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Forbidden subgraph pairs for traceability of block-chains

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Abstract

A block-chain is a graph whose block graph is a path, i.e. it is either a P_1, a P_2, or a 2-connected graph, or a graph of connectivity 1 with exactly two end-blocks. A graph is called traceable if it contains a Hamilton path. A traceable graph is clearly a block-chain, but the reverse does not hold in general. In this paper we characterize all pairs of connected graphs $\{R, S\}$ such that every $\{R, S\}$-free block-chain is traceable.

Keywords: forbidden subgraph, induced subgraph, block-chain, traceable graph, closure, line graph
Mathematics Subject Classification : 05C45

1. Introduction

We use Bondy and Murty [1] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G. For a given graph H, we say that G is H*-free* if G does not contain an induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. Note that if H_1 is an induced subgraph of H_2, then an H_1-free graph is also H_2-free.

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The graph $K_{1,3}$ is called a *claw*; its only vertex with degree 3 is called the *center* of the claw, and the other vertices are called the *end vertices* of the claw. In this paper, instead of $K_{1,3}$-free, we use the term *claw-free*.

Let P_i be the path on $i \geq 1$ vertices, and C_i the cycle on $i \geq 3$ vertices. We use Z_i to denote the graph obtained by identifying a vertex of a C_3 with an end vertex of a P_{i+1} ($i \geq 1$), $B_{i,j}$ for the graph obtained by identifying two vertices of a C_3 with the end vertices of a P_{i+1} ($i \geq 1$) and a P_{j+1} ($j \geq 1$), respectively, and $N_{i,j,k}$ for the graph obtained by identifying the three vertices of a C_3 with the end vertices of a P_{i+1} ($i \geq 1$), P_{j+1} ($j \geq 1$) and P_{k+1} ($k \geq 1$), respectively. In particular, we let $B = B_{1,1}$ (this graph is sometimes called a *bull*) and $N = N_{1,1,1}$ (this graph is sometimes called a *net*).

If a graph is P_2-free, then it is an empty graph (contains no edges). To avoid the discussion of this trivial case, in the following, we throughout assume that our forbidden subgraphs have at least three vertices.

A graph is called *traceable* if it contains a Hamilton path. If a graph is connected and P_3-free, then it is a complete graph and it is trivially traceable. In fact, it is not difficult to show that P_3 is the only single subgraph H such that every connected H-free graph is traceable. Moving to the more interesting case of pairs of subgraphs, the following theorem on forbidden pairs for traceability is well-known.

Theorem 1.1 (Duffus, Gould and Jacobson [5]). *If G is a connected $\{K_{1,3}, N\}$-free graph, then G is traceable.*

Obviously, if H is an induced subgraph of N, then the pair $\{K_{1,3}, H\}$ is also a forbidden pair that guarantees the traceability of every connected graph. In fact, Faudree and Gould proved that these are the only forbidden pairs with this property.

Theorem 1.2 (Faudree and Gould [6]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$, P_4, Z_1, B or N (See Fig. 1).*

Let G be a graph. A maximal nonseparable subgraph (2-connected or P_1 or P_2) of G is called a *block* of G. A block containing exactly one cut vertex of G is called an *end-block*. Adopting the terminology of [7], we say that a graph is a *block-chain* if it is nonseparable or it has connectivity 1 and has exactly two end-blocks. Note that every traceable graph is necessarily a block-chain, but that the reverse does not hold in general. Also note that it is easy to check by a polynomial algorithm whether a given graph is a block-chain or not. In the 'only-if' part of the proof of Theorem 1.2 many graphs are used that are not block-chains (and are therefore trivially non-traceable). A natural extension is to consider forbidden subgraph conditions for a block-chain to be traceable. In this paper, we characterize all the pairs of subgraphs with this property. First note that, similarly as in the above analysis, it is easy to check that any P_3-free block-chain is traceable. We will show that P_3 is the only single forbidden subgraph with this property.

Theorem 1.3. *The only connected graph S such that a block-chain being S-free implies it is traceable is P_3.*

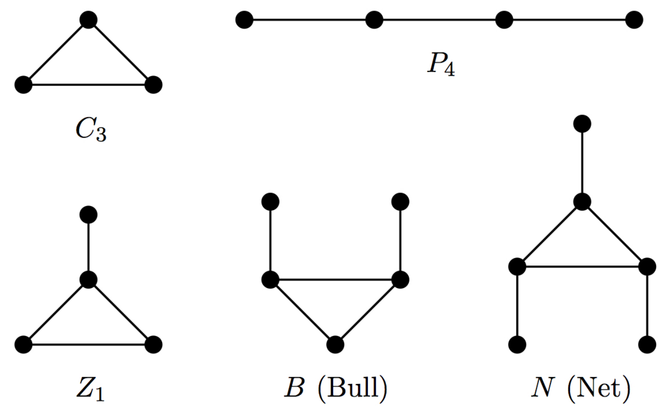

Figure 1. The graphs C_3, P_4, Z_1, B and N

Next we will prove the following characterization of all pairs of connected graphs R and S other than P_3 guaranteeing that every $\{R, S\}$-free block-chain is traceable.

Theorem 1.4. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a block-chain. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$.*

It is interesting to note that one of the pairs does not include the claw, in contrast to all existing characterizations of pairs of forbidden subgraphs for hamiltonian properties we encountered.

In Section 2, we prove the 'only if' part of Theorems 1.3 and 1.4. For the 'if' part of Theorem 1.4, it is sufficient to prove the following results.

Theorem 1.5. *If G is a $\{K_{1,4}, P_4\}$-free block-chain, then G is traceable.*

Theorem 1.6. *If G is a $\{K_{1,3}, N_{1,1,3}\}$-free block-chain, then G is traceable.*

We prove Theorems 1.5 and 1.6 in Sections 4 and 5, respectively.

2. The 'only if' part of Theorems 1.3 and 1.4

We first sketch some families of graphs that are block-chains but not traceable (see Fig. 2). When we say that a graph is of *type* G_i we mean that it is one particular, but arbitrarily chosen member of the family indicated by G_i in Fig. 2.

If S is a connected graph such that every S-free block-chain is traceable, then S must be a common induced subgraph of all graphs of type G_1, G_2 and G_4. Note that the only largest common induced connected subgraph of graphs of type G_1, G_2 and G_4 is a P_3, so we have $S = P_3$. This completes the proof of the 'only if' part of the statement of Theorem 1.3.

Let R and S be two connected graphs other than P_3 such that every $\{R, S\}$-free block-chain is traceable. Then R or S must be an induced subgraph of all graphs of type G_1. Without loss of

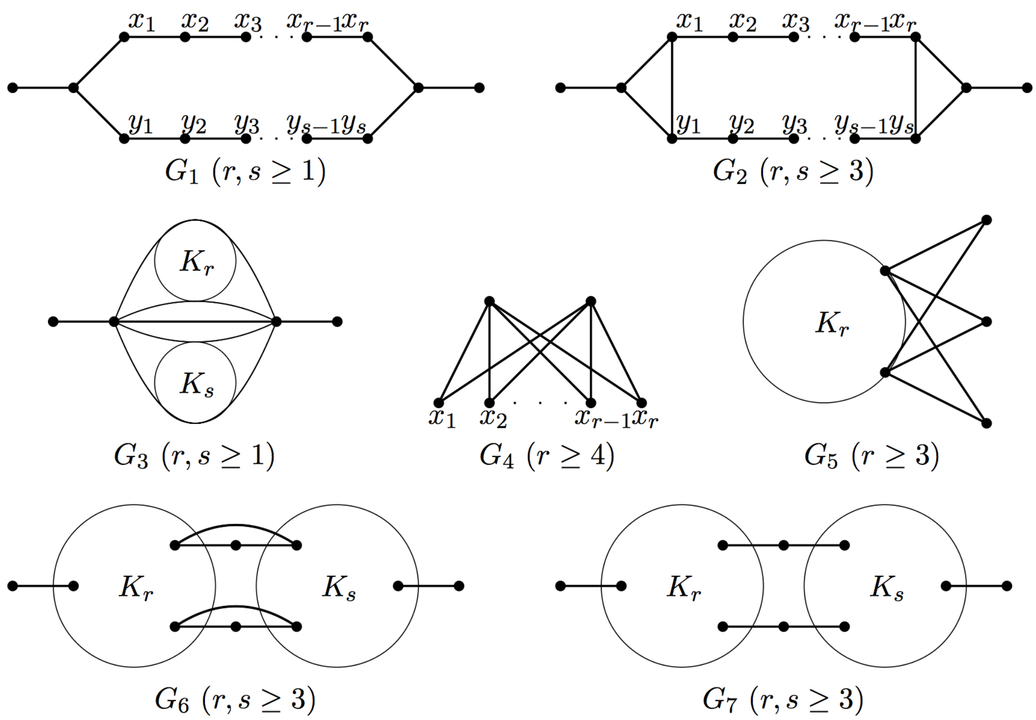

Figure 2. Some block-chains that are not traceable

generality, we assume that R is an induced subgraph of all graphs of type G_1. If $R \neq K_{1,3}$, then R must contain an induced P_4. Note that the graphs of type G_4 and G_5 are all P_4-free, so they must contain S as an induced subgraph. Since the only common induced connected subgraph of the graphs of type G_3 and G_4 other than P_3 is $K_{1,3}$ or $K_{1,4}$, we have that $S = K_{1,3}$ or $K_{1,4}$. This implies that R or S must be $K_{1,3}$ or $K_{1,4}$. Without loss of generality, we assume that $R = K_{1,3}$ or $K_{1,4}$.

Suppose first that $R = K_{1,4}$. Noting that the graphs of type G_1, G_2 and G_3 are all $K_{1,4}$-free, S must be a common induced subgraph of the graphs of type G_1, G_2 and G_3. Since the only common induced connected subgraph of the graphs of type G_1, G_2 and G_3 other than P_3 is P_4, we have $S = P_4$.

Suppose now that $R = K_{1,3}$. Note that the graphs of type G_2 are claw-free. So S must be an induced subgraph of all graphs of type G_2. The common induced connected subgraphs of such graphs have the form P_i, Z_i, $B_{i,j}$ or $N_{i,j,k}$. Note that graphs of type G_6 are claw-free and do not contain an induced P_7 or Z_4, and that graphs of type G_7 are claw-free and do not contain an induced $B_{2,2}$. So R must be an induced connected subgraph of P_6, Z_3, $B_{1,3}$ or $N_{1,1,3}$. Since P_6, Z_3 and $B_{1,3}$ are induced subgraphs of $N_{1,1,3}$, R must be an induced connected subgraph of $N_{1,1,3}$. This completes the proof of the 'only if' part of the statement of Theorem 1.4.

3. Preliminaries

Let G be a graph. For a subgraph B of G, when no confusion can occur, we also use B to denote its vertex set; similarly, for a subset C of $V(G)$, we also use C to denote the subgraph induced by C.

We use $\kappa(G)$ to denote the connectivity of G and $\alpha(G)$ to denote the *independence number* of G, i.e., the maximum number of vertices no two of which are adjacent. The following theorem on hamiltonian and traceable graphs is well-known and will be used in the sequel.

Theorem 3.1 (Chvátal and Erdös [4]). *Let G be a graph on at least three vertices. If $\alpha(G) \leq \kappa(G)$, then G is hamiltonian. If $\alpha(G) \leq \kappa(G) + 1$, then G is traceable.*

For the proof of Theorem 1.6 we will make use of the closure theory developed by Ryjáček in [10]. This requires a short introduction and repetition of the relevant concepts and results.

The *line graph* of a graph H, denoted by $L(H)$, is the graph with vertex set $V(L(H)) = E(H)$, and two distinct vertices are adjacent in $L(H)$ if and only if the two corresponding edges have a vertex in common in H. Note that if $G = L(H)$ and $R = L(S)$, then G is R-free if and only if H does not contain S as a (not necessarily induced) subgraph.

To study the hamiltonian properties of claw-free graphs, in particular in order to show that the conjectures on hamiltonicity of 4-connected claw-free graphs and of 4-connected line graphs are equivalent, Ryjáček developed his closure theory, as follows.

Let G be a claw-free graph and let x be a vertex of G. We call x an *eligible vertex* of G if $N(x)$ induces a connected graph, but not a complete graph, in G. The *completion* of G at x, denoted by G'_x, is the graph obtained from G by adding all missing edges uv with $u, v \in N(x)$. The *closure* of G, denoted by $cl(G)$, is the graph defined by a sequence of graphs $G_1, G_2, \ldots, G_t$, and vertices $x_1, x_2, \ldots, x_{t-1}$ such that

(1) $G_1 = G$, $G_t = cl(G)$;

(2) x_i is an eligible vertex of G_i, $G_{i+1} = (G_i)'_{x_i}$, $1 \leq i \leq t-1$; and

(3) $cl(G)$ has no eligible vertices.

A claw-free graph is said to be *closed* if it has no eligible vertices. Next we list some useful properties on the closure of claw-free graphs in the following lemmas.

Let G be a claw-free graph.

Lemma 3.1 (Ryjáček [10]).
(1) The closure $cl(G)$ is well-defined;
(2) $cl(G)$ is claw-free;
(3) there is a triangle-free graph H such that $cl(G) = L(H)$.

Lemma 3.2 (Brandt, Favaron and Ryjáček [2]). *G is traceable if and only if $cl(G)$ is traceable.*

Lemma 3.3 (Brousek, Favaron and Ryjáček [3]). *If G is $N_{i,j,k}$-free, $i, j, k \geq 1$, then $cl(G)$ is also $N_{i,j,k}$-free.*

Moreover, it is easy to observe that if G is a block-chain, then so is $cl(G)$.

Let G be a graph and let G' be a subgraph of G. For an edge e of G, we say that G' *dominates* e if at least one end vertex of e is in $V(G')$. G' is called *dominating* if it dominates every edge of G.

Alternatively, G' is dominating if and only if every component of $G - V(G')$ is an isolated vertex. For traceability of a line graph $G = L(H)$ the crucial equivalence is the existence of a dominating *trail* in H, i.e., a walk $v_0e_1v_1 \dots v_{\ell-1}e_\ell v_\ell$ (with $e_i = v_{i-1}v_i \in E(H)$ for $= 1, \dots, \ell$) in which all the edge terms are different, that dominates all edges of H.

Lemma 3.4 (Li, Lai and Zhan [9]). *If $G = L(H)$, then G is traceable if and only if H has a dominating trail.*

A graph G is said to be *homogeneously traceable*, if for every vertex x of G, there is a Hamilton path starting from x. We will use the following recent theorem on homogeneously traceable graphs.

Theorem 3.2 (Li, Broersma and Zhang [8]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is homogeneously traceable, if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$.*

4. Proof of Theorem 1.5

Let G be a $\{K_{1,4}, P_4\}$-free block-chain. We are going to prove that G is traceable.

If G contains only one or two vertices, then it is trivially traceable. So we assume that G has at least three vertices. If G is complete, then the result is trivially true. So we assume that G is not complete. Let X be a minimum vertex cut of G.

Clearly each vertex of X has a neighbor in each component of $G - X$. Now we claim that each vertex of X is adjacent to every vertex in $G - X$. Let $x \in X$ and $y \in V(H)$, where H is a component of $G - X$. If $xy \notin E(G)$, then let Q be a shortest path from x to y with all internal vertices in H. Let H' be a component of $G - X$ other than H and let y' be a neighbor of x in H'. Then Qxy' is an induced path with at least four vertices. This contradicts that G is P_4-free. Thus as we claimed, each vertex of X is adjacent to every vertex in $G - X$.

Let S be an independent set of G. Then S is either contained in X or in $V(G - X)$. We assume first that $S \subset V(G - X)$. Let x be a vertex of X. If S has at least four vertices, then the subgraph induced by $\{x\} \cup S$ is a $K_{1,t}$ with $t \geq 4$, contradicting that G is $K_{1,4}$-free. Thus we have $|S| \leq 3$. If $S \subset X$, then by a similar argumentation we also get that $|S| \leq 3$. This implies that the independence number $\alpha(G) \leq 3$.

If G is 2-connected, then $\alpha(G) \leq \kappa(G) + 1$. By Theorem 3.1, G is traceable. So we assume that G has a cut vertex.

Let x be a cut vertex of G. Let H be an arbitrary component of $G - x$. We claim that there is a Hamilton path of $H \cup \{x\}$ starting from x. If H contains only one vertex, then the result is trivially true. So we assume that H has at least two vertices. Note that H is connected and x is adjacent to every vertex of H. Hence $H \cup \{x\}$ is 2-connected. If H contains an independent set S with three vertices, then let y be a neighbor of x in H', where H' is a component of $G - x$ other than H. Then the subgraph induced by $\{x, y\} \cup S$ is a $K_{1,4}$, a contradiction. This implies that $\alpha(H \cup \{x\}) \leq 2$. By Theorem 3.1, $H \cup \{x\}$ is hamiltonian. Thus it contains a Hamilton path starting from x.

It is not difficult to see that either $H \cup \{x\}$ is an end-block or H contains an end-block of G. If $G - x$ has at least three components, then there will be at least three end-blocks of G, contradicting that G is a block-chain. Thus $G - x$ has exactly two components. Let H_1 and H_2 be the two components of $G-x$. Let Q_i, $i=1,2$, be the Hamilton path of $H_i\cup\{x\}$ starting from x. Then Q_1xQ_2 is a Hamilton path of G. This completes the proof of Theorem 1.5.

5. Proof of Theorem 1.6

By Lemmas 3.1, 3.2 and 3.3, and the observation following Lemma 3.3, it is sufficient to prove the result for closed block-chains. Let G be a $\{K_{1,3},N_{1,1,3}\}$-free closed block-chain. We are going to prove that G is traceable.

We use induction on $|V(G)|$. If G contains only one or two vertices, then the result is trivially true. So we assume that G contains at least three vertices.

If G is 2-connected, then by Theorem 3.2, G is (homogeneously) traceable. Thus we assume that G has at least one cut vertex. We also assume that G is non-traceable, and will reach a contradiction in all cases.

Claim 1. If x is a cut vertex of G, then at least one of the components of $G-x$ consists of an isolated vertex.

Proof. It is easy to see that there are exactly two components in $G-x$; otherwise x and its three neighbors in three distinct components of $G-x$ will induce a claw. Let H_1 and H_2 be the two components. Suppose that both H_1 and H_2 have at least 2 vertices. For $i=1,2$, let y_i be a neighbor of x in H_i, and let G_i be the subgraph of G induced by $H_i\cup\{x,y_{3-i}\}$. It is not difficult to see that G_i is a block-chain, and that y_{3-i} has only one neighbor x in G_i. By the induction hypothesis, there is a Hamilton path Q_i of G_i (starting from y_{3-i}). Then $Q'_i=Q_i-y_{3-i}$ is a Hamilton path of $H_i\cup\{x\}$ starting from x. Thus $Q'_1xQ'_2$ is a Hamilton path of G, a contradiction. □

Let x be a cut vertex of G, and let y be an isolated vertex of $G-x$. Clearly the subgraph induced by $\{x,y\}$ is an end-block of G. If G has at least three cut vertices, then there will be at least three end-blocks of G, a contradiction. Thus we assume that there are at most two cut vertices in G.

Suppose first that there is only one cut vertex in G, and denote it by x. Let y be an isolated vertex of $G-x$, and let H be the component of $G-x$ not containing y. We claim that there is a Hamilton path of $H\cup\{x\}$ starting from x. If H has only one vertex, the result is trivially true. So we assume that H has at least two vertices. If $H\cup\{x\}$ has a cut vertex (note that x is not a cut vertex of $H\cup\{x\}$), then it is also a cut vertex of G, a contradiction. So we assume that $H\cup\{x\}$ is 2-connected. By Theorem 3.2, $H\cup\{x\}$ is homogeneously traceable. Thus as we claimed, there is a Hamilton path Q of $H\cup\{x\}$ starting from x. So yxQ is a Hamilton path of G. This contradiction shows that G has exactly two cut vertices.

Let r and s be the two cut vertices of G, and let r_0 and s_0 be the isolated vertices of $G-r$ and $G-s$, respectively. Let $B=G-\{r_0,s_0\}$. If B has only two vertices r and s, then clearly G is traceable. So we assume that B has at least one vertex other than r and s. Note that if B has a cut vertex, then it is also a cut vertex of G (clearly r and s are not cut vertices of B), a contradiction. So we assume that B is 2-connected.

Using Lemma 3.1, let H be a triangle-free graph such that $G=L(H)$. Let B' be the subgraph of H corresponding to B and let e,f,e_0,f_0 be the edges of H corresponding to the vertices r,s,r_0,s_0 of G, respectively. Denote $e_0=v_1v'_1$, $e=v'_1v''_1$, $f_0=v_2v'_2$ and $f=v'_2v''_2$.

Recall that G is traceable if and only if H contains a dominating trail, and observe that any dominating trail in H must have v_1/v_1' and v_2/v_2' as end vertices. Let T be a trail in H from v_1 to v_2 such that it dominates a maximum number of edges. By the assumption, T is not a dominating trail. Let D be a non-trivial component of $H - V(T)$.

Claim 2. $|N_T(D)| \geq 2$.

Proof. We suppose to the contrary that there is only one neighbor u of D in T. If u has only one neighbor, say x, in D, then ux is a bridge (cut edge) of H and the vertex of G corresponding to ux is a cut vertex of G other than r, s, a contradiction. Thus we have that u has at least two neighbors, say x, y, in D. Let P be a path of D from x to y. Then $T' = T \cup uxPyu$ is a trail dominating more edges than T, a contradiction to the choice of T. □

Using Claim 2, let u_1, u_2 be two neighbors of D in T. Let P be a path from u_1 to u_2 of length at least 2 with all internal vertices in D. If $u_1u_2 \in E(T)$, then $T' = T - u_1u_2 \cup P$ is a trail dominating more edges than T, a contradiction. Thus we conclude that $u_1u_2 \notin E(T)$. Let Q be a path from u_1 to u_2 with all edges in T. Using that $u_1u_2 \notin E(T)$, $|E(Q)| \geq 2$. We choose Q as long as possible. Note that $C = P \cup Q$ is a cycle of H.

Let Q_1 be a path from v_1 to C, and Q_2 be a path from v_2 to $C \cup Q_1$. Let w_1 and w_2 be the end vertices of Q_1 and Q_2 other than v_1 and v_2, respectively.

Since G is $N_{1,1,3}$-free, H contains no copy of S as a (not necessarily induced) subgraph (see Fig. 3). In the following, we prove that H contains S as a subgraph, thus reaching a contradiction in all cases.

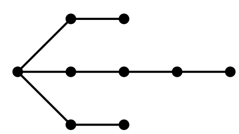

Fig. 3. The graph S.

Case 1. $w_2 \in V(Q_1)$.

In this case, Q_1 is divided by w_2 into two subpaths. Let P_1' be the subpath of Q_1 from v_1 to w_2, and P_1'' from w_2 to w_1 (P_1'' consists of only one vertex w_1 if $w_1 = w_2$).

Since the only neighbors of v_1' in H are v_1 and v_1'', we have that the length of P_1', and similarly of Q_2, is at least 2. Let x_1 be the predecessor of w_2, and x_1' the predecessor of x_1, on P_1', and let x_2 be the predecessor of w_2, and x_2' the predecessor of x_2, on Q_2.

If $|V(P_1'' \cup C) \backslash \{w_2\}| \geq 4$, then there is a path in $P_1'' \cup C$ starting from w_2 of length at least 4. Let then $w_2yy'y''y'''$ be such a path. Then the subgraph formed by $w_2x_1x_1'$, $w_2x_2x_2'$ and $w_2yy'y''y'''$ is an S.

Now we assume that $|V(P_1'' \cup C) \backslash \{w_1\}| = 3$, which implies that $w_1 = w_2$ and the length of C is 4. Note that $u_1u_2 \notin E(C)$. Let $C = u_1yu_2zu_1$, where y is a vertex in D, and let y' be a neighbor of y in D. If $w_1 = u_1$, then the subgraph formed by $u_1x_1x_1'$, $u_1x_2x_2'$ and u_1zu_2yy' is an S.

Next we assume that $w_1 \neq u_1$, and similarly, $w_1 \neq u_2$, which implies that $w_1 = z$. Let u_1' be a neighbor of u_1 on T other than w_1 (u_1' exists since the degree of u_1 in T is even). Since H is triangle-free, $u_1' \neq u_2$. If $u_1' \in V(Q_1 \cup Q_2)\backslash\{w_1\}$, then there will be a path from u_1 to u_2 in T longer than Q, a contradiction. Thus we conclude that $u_1' \notin V(Q_1 \cup Q_2 \cup C)$. Now the subgraph formed by $w_1x_1x_1'$, $w_1x_2x_2'$ and $w_1u_2yu_1u_1'$ is an S.

Case 2. $w_2 \notin V(Q_1)$.

In this case, w_1, w_2 are two distinct vertices in $V(Q)$. We assume without loss of generality that u_1, w_1, w_2, u_2 appear along Q in this order (with possibly $w_1 = u_1$ or $w_2 = u_2$ or both). As in Case 1, let x_1 be the predecessor of w_1, and x_1' the predecessor of x_1 on Q_1, and let x_2 be the predecessor of w_2, and x_2' the predecessor of x_2 on Q_2.

Case 2.1. $w_1w_2 \in E(Q)$.

If the length of C is at least 6, then there is a path in C starting from w_1 of length at least 4 not passing through w_2. Let $w_1yy'y''y'''$ be such a path. Then the subgraph formed by $w_1x_1x_1'$, $w_1w_2x_2$ and $w_1yy'y''y'''$ is an S. So we assume that the length of C is at most 5.

We first assume that the length of C is 5. Let $C = y_1w_1w_2y_2zy_1$. If $z \in V(Q)$, then $V(C) \cap V(D)$ will contain either y_1 or y_2 (but not both). Without loss of generality, we assume that $y_1 \in V(D)$, and let y_1' be a neighbor of y_1 in D. Then the subgraph formed by $w_2x_2x_2'$, $w_2w_1x_1$ and $w_2y_2zy_1y_1'$ is an S.

Now we assume that $z \in V(D)$. We claim that either $u_1 = y_1$ or $u_2 = y_2$; otherwise $u_1 = w_1$ and $u_2 = w_2$ will be adjacent in T. Without loss of generality, we assume that $u_1 = y_1$. Let u_1' be a neighbor of u_1 on T other than w_1 (u_1' exists since the degree of u_1 in T is even). Since H is triangle-free, $u_1' \neq w_2, y_2$. If $u_1' \in V(Q_1 \cup Q_2)\backslash\{w_1, w_2, x_2\}$, then there will be a path from u_1 to u_2 in T longer than Q, a contradiction. If $u_1' = x_2$, then $x_2 \neq v_2'$ and $x_2' \neq v_2$. Let x_2'' be the predecessor of x_2' on Q_2. Then the subgraph formed by $w_1x_1x_1'$, $w_1w_2y_2$ and $w_1u_1x_2x_2'x_2''$ is an S. Now we assume that $u_1' \notin V(Q_1 \cup Q_2 \cup C)$. Then the subgraph formed by $w_2x_2x_2'$, $w_2w_1x_1$ and $w_2y_2zu_1u_1'$ is an S.

For the final subcase, we assume that the length of C is 4. This implies that the length of P and Q are both 2. Let $C = y_1w_1w_2y_2y_1$. Since $w_1, w_2 \notin V(D)$, without loss of generality, we assume that $y_2 \in V(D)$, which implies that $u_1 = y_1$ and $u_2 = w_2$. Let y_2' be a neighbor of y_2 in D. Let u_1' be a neighbor of u_1 in T other than w_1. Since H is triangle-free, $u_1' \neq w_2$. If $u_1' \in V(Q_1 \cup Q_2)\backslash\{w_1, w_2, x_2\}$, then there will be a path from u_1 to u_2 in T longer than Q, a contradiction. If $u_1' = x_2$, then $x_2 \neq v_2'$ and $x_2' \neq v_2$. Let x_2'' be the predecessor of x_2' on Q_2. Then the subgraph formed by $w_1x_1x_1'$, $w_1w_2y_1$ and $w_1u_1x_2x_2'x_2''$ is an S. Thus we assume that $u_1' \notin V(Q_1 \cup Q_2 \cup C)$.

Let u_1'' be a neighbor of u_1' in T other than u_1. Since H is triangle-free, $u_1'' \neq w_1$. If $u_1' \in V(Q_1 \cup Q_2)\backslash\{w_1, w_2\}$, then there will be a path from u_1 to u_2 in T longer than Q, a contradiction. If $u_1'' = w_2$, then the subgraph formed by $w_2x_2x_2'$, $w_2w_1x_1$ and $w_2u_1'u_1y_2y_2'$ is an S. Now we assume that $u_1' \notin V(Q_1 \cup Q_2 \cup C)$. Then the subgraph formed by $w_2x_2x_2'$, $w_2w_1x_1$ and $w_2y_2u_1u_1'u_1''$ is an S.

Case 2.2. $w_1w_2 \notin E(Q)$.

In this case, w_1 and w_2 divide C into two subpaths (from w_1 to w_2), say R_1 and R_2. Clearly the lengths of R_1 and R_2 are both at least 2. Without loss of generality, we assume that the length of R_1 is no less than that of R_2. If the length of C is at least 5, then the length of R_1 will be at least 3.

In this case, there is a path in R_1 from w_1 of length at least 2 not passing through w_2, and there is a path in $R_2w_2Q_2$ from w_1 of length at least 4. Let w_1yy' and $w_1zz'z''z'''$ be two such paths. Then the subgraph formed by $w_1x_1x_1'$, w_1yy' and $w_1zz'z''z'''$ is an S.

Finally, we assume that the length of C is 4. Let $C = u_1yu_2zu_1$, where $y \in V(D)$, and let y' be a neighbor of y in D. This implies that $w_1 = u_1$ and $w_2 = u_2$. Thus the subgraph formed by $w_1x_1x_1'$, w_1yy' and $w_1zw_2x_2x_2'$ is an S.

This completes the proof.

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