiv 9 \pmod{16}$ .

First, we claim that  $P_m \in M_4^{0,3;1}$ . To see this, label the vertices of  $P_m$  by  $1, 0, 2, 0, 1, 0, 2, 0, \dots, 1, 0, 2, 0, 1$  in order. Next, we claim that  $P_n \in M_4^{0,3;3}$ . To see this, label the vertices of  $P_n$  by  $1, 0, 2, 0, 1, 0, 2, 0, \dots, 1, 0, 2, 0, 1$  in order. Finally, apply Lemma 2.1 (16).

Case 4.  $m \equiv 5 \pmod{16}$  and  $n \equiv 5 \pmod{16}$ .

First, we claim that  $P_m \in M_4^{0,1;1}$ . To see this, label the vertices of  $P_m$  by  $0, 0, 1, 0, 0, 0, 1, 0, \dots, 0, 0, 1, 0, 0$  in order. Next, we claim that  $P_n \in M_4^{0,3;2}$ . To see this, label the vertices of  $P_n$  by  $3, 0, 0, 0, 3, 0, 0, 0, \dots, 3, 0, 0, 0, 3$  in order. Finally, apply Lemma 2.1 (30).

Case 5.  $m \equiv 5 \pmod{16}$  and  $n \equiv 13 \pmod{16}$ .

First, we claim that  $P_m \in M_4^{0,1;1}$ . To see this, label the vertices of  $P_m$  by  $0, 0, 1, 0, 0, 0, 1, 0, \dots, 0, 0, 1, 0, 0$  in order. Next, we claim that  $P_n \in M_4^{0,3;2}$ . To see this, label the vertices of  $P_n$  by  $1, 0, 2, 0, 1, 0, 2, 0, \dots, 1, 0, 2, 0, 1$  in order. Finally, apply Lemma 2.1 (30).

Case 6.  $m \equiv 9 \pmod{16}$  and  $n \equiv 9 \pmod{16}$ .

First, we claim that  $P_m \in M_4^{0,3;2}$ . To see this, label the vertices of  $P_m$  by  $0, 0, 3, 0, 0, 0, 3, 0, \dots, 0, 0, 3, 0, 0$  in order. Next, we claim that  $P_n \in M_4^{0,1;1}$ . To see this, label the vertices of  $P_n$  by  $3, 0, 2, 0, 3, 0, 2, 0, \dots, 3, 0, 2, 0, 3$  in order. Finally, apply Lemma 2.1 (30).

Case 7.  $m \equiv 13 \pmod{16}$  and  $n \equiv 13 \pmod{16}$ .

First, we claim that  $P_m \in M_4^{0,1;0}$ . To see this, label the vertices of  $P_m$  by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1$  in order. Next, we claim that  $P_n \in M_4^{0,3;1}$ . To see this, label the vertices of  $P_n$  by  $0, 0, 3, 0, 0, 0, 3, 0, \dots, 0, 0, 3, 0, 0$  in order. Finally, apply Lemma 2.1 (29).

Cases 8.1.  $m \equiv 1 \pmod{16}$  and  $n \equiv 5 \pmod{16}$ ;

8.2.  $m \equiv 13 \pmod{16}$  and  $n \equiv 1 \pmod{16}$ ;

8.3.  $m \equiv 5 \pmod{16}$  and  $n \equiv 9 \pmod{16}$ ;

8.4.  $m \equiv 9 \pmod{16}$  and  $n \equiv 13 \pmod{16}$ .

First, label the vertices of  $P_m$  by  $3, 0, 2, 0, 3, 0, 2, 0, \dots, 3, 0, 2, 0, 3$  in order. This shows that  $P_m \in M_4^{0,1;3}$  if  $m \equiv 1 \pmod{16}$ ,  $P_m \in M_4^{0,1;0}$  if  $m \equiv 5 \pmod{16}$ ,  $P_m \in M_4^{0,1;1}$  if  $m \equiv 9 \pmod{16}$ , and  $P_m \in M_4^{0,1;2}$  if  $m \equiv 13 \pmod{16}$ .

Next, label the vertices of  $P_n$  by  $0, 0, 1, 0, 0, 0, 1, 0, \dots, 0, 0, 1, 0, 0$  in order. This implies that  $P_n \in M_4^{0,1;0}$  if  $n \equiv 1 \pmod{16}$ ,  $P_n \in M_4^{0,1;1}$  if  $n \equiv 5 \pmod{16}$ ,  $P_n \in M_4^{0,1;2}$  if  $n \equiv 9 \pmod{16}$ , and  $P_n \in M_4^{0,1;3}$  if  $n \equiv 13 \pmod{16}$ .

If  $m \equiv 13 \pmod{16}$  and  $n \equiv 1 \pmod{16}$  or if  $m \equiv 5 \pmod{16}$  and  $n \equiv 9 \pmod{16}$ , then apply Lemma 2.1 (13). If  $m \equiv 1 \pmod{16}$  and  $n \equiv 5 \pmod{16}$ , or if  $m \equiv 9 \pmod{16}$  and  $n \equiv 13 \pmod{16}$ , then apply Lemma 2.1 (14). □

2.3. Join of a path and an even cycle

**Theorem 2.2.** For  $m \geq 2$  and  $n \geq 2$ ,  $mc(P_m \vee C_{2n}) = 4$ .

*Proof.* First, label the vertices of  $C_{2n}$ ,  $n \equiv 0 \pmod{2}$ , by  $0, 1, 3, 0, 0, 1, 3, 0, \dots, 0, 1, 3, 0$  in cyclic order. This shows that  $C_{2n} \in M_4^{1,3;0}$  if  $n \equiv 0 \pmod{2}$ .

Next, label the vertices of  $C_{2n}$ ,  $n \equiv 1 \pmod{2}$ , by  $1, 0, 1, 0, 1, 0, 1, 0, \dots, 1, 0, 1, 0, 1, 0$  in cyclic order. This shows that  $C_{2n} \in M_4^{0,2;1}$  if  $n \equiv 1 \pmod{4}$  and  $C_{2n} \in M_4^{0,2;3}$  if  $n \equiv 3 \pmod{4}$ .

Finally, for  $m \equiv 0 \pmod{4}$ , label the vertices of  $P_m$  by  $0, 0, 2, 0, 0, 0, 2, 0, \dots, 0, 0, 2, 0$  in order; for  $m \equiv 1 \pmod{4}$ , label the vertices of  $P_m$  by  $0, 0, 2, 0, 0, 0, 2, 0, \dots, 0, 0, 2, 0, 0$  in order; for  $m \equiv 2 \pmod{4}$ , label the vertices of  $P_m$  by  $2, 0, 0, 0, 2, 0, 0, 0, \dots, 2, 0, 0, 0, 2, 0$  in order; and for  $m \equiv 3 \pmod{4}$ , label the vertices of  $P_m$  by  $0, 0, 2, 0, 0, 0, 2, 0, \dots, 0, 0, 2, 0, 0, 0, 2$  in order. This shows that  $P_m \in M_4^{0,2;0} \cup M_4^{0,2;2}$ .

If  $n \equiv 0 \pmod{2}$ , then apply Lemma 2.1 (5) and (39). If  $n \equiv 1 \pmod{4}$ , then apply Lemma 2.1 (21) and (22). If  $n \equiv 3 \pmod{4}$ , then apply Lemma 2.1 (23) and (24). □

2.4. Join of a path and a complete bipartite graph

**Theorem 2.3.** For integers  $n \geq 2$ ,  $r \geq 1$ , and  $s \geq 1$ ,  $mc(P_n \vee K_{r,s}) = 4$ .

*Proof.* Let  $P_n := u_1 u_2 \dots u_n$ ,  $X = \{x_1, x_2, \dots, x_r\}$ ,  $Y = \{y_1, y_2, \dots, y_s\}$ , and the bipartition of  $K_{r,s}$  be  $(X, Y)$ . We consider four cases.

Case 1.  $n \equiv i \pmod{16}$ ,  $i \in \{1, 2, 3, 4, 5\}$ .

First, we claim that  $K_{r,s} \in M_4^{2,3;1}$ . To see this, label the vertex  $x_1$  of  $K_{r,s}$  by 2, the vertex  $y_1$  of  $K_{r,s}$  by 3 and all other vertices of  $K_{r,s}$  by 0. Next, we claim that  $P_n \in M_4^{0,1;1}$ . To see this, for  $n \equiv 1 \pmod{16}$ , label the vertices of  $P_n$  by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1$  in order; for  $n \equiv 2 \pmod{16}$ , label the vertices of  $P_n$  by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1, 0$  in order; for  $n \equiv 3 \pmod{16}$ , label the vertices of  $P_n$  by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1, 0, 0$  in order; for  $n \equiv 4 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 0, 1, 0, 0, 0, 1, 0, \dots, 0, 0, 1, 0$  in order; for  $n \equiv 5 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 0, 1, 0, 0, 0, 1, 0, \dots, 0, 0, 1, 0, 0$  in order. Finally, apply Lemma 2.1 (2).

Case 2.  $n \equiv i \pmod{16}$ ,  $i \in \{6, 7, 8\}$ .

First, we claim that  $K_{r,s} \in M_4^{0,3;3}$ . To see this, label the vertex  $y_1$  of  $K_{r,s}$  by 3 and all other vertices of  $K_{r,s}$  by 0. Next, we claim that  $P_n \in M_4^{0,1;2}$ . To see this, for  $n \equiv 6 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 1, 0, 0, 0, 1, 0, 0, \dots, 0, 1, 0, 0, 0, 1$  in order; for  $n \equiv 7 \pmod{16}$ , label the

vertices of  $P_n$  by  $0, 1, 0, 0, 0, 1, 0, 0, \dots, 0, 1, 0, 0, 0, 1, 0$  in order; for  $n \equiv 8 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 1, 0, 0, 0, 1, 0, 0, \dots, 0, 1, 0, 0$  in order. Finally, apply Lemma 2.1 (31).

Case 3.  $n \equiv i \pmod{16}, i \in \{9, 10, 11, 12, 13\}$ .

First, we claim that  $K_{r,s} \in M_4^{0,1;1}$ . To see this, label the vertex  $y_1$  of  $K_{r,s}$  by 1 and all other vertices of  $K_{r,s}$  by 0. Next, we claim that  $P_n \in M_4^{0,1;3}$ . To see this, for  $n \equiv 9 \pmod{16}$ , label the vertices of  $P_n$  by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1$  in order; for  $n \equiv 10 \pmod{16}$ , label the vertices of  $P_n$  by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1, 0$  in order; for  $n \equiv 11 \pmod{16}$ , label the vertices of  $P_n$  by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1, 0, 0$  in order; for  $n \equiv 12 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 0, 1, 0, 0, 0, 1, 0, \dots, 0, 0, 1, 0$  in order; for  $n \equiv 13 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 0, 1, 0, 0, 0, 1, 0, \dots, 0, 0, 1, 0, 0$  in order. Finally, apply Lemma 2.1 (14).

Case 4.  $n \equiv i \pmod{16}, i \in \{0, 14, 15\}$ .

First, we claim that  $K_{r,s} \in M_4^{1,2;3}$ . To see this, label the vertex  $x_1$  of  $K_{r,s}$  by 1, the vertex  $y_1$  of  $K_{r,s}$  by 2 and all other vertices of  $K_{r,s}$  by 0. Next, we claim that  $P_n \in M_4^{0,1;0}$ . To see this, for  $n \equiv 14 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 1, 0, 0, 0, 1, 0, 0, \dots, 0, 1, 0, 0, 0, 1$  in order; for  $n \equiv 15 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 1, 0, 0, 0, 1, 0, 0, \dots, 0, 1, 0, 0, 0, 1, 0$  in order; for  $n \equiv 0 \pmod{16}$ , label the vertices of  $P_n$  by  $0, 1, 0, 0, 0, 1, 0, 0, \dots, 0, 1, 0, 0$  in order. Finally, apply Lemma 2.1 (33).  $\square$

### 2.5. Join of an even cycle and a complete bipartite graph

Let  $X = \{x_1, x_2, \dots, x_r\}, Y = \{y_1, y_2, \dots, y_s\}$ , and the bipartition of  $K_{r,s}$  be  $(X, Y)$ .

**Theorem 2.4.** For integers  $n \geq 1, r \geq 1$ , and  $s \geq 1, mc(C_{4n} \vee K_{r,s}) = 4$ .

*Proof.* Let  $C_{4n} := u_1 u_2 \dots u_{4n} u_1$ . Label the vertices of  $C_{4n}$  by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0$  in cyclic order. We consider four cases.

Case 1.  $n \equiv 1 \pmod{4}$ .

$C_{4n} \in M_4^{0,1;1}$  and by the proof of Theorem 2.3,  $K_{r,s} \in M_4^{2,3;1}$ . Apply Lemma 2.1 (2).

Case 2.  $n \equiv 2 \pmod{4}$ .

$C_{4n} \in M_4^{0,1;2}$  and by the proof of Theorem 2.3,  $K_{r,s} \in M_4^{0,3;3}$ . Apply Lemma 2.1 (31).

Case 3.  $n \equiv 3 \pmod{4}$ .

$C_{4n} \in M_4^{0,1;3}$  and by the proof of Theorem 2.3,  $K_{r,s} \in M_4^{0,1;1}$ . Apply Lemma 2.1 (14).

Case 4.  $n \equiv 0 \pmod{4}$ .

$C_{4n} \in M_4^{0,1;0}$  and by the proof of Theorem 2.3,  $K_{r,s} \in M_4^{1,2;3}$ . Apply Lemma 2.1 (33).  $\square$

**Theorem 2.5.** For integers  $n \geq 1, r \geq 1$ , and  $s \geq 1, mc(C_{4n+2} \vee K_{r,s}) = 4$ .

*Proof.* Label the vertex  $x_1$  of  $K_{r,s}$  by 2 and all other vertices of  $K_{r,s}$  by 0. This shows that  $K_{r,s} \in M_4^{0,2;2}$ . Let  $C_{4n+2} := u_1 u_2 \dots u_{4n+2} u_1$ . Label the vertices of  $C_{4n+2}$  by  $1, 0, 1, 0, 1, 0, 1, 0, \dots, 1, 0, 1, 0, 1, 0$  in cyclic order.

If  $n \equiv 1 \pmod{2}$ , then  $C_{4n+2} \in M_4^{0,2;3}$ . Now apply Lemma 2.1 (23).

If  $n \equiv 0 \pmod{2}$ , then  $C_{4n+2} \in M_4^{0,2;1}$ . Now apply Lemma 2.1 (22).  $\square$

2.6. Join of two regular bipartite graphs

**Theorem 2.6.** Let  $G$  be a  $k$ -regular bipartite graph with  $k \equiv 1 \pmod{2}$  and let  $H$  be an  $\ell$ -regular bipartite graph with  $\ell \equiv 1 \pmod{2}$ . We have  $mc(G \vee H) = 4$ .

*Proof.* Let  $(X, Y)$  be the bipartition of  $G$  with  $|X| = |Y| = m$ , and let  $(U, V)$  be the bipartition of  $H$  with  $|U| = |V| = n$ . Define  $c : V(G \vee H) \rightarrow \mathbb{Z}_4$  by  $c(x) = 0$  for  $x \in X$ ,  $c(y) = 2$  for  $y \in Y$ ,  $c(u) = 1$  for  $u \in U$ ,  $c(v) = 3$  for  $v \in V$ . Then,  $\sigma(x) = (n + 3n + 2k) \pmod{4} = 2$  for  $x \in X$  and  $\sigma(y) = (n + 3n) \pmod{4} = 0$  for  $y \in Y$ . We consider 2 cases.

*Case 1.* Either  $m \equiv 0 \pmod{2}$  and  $\ell \equiv 1 \pmod{4}$  or  $m \equiv 1 \pmod{2}$  and  $\ell \equiv 3 \pmod{4}$ .

In this case,  $\sigma(u) = (2m + 3\ell) \pmod{4} = 3$  for  $u \in U$  and  $\sigma(v) = (2m + \ell) \pmod{4} = 1$  for  $v \in V$ .

*Case 2.* Either  $m \equiv 0 \pmod{2}$  and  $\ell \equiv 3 \pmod{4}$  or  $m \equiv 1 \pmod{2}$  and  $\ell \equiv 1 \pmod{4}$ .

In this case,  $\sigma(u) = (2m + 3\ell) \pmod{4} = 1$  for  $u \in U$  and  $\sigma(v) = (2m + \ell) \pmod{4} = 3$  for  $v \in V$ . □

**Theorem 2.7.** Let  $G$  be a  $k$ -regular bipartite graph with  $k \equiv 2 \pmod{4}$  and let  $H$  be an  $\ell$ -regular bipartite graph with  $\ell \equiv 1 \pmod{2}$ . We have  $mc(G \vee H) = 4$ .

*Proof.* Let  $(X, Y)$  be the bipartition of  $G$  with  $|X| = |Y| = m$ , and let  $(U, V)$  be the bipartition of  $H$  with  $|U| = |V| = n$ . We consider two cases.

*Case 1.*  $m \equiv 1 \pmod{2}$ .

Define  $c : V(G \vee H) \rightarrow \mathbb{Z}_4$  by  $c(x) = 0$  for  $x \in X$ ,  $c(y) = 1$  for  $y \in Y$ ,  $c(u) = 0$  for  $u \in U$ ,  $c(v) = 2$  for  $v \in V$ . Then,  $\sigma(x) = (2n + k) \pmod{4} = (2n + 2) \pmod{4}$  for  $x \in X$ ,  $\sigma(y) = 2n \pmod{4}$  for  $y \in Y$ ,  $\sigma(u) = (m + 2\ell) \pmod{4} = (m + 2) \pmod{4}$  for  $u \in U$  and  $\sigma(v) = m \pmod{4}$  for  $v \in V$ . Note that  $\{2n, 2n + 2\} \pmod{4} = \{0, 2\}$  and as  $m \equiv 1 \pmod{2}$ ,  $\{m, m + 2\} \pmod{4} = \{1, 3\}$ .

*Case 2.*  $m \equiv 0 \pmod{2}$ .

Define  $c : V(G \vee H) \rightarrow \mathbb{Z}_4$  by  $c(x) = 0$  for  $x \in X$ ,  $c(y) = 1$  for  $y \in Y$ ,  $c(u) = 1$  for  $u \in U$ ,  $c(v) = 3$  for  $v \in V$ . Then,  $\sigma(x) = (4n + k) \pmod{4} = 2$  for  $x \in X$ ,  $\sigma(y) = 4n \pmod{4} = 0$  for  $y \in Y$ ,  $\sigma(u) = (m + 3\ell) \pmod{4}$  for  $u \in U$  and  $\sigma(v) = (m + \ell) \pmod{4}$  for  $v \in V$ .

Note that (i) if either  $m \equiv 0 \pmod{4}$  and  $\ell \equiv 1 \pmod{4}$  or  $m \equiv 2 \pmod{4}$  and  $\ell \equiv 3 \pmod{4}$ , then  $\sigma(u) = 3$  for  $u \in U$  and  $\sigma(v) = 1$  for  $v \in V$ , (ii) if either  $m \equiv 2 \pmod{4}$  and  $\ell \equiv 1 \pmod{4}$  or  $m \equiv 0 \pmod{4}$  and  $\ell \equiv 3 \pmod{4}$ , then  $\sigma(u) = 1$  for  $u \in U$  and  $\sigma(v) = 3$  for  $v \in V$ . □

We propose:

**Problem 1.** Let  $G$  be a  $k$ -regular bipartite graph with  $k \equiv 0 \pmod{4}$  and let  $H$  be an  $\ell$ -regular bipartite graph with  $\ell \equiv 1 \pmod{2}$ . Find  $mc(G \vee H)$ .

**Problem 2.** Let  $G$  be a  $k$ -regular bipartite graph with  $k \equiv 0 \pmod{2}$  and let  $H$  be an  $\ell$ -regular bipartite graph with  $\ell \equiv 0 \pmod{2}$ . Find  $mc(G \vee H)$ .

2.7. Join of two even cycles

**Theorem 2.8.** For  $m \geq 1$  and  $n \geq 1$ ,  $mc(C_{4m} \vee C_{4n}) = 4$ .

*Proof.* First, label the vertices of  $C_{4m}$  by  $0, 0, 2, 0, 0, 0, 2, 0, \dots, 0, 0, 2, 0$  in cyclic order. This shows that  $C_{4m} \in M_4^{0,2;0} \cup M_4^{0,2;2}$ . Next, label the vertices of  $C_{4n}$  by  $1, 3, 0, 0, 1, 3, 0, 0, \dots, 1, 3, 0, 0$  in cyclic order. This implies that  $C_{4n} \in M_4^{1,3;0}$ . Finally, apply Lemma 2.1 (5) and (39).  $\square$

**Theorem 2.9.** For  $m \geq 1$  and  $n \geq 1$ ,  $mc(C_{4m} \vee C_{4n+2}) = 4$ .

*Proof.* By the proof of Theorem 2.3,  $C_{4m} \in M_4^{0,2;0} \cup M_4^{0,2;2}$ . Labelling the vertices of  $C_{4n+2}$  by  $1, 0, 1, 0, 1, 0, 1, 0, \dots, 1, 0, 1, 0, 1, 0$  in cyclic order shows that  $C_{4n+2} \in M_4^{0,2;1} \cup M_4^{0,2;3}$ . Application of Lemma 2.3 (21), (22), (23) and (24) completes the proof.  $\square$

**Lemma 2.2.** Let  $\ell \geq 1$ . If  $C_{4\ell+2} \in M_4^{i,j;k}$ , then  $\{i, j\} = \{0, 2\}$  and  $k \in \{1, 3\}$ .

*Proof.* Let  $C_{4\ell+2} = z_1 z_2 \dots z_{4\ell+2} z_1$ .  $C_{4\ell+2} \in M_4^{i,j;k}$  implies  $C_{4\ell+2}$  has a modular 4-coloring  $c$  such that  $\sigma(z_{2p-1}) = i$  and  $\sigma(z_{2p}) = j$  for  $p \in \{1, 2, \dots, 2\ell + 1\}$ , and  $\sum_{q=1}^{4\ell+2} c(z_q) \equiv k \pmod 4$ . We consider four cases.

*Case 1.*  $i = 0$  and  $j = 1$ .

The  $\sigma$ -value 1 for the vertices  $z_2, z_4, z_6, \dots, z_{4\ell}$  in order implies  $(c(z_1), c(z_3), c(z_5), c(z_7), \dots, c(z_{4\ell+1}))$  is one of the following:  $(0, 1, 0, 1, \dots, 0)$ ,  $(1, 0, 1, 0, \dots, 1)$ ,  $(2, 3, 2, 3, \dots, 2)$ ,  $(3, 2, 3, 2, \dots, 3)$ . But then  $\sigma(z_{4\ell+2}) \neq 1$ , a contradiction.

*Case 2.*  $i \in \{0, 2\}$  and  $j = 3$ .

The  $\sigma$ -value 3 for the vertices  $z_2, z_4, z_6, \dots, z_{4\ell}$  in order implies  $(c(z_1), c(z_3), c(z_5), c(z_7), \dots, c(z_{4\ell+1}))$  is one of the following:  $(0, 3, 0, 3, \dots, 0)$ ,  $(3, 0, 3, 0, \dots, 3)$ ,  $(1, 2, 1, 2, \dots, 1)$ ,  $(2, 1, 2, 1, \dots, 2)$ . But then  $\sigma(z_{4\ell+2}) \neq 3$ , a contradiction.

*Case 3.*  $i = 1$  and  $j \in \{2, 3\}$ .

The  $\sigma$ -value 1 for the vertices  $z_3, z_5, z_7, \dots, z_{4\ell+1}$  in order implies  $(c(z_2), c(z_4), c(z_6), c(z_8), \dots, c(z_{4\ell+2}))$  is one of the following:  $(0, 1, 0, 1, \dots, 0)$ ,  $(1, 0, 1, 0, \dots, 1)$ ,  $(2, 3, 2, 3, \dots, 2)$ ,  $(3, 2, 3, 2, \dots, 3)$ . But then  $\sigma(z_1) \neq 1$ , a contradiction.

*Case 4.*  $i = 0$  and  $j = 2$ .

The  $\sigma$ -value 0 for the vertices  $z_3, z_5, z_7, \dots, z_{4\ell+1}$  in order implies  $(c(z_2), c(z_4), c(z_6), c(z_8), \dots, c(z_{4\ell+2}))$  is one of the following:  $(0, 0, 0, 0, \dots, 0)$ ,  $(2, 2, 2, 2, \dots, 2)$ ,  $(1, 3, 1, 3, \dots, 1)$ ,  $(3, 1, 3, 1, \dots, 3)$ .  $\sigma(z_1) = 0$  implies that  $(c(z_2), c(z_4), c(z_6), c(z_8), \dots, c(z_{4\ell+2})) = (0, 0, 0, 0, \dots, 0)$  or  $(2, 2, 2, 2, \dots, 2)$ .

The  $\sigma$ -value 2 for the vertices  $z_2, z_4, z_6, \dots, z_{4\ell}$  in order implies  $(c(z_1), c(z_3), c(z_5), c(z_7), \dots, c(z_{4\ell+1}))$  is one of the following:  $(0, 2, 0, 2, \dots, 0)$ ,  $(2, 0, 2, 0, \dots, 2)$ ,  $(1, 1, 1, 1, \dots, 1)$ ,  $(3, 3, 3, 3, \dots, 3)$ .  $\sigma(z_{4\ell+2}) = 2$  implies that  $(c(z_1), c(z_3), c(z_5), c(z_7), \dots, c(z_{4\ell+1})) = (1, 1, 1, 1, \dots, 1)$  or  $(3, 3, 3, 3, \dots, 3)$ .

Hence the  $c$ -values for the vertices  $z_1, z_2, \dots, z_{4\ell+2}$  in cyclic order are:  $0, 1, 0, 1, \dots, 0, 1; 0, 3, 0, 3, \dots, 0, 3; 2, 1, 2, 1, \dots, 2, 1; 2, 3, 2, 3, \dots, 2, 3$ .  $\square$

**Theorem 2.10.** For  $m \geq 1$  and  $n \geq 1$ ,  $mc(C_{4m+2} \vee C_{4n+2}) = 5$ .



*Proof.* Let  $C_{4m+2} = x_1x_2 \dots x_{4m+2}x_1$  and  $C_{4n+2} = y_1y_2 \dots y_{4n+2}y_1$ . Define  $c : V(C_{4m+2} \vee C_{4n+2}) \rightarrow \mathbb{Z}_5$  by  $c(x_{2p}) = 1$  and  $c(x_{2p-1}) = 4$  for  $p \in \{1, 2, \dots, 2m + 1\}$  and  $c(y_{2q}) = 2$  and  $c(y_{2q-1}) = 3$  for  $q \in \{1, 2, \dots, 2n + 1\}$ . Then,  $\sigma(x_{2p}) = 3$  and  $\sigma(x_{2p-1}) = 2$  for  $p \in \{1, 2, \dots, 2m + 1\}$  and  $\sigma(y_{2q}) = 1$  and  $\sigma(y_{2q-1}) = 4$  for  $q \in \{1, 2, \dots, 2n + 1\}$ . Hence  $c$  is a modular 5-coloring and therefore  $mc(C_{4m+2} \vee C_{4n+2}) \leq 5$ .

Suppose there exists a modular 4-coloring  $c$  for  $C_{4m+2} \vee C_{4n+2}$ . This induces a modular 4-coloring  $c' = c|_{\{x_1, x_2, \dots, x_{4m+2}\}}$  such that  $\sigma'(x_{2p-1}) = i'$  and  $\sigma'(x_{2p}) = j'$  for  $p \in \{1, 2, \dots, 2m + 1\}$ , and  $\sum_{q=1}^{4m+2} c'(x_q) \equiv k' \pmod{4}$  for some  $i', j', k'$ ; and a modular 4-coloring  $c'' = c|_{\{y_1, y_2, \dots, y_{4n+2}\}}$  such that  $\sigma''(y_{2p-1}) = i''$  and  $\sigma''(y_{2p}) = j''$  for  $p \in \{1, 2, \dots, 2n + 1\}$ , and  $\sum_{q=1}^{4n+2} c''(y_q) \equiv k'' \pmod{4}$  for some  $i'', j'', k''$ .

By the proof of Lemma 2.2, both the sequences  $\{c'(x_p)\}_{p=1}^{4m+2}$  and  $\{c''(y_p)\}_{p=1}^{4n+2}$  are in  $\{(0, 1, 0, 1, \dots, 0, 1), (0, 3, 0, 3, \dots, 0, 3), (2, 1, 2, 1, \dots, 2, 1), (2, 3, 2, 3, \dots, 2, 3)\}$ .

If  $\{c'(x_p)\}_{p=1}^{4m+2} = \{c''(y_p)\}_{p=1}^{4n+2}$  is one of:  $(0, 1, 0, 1, \dots, 0, 1), (0, 3, 0, 3, \dots, 0, 3), (2, 1, 2, 1, \dots, 2, 1), (2, 3, 2, 3, \dots, 2, 3)$ , then  $\{\sigma(x_p)\}_{p=1}^{4m+2} = \{\sigma(y_p)\}_{p=1}^{4n+2} = (1, 3, 1, 3, \dots, 1, 3)$ , a contradiction.

If  $\{\{c'(x_p)\}_{p=1}^{4m+2}, \{c''(y_p)\}_{p=1}^{4n+2}\} \in \{(0, 1, 0, 1, \dots, 0, 1), (0, 3, 0, 3, \dots, 0, 3)\}, \{(0, 1, 0, 1, \dots, 0, 1), (2, 1, 2, 1, \dots, 2, 1)\}, \{(0, 1, 0, 1, \dots, 0, 1), (2, 3, 2, 3, \dots, 2, 3)\}, \{(0, 3, 0, 3, \dots, 0, 3), (2, 1, 2, 1, \dots, 2, 1)\}, \{(0, 3, 0, 3, \dots, 0, 3), (2, 3, 2, 3, \dots, 2, 3)\}, \{(2, 1, 2, 1, \dots, 2, 1), (2, 3, 2, 3, \dots, 2, 3)\}$ , then  $\{\sigma(x_p)\}_{p=1}^{4m+2} = \{\sigma(y_p)\}_{p=1}^{4n+2} = (1, 3, 1, 3, \dots, 1, 3)$ , again a contradiction.

From these two contradictions, we have  $mc(C_{4m+2} \vee C_{4n+2}) \geq 5$ . □

### 2.8. Join of a regular bipartite graph and an empty graph

Okamoto, Salehi and Zhang [2] have shown that for any integer  $n \geq 3$ ,  $mc(C_n \vee K_1) = \chi(C_n \vee K_1)$ . Using the proof technique of this result, we prove Theorem 2.11.

**Theorem 2.11.** *Let  $G$  be an  $r$ -regular bipartite graph with  $r \equiv 1$  or  $2 \pmod{3}$ . Then, for any positive integer  $s$ ,  $mc(G \vee sK_1) = 3$ .*

*Proof.* Let  $(X, Y)$  be the bipartition of  $G$ . Construct  $G \vee sK_1$  from  $G$  by joining new vertices  $w_1, w_2, \dots, w_s$  to every vertex of  $G$ . Define  $c : V(G \vee sK_1) \rightarrow \mathbb{Z}_3$  by  $c(w_i) = 0$  for  $i \in \{1, 2, \dots, s\}$ ,  $c(x) = 1$  if  $x \in X$  and  $c(y) = 2$  if  $y \in Y$ . Then, for  $i \in \{1, 2, \dots, s\}$ ,  $\sigma(w_i) = |X| + 2|Y| = |X| + 2|X| \equiv 0 \pmod{3}$ ; for  $x \in X$ ,  $\sigma(x) = 2r$  is 2 if  $r \equiv 1 \pmod{3}$  and it is 1 if  $r \equiv 2 \pmod{3}$ ; for  $y \in Y$ ,  $\sigma(y) = r$  is 1 if  $r \equiv 1 \pmod{3}$  and it is 2 if  $r \equiv 2 \pmod{3}$ . Hence,  $c$  is a modular 3-coloring of  $G$ , and therefore  $mc(G \vee sK_1) \leq 3$ . But,  $mc(G \vee sK_1) \geq \chi(G \vee sK_1) = 3$ . Thus,  $mc(G \vee sK_1) = 3$ . □

*Remark 2.1.* Let  $G$  be an  $r$ -regular bipartite graph with  $r \equiv 0 \pmod{3}$  and let  $(X, Y)$  be the bipartition of  $G$ . Suppose there exist partitions  $\{X_1, X_2\}$  of  $X$  and  $\{Y_1, Y_2\}$  of  $Y$  and integers  $t'$  and  $t''$  both not congruent to  $0 \pmod{3}$  such that  $t'$  is not congruent to  $t'' \pmod{3}$ , the subgraph induced by  $X_1 \cup Y_1$  is  $t'$ -regular and the subgraph induced by  $X_2 \cup Y_2$  is  $t''$ -regular, then  $mc(G \vee sK_1) = 3$ .

To see this, define  $c : V(G \vee sK_1) \rightarrow \mathbb{Z}_3$  by  $c(w_i) = 0$  for  $i \in \{1, 2, \dots, s\}$ ,  $c(x) = 1$  if  $x \in X_1$ ,  $c(x) = 2$  if  $x \in X_2$ ,  $c(y) = 2$  if  $y \in Y_1$ , and  $c(y) = 1$  if  $y \in Y_2$ . Then, for  $i \in \{1, 2, \dots, s\}$ ,  $\sigma(w_i) = |X_1| + 2|X_2| + 2|Y_1| + |Y_2| = 3|X| = 0 \pmod 3$ ; for  $x \in X_1$ ,  $\sigma(x) = 2t' + (r - t') = r + t'$ ; for  $x \in X_2$ ,  $\sigma(x) = t'' + 2(r - t'') = 2r - t''$ ; for  $y \in Y_1$ ,  $\sigma(y) = t' + 2(r - t') = 2r - t'$ ; and for  $y \in Y_2$ ,  $\sigma(y) = 2t'' + (r - t'') = r + t''$ . By hypothesis, in  $\mathbb{Z}_3$ ,  $0 \notin \{r + t', r + t'', 2r - t', 2r - t''\}$  and  $\{r + t', 2r - t''\} \cap \{r + t'', 2r - t'\} = \emptyset$ .

### 3. Join of a path and a complete graph

**Theorem 3.1.** For integers  $n \geq 3$  and  $p \geq 3$ ,  $mc(P_n \vee K_p) = p + 2$ .

*Proof.* Let  $P_n := u_1u_2 \dots u_n$  and  $V(K_p) = \{v_1, v_2, \dots, v_p\}$ . Define  $c : V(P_n \vee K_p) \rightarrow \mathbb{Z}_{p+2}$  as follows: Label the vertices of  $P_n$ , in order, by  $0, 1, 0, 0, 0, 1, 0, 0, \dots, 0, 1, 0, 0$  for  $n \equiv 0 \pmod 4$ ; by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1$  for  $n \equiv 1 \pmod 4$ ; by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1, 0$  for  $n \equiv 2 \pmod 4$ ; and by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0, 1, 0, 0$  for  $n \equiv 3 \pmod 4$ . Label the  $p$  vertices of  $K_p$  by the  $p$  numbers in

$$\{0, 1, 2, \dots, p + 1\} \setminus \left\{ \left( \left\lceil \frac{n}{4} \right\rceil - 1 \right) \pmod{(p + 2)}, \left\lceil \frac{n}{4} \right\rceil \pmod{(p + 2)} \right\}.$$

The  $\sigma$ -values of the vertices of  $P_n$  in  $P_n \vee K_p$  are, alternately,  $\left( \frac{(p+1)(p+2)}{2} - 2 \left\lceil \frac{n}{4} \right\rceil + 1 \right) \pmod{(p+2)}$  and  $\left( \frac{(p+1)(p+2)}{2} - 2 \left\lceil \frac{n}{4} \right\rceil + 2 \right) \pmod{(p+2)}$ . The  $\sigma$ -values of the  $p$  vertices of  $K_p$  in  $P_n \vee K_p$  are the  $p$  numbers in  $\left\{ \left( \frac{(p+1)(p+2)}{2} - \left\lceil \frac{n}{4} \right\rceil + 1 - i \right) \pmod{(p+2)} : i \in \{0, 1, 2, \dots, p + 1\} \setminus \left\{ \left( \left\lceil \frac{n}{4} \right\rceil - 1 \right) \pmod{(p+2)}, \left\lceil \frac{n}{4} \right\rceil \pmod{(p+2)} \right\} \right\}$ . Note that the  $\sigma$ -value of any vertex of  $P_n$  in  $P_n \vee K_p$  is different from that of any vertex of  $K_p$  in  $P_n \vee K_p$ . This completes the proof.  $\square$

### 4. Join of an even cycle and a complete graph

**Theorem 4.1.** For integers  $n \geq 2$  and  $p \geq 3$ ,  $mc(C_{2n} \vee K_p) = p + 2$ .

*Proof.* Let  $C_{2n} := u_1u_2 \dots u_{2n}u_1$  and  $V(K_p) = \{v_1, v_2, \dots, v_p\}$ . Define  $c : V(C_{2n} \vee K_p) \rightarrow \mathbb{Z}_{p+2}$  as follows: We consider two cases:

Case 1.  $n \equiv 0 \pmod 2$ .

Label the vertices of  $C_{2n}$ , in cyclic order, by  $1, 0, 0, 0, 1, 0, 0, 0, \dots, 1, 0, 0, 0$ . Label the  $p$  vertices of  $K_p$  by the  $p$  numbers in

$$\{0, 1, 2, \dots, p + 1\} \setminus \left\{ \left( \frac{n}{2} - 1 \right) \pmod{(p + 2)}, \frac{n}{2} \pmod{(p + 2)} \right\}.$$

The  $\sigma$ -values of the vertices of  $C_{2n}$  in  $C_{2n} \vee K_p$  are, alternately,  $\left( \frac{(p+1)(p+2)}{2} - n + 1 \right) \pmod{(p+2)}$  and  $\left( \frac{(p+1)(p+2)}{2} - n + 2 \right) \pmod{(p+2)}$ . The  $\sigma$ -values of the  $p$  vertices of  $K_p$  in  $C_{2n} \vee K_p$  are the  $p$  numbers in  $\left\{ \left( \frac{(p+1)(p+2)}{2} - \frac{n}{2} + 1 - i \right) \pmod{(p+2)} : i \in \{0, 1, 2, \dots, p + 1\} \setminus \left\{ \left( \frac{n}{2} - 1 \right) \pmod{(p+2)}, \frac{n}{2} \pmod{(p+2)} \right\} \right\}$ . Observe that the  $\sigma$ -value of any vertex of  $C_{2n}$  in  $C_{2n} \vee K_p$  is different from that of any vertex of  $K_p$  in  $C_{2n} \vee K_p$ .

Case 2.  $n \equiv 1 \pmod 2$ .

Label the vertices of  $C_{2n}$ , in cyclic order, by  $1, 0, 1, 0, 1, 0, 1, 0, \dots, 1, 0, 1, 0$ . Label the  $p$  vertices of  $K_p$  by the  $p$  numbers in  $\{0, 1, 2, \dots, p+1\} \setminus \{(n-2) \pmod{(p+2)}, n \pmod{(p+2)}\}$ . The  $\sigma$ -values of the vertices of  $C_{2n}$  in  $C_{2n} \vee K_p$  are, alternately,  $(\frac{(p+1)(p+2)}{2} - 2n+2) \pmod{(p+2)}$  and  $(\frac{(p+1)(p+2)}{2} - 2n+4) \pmod{(p+2)}$ . The  $\sigma$ -values of the  $p$  vertices of  $K_p$  in  $C_{2n} \vee K_p$  are the  $p$  numbers in  $\{(\frac{(p+1)(p+2)}{2} - n + 2 - i) \pmod{(p+2)} : i \in \{0, 1, 2, \dots, p+1\} \setminus \{(n-2) \pmod{(p+2)}, n \pmod{(p+2)}\}\}$ . Clearly, the  $\sigma$ -value of any vertex of  $C_{2n}$  in  $C_{2n} \vee K_p$  is different from that of any vertex of  $K_p$  in  $C_{2n} \vee K_p$ .

This completes the proof. □

## 5. Conclusion

We have seen that  $mc(G \vee H) = \chi(G) + \chi(H)$  for every join graph  $G \vee H$ , except for the join graph  $C_{4m+2} \vee C_{4n+2}$ , that we have encountered in this paper.

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