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On scores, losing scores and total scores in k-hypertournaments

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Abstract

A k-hypertournament is a complete k-hypergraph with each k-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the edge. In a k-hypertournament, the score s_i (losing score r_i) of a vertex v_i is the number of arcs containing v_i in which v_i is not the last element (in which v_i is the last element). The total score of v_i is defined as $t_i = s_i - r_i$. In this paper we obtain stronger inequalities for the quantities $\sum_{i\in I} r_i$, $\sum_{i\in I} s_i$ and $\sum_{i\in I} t_i$, where $I \subseteq \{1, 2, \ldots, n\}$. Furthermore, we discuss the case of equality for these inequalities. We also characterize total score sequences of strong k-hypertournaments.

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1. Introduction

A tournament is a complete oriented graph. In a tournament the score of a vertex is its out-degree and the sequence of scores listed in non-decreasing order is called the score sequence. Landau's theorem [10] gives a necessary and sufficient condition for a sequence of non-negative

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integers to be the score sequence of some tournament. More results on tournament scores can be found in [7, 13, 14, 15, 16, 17, 19]

A k-hypergraph is a pair $H = (V, E)$, where V is the set of vertices and E is the set of k-subsets of V, called edges [2]. Hypertournaments are generalizations of tournaments. Given two non-negative integers n and k with $n \geq k > 1$, a k-hypertournament on n vertices is a pair (V, A), where V is the set of vertices with $|V| = n$, and A is the set of k-tuples of vertices, called arcs, such that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. Several authors have generalized concepts and results from tournaments to hypertournaments. The recent work on reconstruction of complete interval tournaments due to Ivanyi [5, 6] can be extended to hypertournaments. The concept of kings in tournaments has been introduced in hypertournaments by Brcanov and Petrovic [4], but is still in its infancy. Zhou et al. [24] extended the concept of scores in tournaments to that of scores and losing scores in hypertournaments, and derived a result analogous to Landau's theorem on tournaments. The score $s(v_i)$ or s_i of a vertex v_i is the number of arcs containing v_i in which v_i is not the last element, and the losing score $r(v_i)$ or r_i of a vertex v_i is the number of arcs containing v_i in which v_i is the last element. The score sequence (losing score sequence) is formed by listing the scores (losing scores) in non-decreasing order.

The following characterizations of score sequences and losing score sequences of k-hypertournaments can be found in [24].

Proposition 1.1. *Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $R = [r_1, r_2, \cdots, r_n]$ of non-negative integers is a losing score sequence of some k-hypertournament if and only if for each $1 \leq j \leq n$,*

$$\sum_{i=1}^{j} r_i \geq \binom{j}{k},$$

with equality when $j = n$.

Proposition 1.2. *Given non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $S = [s_1, s_2, \cdots, s_n]$ of non-negative integers is a score sequence of some k-hypertournament if and only if for each $1 \leq j \leq n$,*

$$\sum_{i=1}^{j} s_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k},$$

with equality when $j = n$.

Bang and Sharp [1] proved Landau's theorem using Hall's theorem on a system of distinct representatives of a collection of sets. Based on Bang and Sharp's ideas, Koh and Ree [9] have given a different proof of Proposition 1.1 and 1.2. Some more results on scores of k-hypertournaments can be found in [8, 11, 12, 18, 20, 21, 22, 23].

Brualdi and Shen [3] have strengthened the inequalities on scores in a tournament. In section 2 we extend their results to losing scores and scores in k-hypertournaments and obtain bounds for $\sum_{i\in I} r_i$, $\sum_{i\in I} s_i$ that are stronger than those given in Proposition 1.1 and 1.2 We discuss the case of equality for several inequalities derived in this section.

The total score of vertex v_i is defined as $t_i = s_i - r_i$. The total score sequence is the sequence of total scores arranged in non-increasing order. Koh and Ree [9] characterized total score sequences in k-hypertournaments.

Proposition 1.3. *A non-increasing sequence of integers $t_1 \geq t_2 \geq \cdots \geq t_n$ is a total score sequence of a k-hypertournament of order n if and only if t_i has the same parity as that of $\binom{n-1}{k-1}$ for each $i = 1, 2, \cdots n$,*

$$\sum_{i=1}^{j} t_i \leq j\binom{n-1}{k-1} - 2\binom{j}{k}$$

with equality when $j = n$.

In Section 3, we improve on the bounds for total scores given by Proposition 1.3. Moreover we give necessary and sufficient conditions for a non-increasing sequence of integers to be the total score sequence of a strong k-hypertournament.

We adopt standard notations. The set of first n positive integers is denoted by $[n]$, $|I|$ stands for the cardinality of set I and $[x_i]_1^n$ represents an n-term sequence.

2. Stronger inequalities on losing scores and scores

Brualdi and Shen [3] obtained stronger bounds for scores in tournaments, which indeed give better necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of a tournament. The bounds obtained in Theorems 2.1, 2.2 and 2.3 are generalizations of those on tournament scores given in [3]. The following result gives a lower bound for $\sum_{i\in I} r_i$, where $I \subseteq [n]$.

Theorem 2.1. *Given two non-negative integers n and k with $n \geq k > 1$, a sequence $R = [r_i]_1^n$ of non-negative integers in non-decreasing order is a losing score sequence of some k-hypertournament if and only if for every subset $I \subseteq [n]$,*

$$\sum_{i\in I} r_i \geq \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{|I|}{k}, \tag{1}$$

with equality when $|I| = n$.

Proof. Sufficiency. Let the sequence $R = [r_i]_1^n$ of non-negative integers satisfy conditions (1). For any subset $I \subseteq [n]$, we have

$$\sum_{i\in I}\binom{i-1}{k-1} \geq \sum_{i=1}^{|I|}\binom{i-1}{k-1} = \sum_{i=1}^{|I|}\left[\binom{i}{k} - \binom{i-1}{k}\right] = \binom{|I|}{k}.$$

Therefore, from conditions (1) we have

$$\sum_{i\in I} r_i \geq \frac{1}{2}\binom{|I|}{k} + \frac{1}{2}\binom{|I|}{k} = \binom{|I|}{k}.$$

Hence, by Proposition 1.1, R is a losing score sequence.

Necessity. Assume $R = [r_i]_1^n$ is a losing score sequence of some k-hypertournament. For any subset $I \subseteq [n]$, define

$$f(I) = \sum_{i\in I} r_i - \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} - \frac{1}{2}\binom{|I|}{k}.$$

Suppose among all I minimizing $f(I)$, we select one that minimizes $|I|$. We claim that $I = \{i : 1 \leq i \leq |I|\}$. If not, then there exist $i_0 \notin I$ and $j \in I$ such that $j = i_0 + 1$. So $r_{i_0} \leq r_j$ and we have

$$\begin{aligned} f(I) &= \sum_{t\in I} r_t - \frac{1}{2}\sum_{t\in I}\binom{t-1}{k-1} - \frac{1}{2}\binom{|I|}{k} \\ &= \sum_{t\in I, t\neq j} r_t + r_j - \frac{1}{2}\left[\sum_{t\in I, t\neq j}\binom{t-1}{k-1} + \binom{j-1}{k-1}\right] - \frac{1}{2}\binom{|I|}{k}. \end{aligned}$$

Therefore,

$$f(I) - f(I - \{j\}) = r_j - \frac{1}{2}\binom{j-1}{k-1} - \frac{1}{2}\binom{|I|-1}{k-1}.$$

By the choice of I we have $f(I) - f(I - \{j\}) < 0$. Therefore

$$r_j - \frac{1}{2}\binom{j-1}{k-1} - \frac{1}{2}\binom{|I|-1}{k-1} < 0,$$

or

$$r_j < \frac{1}{2}\binom{j-1}{k-1} + \frac{1}{2}\binom{|I|-1}{k-1}.$$

Again,

$$f(I \cup \{i_0\}) = \sum_{t\in I} r_t + r_{i_0} - \frac{1}{2}\left[\sum_{t\in I}\binom{t-1}{k-1} + \binom{i_0-1}{k-1}\right] - \frac{1}{2}\binom{|I|+1}{k}.$$

So,

$$f(I \cup \{i_0\}) - f(I) = r_{i_0} - \frac{1}{2}\binom{i_0-1}{k-1} - \frac{1}{2}\binom{|I|}{k-1}.$$

Since $f(I \cup \{i_0\}) - f(I) \geq 0$, therefore,

$$r_{i_0} - \frac{1}{2}\binom{i_0-1}{k-1} - \frac{1}{2}\binom{|I|}{k-1} \geq 0,$$

or

$$r_{i_0} \geq \frac{1}{2}\binom{i_0-1}{k-1} + \frac{1}{2}\binom{|I|}{k-1}.$$

Hence,

$$\frac{1}{2}\binom{i_0-1}{k-1} + \frac{1}{2}\binom{|I|}{k-1} \leq r_{i_0} \leq r_j < \frac{1}{2}\binom{j-1}{k-1} + \frac{1}{2}\binom{|I|-1}{k-1},$$

or

$$\binom{i_0-1}{k-1} + \binom{|I|}{k-1} < \binom{i_0}{k-1} + \binom{|I|-1}{k-1}$$

since $j = i_0 + 1$, or

$$\binom{|I|}{k-1} - \binom{|I|-1}{k-1} < \binom{i_0}{k-1} - \binom{i_0-1}{k-1},$$

or

$$\binom{|I|-1}{k-2} < \binom{i_0-1}{k-2},$$

which is a contradiction, and the claim is proved.
Hence,

$$\begin{aligned} f(I) &= \sum_{i=1}^{|I|} r_i - \frac{1}{2}\sum_{i=1}^{|I|}\binom{i-1}{k-1} - \frac{1}{2}\binom{|I|}{k} \\ &= \sum_{i=1}^{|I|} r_i - \frac{1}{2}\binom{|I|}{k} - \frac{1}{2}\binom{|I|}{k} \geq \binom{|I|}{k} - \binom{|I|}{k} \qquad \text{(by Proposition 1.1)} \\ &= 0. \end{aligned}$$

Thus,

$$\sum_{i\in I} r_i - \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} - \frac{1}{2}\binom{|I|}{k} \geq 0,$$

or

$$\sum_{i\in I} r_i \geq \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{|I|}{k},$$

which proves the necessity as I has been chosen to minimize $f(I)$. $\square$

Since $\sum_{i\in I}\binom{i-1}{k-1} \geq \sum_{i=1}^{j}\binom{i-1}{k-1} = \binom{j}{k}$, the lower bounds proved in Theorem 2.1 are individually stronger than the ones given in Proposition 1.1. However, as a whole Theorem 1.1 is equivalent to Proposition 1.1. The next result gives a set of upper bounds for $\sum_{i\in I} r_i$.

Theorem 2.2. *Given two non-negative integers n and k with $n \geq k > 1$, a sequence $R = [r_i]_1^n$ of non-negative integers in non-decreasing order is a losing score sequence of some k-hypertournament if and only if for any subset $I \subseteq [n]$,*

$$\sum_{i\in I} r_i \leq \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{n}{k} - \frac{1}{2}\binom{n-|I|}{k},$$

with equality when $|I| = n$.

Proof. Let $J = [n] - I$, so that $I \cup J = [n]$ and $|J| + |I| = n$. Then, by Theorem 2.1, R is a losing score sequence if and only if

$$\sum_{i\in[n]} r_i = \binom{n}{k} \quad and \quad \sum_{i\in J} r_i \geq \frac{1}{2}\sum_{i\in J}\binom{i-1}{k-1} + \frac{1}{2}\binom{|J|}{k}$$

or equivalently if

$$\sum_{i\in I} r_i + \sum_{i\in J} r_i = \binom{n}{k} \quad and \quad \sum_{i\in J} r_i \geq \frac{1}{2}\sum_{i\in J}\binom{i-1}{k-1} + \frac{1}{2}\binom{|J|}{k}$$

or equivalently if

$$\sum_{i\in I} r_i = \binom{n}{k} - \sum_{i\in J} r_i \leq \binom{n}{k} - \frac{1}{2}\sum_{i\in J}\binom{i-1}{k-1} - \frac{1}{2}\binom{|J|}{k}$$

$$= \binom{n}{k} - \frac{1}{2}\left[\binom{n}{k} - \sum_{i\in I}\binom{i-1}{k-1}\right] - \frac{1}{2}\binom{n-|I|}{k},$$

because $\sum_{i\in I}\binom{i-1}{k-1} + \sum_{i\in J}\binom{i-1}{k-1} = \binom{n}{k}$ and $|I| + |J| = n$.

Hence,

$$\sum_{i\in I} r_i \leq \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{n}{k} - \frac{1}{2}\binom{n-|I|}{k},$$

which proves the result. □

The next result follows from Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Let n and k be two non-negative integers with $n \geq k > 1$. If $R = [r_i]_1^n$ is a losing score sequence of a k-hypertournament, then for each $1 \leq i \leq n$ we have*

$$\frac{1}{2}\binom{i-1}{k-1} \leq r_i \leq \frac{1}{2}\binom{i-1}{k-1} + \frac{1}{2}\binom{n-1}{k-1}.$$

Proof. Let $I = \{i\}$ in Theorem 2.1 and Theorem 2.2. Then,

$$\sum_{i\in I} r_i \geq \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{|I|}{k}$$

implies that

$$r_i \geq \frac{1}{2}\binom{i-1}{k-1} + \frac{1}{2}\binom{1}{k} = \frac{1}{2}\binom{i-1}{k-1},$$

and

$$\sum_{i\in I} r_i \leq \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{n}{k} - \frac{1}{2}\binom{n-|I|}{k}$$

implies that

$$r_i \leq \frac{1}{2}\binom{i-1}{k-1} + \frac{1}{2}\binom{n}{k} - \frac{1}{2}\binom{n-1}{k} = \frac{1}{2}\binom{i-1}{k-1} + \frac{1}{2}\binom{n-1}{k-1}.$$

Therefore,

$$\frac{1}{2}\binom{i-1}{k-1} \leq r_i \leq \frac{1}{2}\binom{i-1}{k-1} + \frac{1}{2}\binom{n-1}{k-1},$$

completing the proof. □

Since $s_{n+1-i} + r_i = \binom{n-1}{k-1}$, for $I \subseteq [n] = \{1, 2, ..., n\}$, we have $\sum_{i\in I} s_{n+1-i} + \sum_{i\in I} r_i = \sum_{i\in I}\binom{n-1}{k-1}$ or $\sum_{i\in I} s_{n+1-i} = |I|\binom{n-1}{k-1} - \sum_{i\in I} r_i$. Hence by using Theorem 2.1 and Theorem 2.2, we obtain respectively the following results.

Lemma 2.1. *Given two non-negative integers n and k with $n \geq k > 1$, a sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of some k-hypertournament if and only if for every subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i\in I} s_{n+1-i} \leq |I|\binom{n-1}{k-1} - \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} - \frac{1}{2}\binom{|I|}{k}$$

with equality when $|I| = n$.

Lemma 2.2. *Given two non-negative integers n and k with $n \geq k > 1$, a sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of some k-hypertournament if and only if for every subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i\in I} s_{n+1-i} \geq |I|\binom{n-1}{k-1} - \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} - \frac{1}{2}\binom{n}{k} + \frac{1}{2}\binom{n-|I|}{k}$$

with equality when $|I| = n$.

The following is the consequence of Lemma 2.4 and 2.5.

Theorem 2.4. *Let n and k be two non-negative integers with $n \geq k > 1$. If $S = [s_i]_1^n$ is a score sequence of a k-hypertournament, then for each $1 \leq i \leq n$ we have*

$$\frac{1}{2}\binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1} \leq s_{n+1-i} \leq \binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1}.$$

Proof. Let $I = \{i\}$ in Lemma 2.4 and 2.5. Then,

$$\sum_{i\in I} s_{n+1-i} \leq |I|\binom{n-1}{k-1} - \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} - \frac{1}{2}\binom{|I|}{k}$$

implies that

$$s_{n+1-i} \leq \binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1},$$

and

$$\sum_{i\in I} s_{n+1-i} \geq |I|\binom{n-1}{k-1} - \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} - \frac{1}{2}\binom{n}{k} + \frac{1}{2}\binom{n-|I|}{k}$$

implies that

$$s_{n+1-i} \geq \binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1} - \frac{1}{2}\binom{n}{k} + \frac{1}{2}\binom{n-1}{k},$$

or

$$s_{n+1-i} \geq \binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1} - \frac{1}{2}\binom{n-1}{k-1},$$

or

$$s_{n+1-i} \geq \frac{1}{2}\binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1}.$$

Therefore,

$$\frac{1}{2}\binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1} \leq s_{n+1-i} \leq \binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1}.$$

□

3. Total scores and strong hypertournaments

Let $[s_i]_1^n$ in non-increasing order and $[r_i]_1^n$ in non-decreasing order be respectively the score and losing score sequences of a k-hypertournament. The total score t_i of a vertex v_i is defined as $t_i = s_i - r_i$. So $T = [t_i]_1^n$, called the total score sequence, is a non-increasing sequence of integers. Using the improved bounds for scores and losing scores proved earlier we now derive stronger upper and lower bounds for total scores in hypertournaments.

Theorem 3.1. *A non-increasing sequence of integers $T = [t_i]_1^n$ is a total score sequence of a k-hypertournament of order n, with $n \geq k > 1$, if and only if t_i has the same parity as $\binom{n-1}{k-1}$ for each $i = 1, 2, \ldots n$, and for every $I \subseteq [n]$*

$$|I|\binom{n-1}{k-1} - \sum_{i\in I}\binom{i-1}{k-1} - \binom{n}{k} + \binom{n-|I|}{k} \leq \sum_{i\in I} t_i \tag{2}$$

$$\leq |I|\binom{n-1}{k-1} - \sum_{i\in I}\binom{i-1}{k-1} - \binom{|I|}{k}$$

with equality when $I = [n]$.

Proof. Suppose $t_1 \geq t_2 \geq \cdots \geq t_n$ is the total score sequence of a k-hypertournament H of order n. Then, there exist score and losing score sequences $s_1 \geq s_2 \geq \cdots \geq s_n$ and $r_1 \leq r_2 \leq \cdots \leq r_n$ with $t_i = s_i - r_i$. Therefore, Theorems 2.1, 2.2 and Lemma 2.4, 2.5 together imply conditions (2). Furthermore, since $t_i = s_i - r_i$ and $\binom{n-1}{k-1} = s_i + r_i$, therefore $t_i = \binom{n-1}{k-1} - 2r_i$ has the same parity as $\binom{n-1}{k-1}$ for $i = 1, 2, \ldots, n$.
For the converse, suppose that a non-increasing sequence of integers $t_1 \geq t_2 \geq \cdots \geq t_n$ satisfies conditions (2). For each $i = 1, 2, \ldots, n$, define

$$s_i = \frac{1}{2}\left[\binom{n-1}{k-1} + t_i\right] \text{ and } r_i = \frac{1}{2}\left[\binom{n-1}{k-1} - t_i\right]. \tag{3}$$

Then, $t_1 \leq \binom{n-1}{k-1}$, and

$$\begin{aligned} t_n &= \sum_{i\in[n]} t_i - \sum_{i\in[n-1]} t_i \\ &\geq \left[n\binom{n-1}{k-1} - \sum_{i\in[n]}\binom{i-1}{k-1} - \binom{n}{k}\right] \\ &\quad - \left[(n-1)\binom{n-1}{k-1} - \sum_{i\in[n-1]}\binom{i-1}{k-1} - \binom{n-1}{k}\right] \\ &= -\binom{n-1}{k-1}. \end{aligned}$$

So,

$$-\binom{n-1}{k-1} \leq t_i \leq \binom{n-1}{k-1}$$

and hence $s_i \geq 0$ and $r_i \geq 0$ for all i. The sequence $[s_i]_1^n$ is non-increasing. So, for $1 \leq j \leq n$,

$$\sum_{i\in I} s_i = \sum_{i\in I}\frac{1}{2}\left[\binom{n-1}{k-1} + t_i\right]$$

$$\leq \frac{1}{2}\left[|I|\binom{n-1}{k-1} + |I|\binom{n-1}{k-1} - \sum_{i\in I}\binom{i-1}{k-1} - \binom{|I|}{k}\right]$$

$$= |I|\binom{n-1}{k-1} - \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} - \frac{1}{2}\binom{|I|}{k}.$$

If we arrange $[s_i]_1^n$ in non-decreasing order, then

$$\sum_{i\in I} s_{n+1-i} \leq |I|\binom{n-1}{k-1} - \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} - \frac{1}{2}\binom{|I|}{k}.$$

Similarly for the non-decreasing sequence $[r_i]_1^n$,

$$\sum_{i\in I} r_i = \frac{1}{2}\left[|I|\binom{n-1}{k-1} - \sum_{i\in I} t_i\right]$$

$$\geq \frac{1}{2}\left[|I|\binom{n-1}{k-1} - |I|\binom{n-1}{k-1} + \sum_{i\in I}\binom{i-1}{k-1} + \binom{|I|}{k}\right]$$

$$= \frac{1}{2}\sum_{i\in I}\binom{i-1}{k-1} + \frac{1}{2}\binom{|I|}{k}.$$

Thus, by Lemma 2.4 and Theorem 2.1, there exists a k-hypertournament with $[s_i]_1^n$ and $[r_i]_1^n$ as its score and losing score sequences,and hence $t_i = s_i - r_i$, for $i = 1, 2, \ldots, n$ as its total scores. □

Theorem 3.2. *If a non-increasing sequence of integers $T = [t_i]_1^n$ is a total score sequence of a k-hypertournament of order n, with $n \geq k > 1$, then for $1 \leq i \leq n$,*

$$-\binom{i-1}{k-1} \leq t_i \leq \binom{n-1}{k-1} - \binom{i-1}{k-1}$$

Proof. If the scores are arranged in non-increasing order, then from Theorem 2.7

$$\frac{1}{2}\binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1} \leq s_i \leq \binom{n-1}{k-1} - \frac{1}{2}\binom{i-1}{k-1}.$$

This together with Theorem 2.3 proves the result. □

Alternatively, we can prove Theorem 3.2 by substituting $I = \{i\}$ in Theorem 3.1.

An (x, y)-path in a k-hypertournament H is a sequence

$$(x =)v_1 e_1 v_2 e_2 v_3 \cdots v_{t-1} e_{t-1} v_t (= y)$$

of distinct vertices $v_1, v_2, \cdots, v_t$, $t \geq 1$ and distinct arcs $e_1, e_2, \cdots, e_{t-1}$ such that v_{i+1} lies on the last entry in e_i, $1 \leq i \leq t-1$. A k-hypertournament H is strong if for any two vertices $x \in V$ and $y \in V$, H contains both an (x, y)-path and a (y, x)-path. A strong component of a k-hypertournament H is a maximal strong subhypertournament of H. The following results due to Zhou et al. [24] characterizes the score and losing score sequences of a strong k-hypertournament.

Theorem 3.3. *A sequence $R = [r_i]_1^n$ of non-negative integers in non-decreasing order is a losing score sequence of a strong k-hypertournament with $n > k$ if and only if*

$$\sum_{i=1}^{j} r_i > \binom{j}{k}, \text{ for } k \leq j \leq n-1 \text{ and } \sum_{i=1}^{n} r_i = \binom{n}{k}. \tag{4}$$

Theorem 3.4. *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a strong k-hypertournament with $n > k$ if and only if*

$$\sum_{i=1}^{j} s_i > j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k}, \text{ for } k \leq j \leq n-1 \text{ and } \sum_{i=1}^{n} s_i = (k-1)\binom{n}{k}.$$

Rearranging the score sequence in non-increasing order, $s_1 \geq s_2 \geq \cdots \geq s_n$, we obtain

$$\sum_{i=1}^{j} s_i < j\binom{n-1}{k-1} - \binom{j}{k}, \text{ for } k \leq j \leq n-1 \text{ and } \sum_{i=1}^{n} s_i = (k-1)\binom{n}{k}. \tag{5}$$

We conclude this section by characterizing total score sequences of strong k-hypertournaments.

Theorem 3.5. *A non-increasing sequence of integers $t_1 \geq t_2 \geq \cdots \geq t_n$ is a total score sequence of a strong k-hypertournament of order n with $n > k$ if and only if t_i has the same parity as that of $\binom{n-1}{k-1}$ for each $i = 1, 2, \cdots n$,*

$$\sum_{i=1}^{j} t_i < j\binom{n-1}{k-1} - 2\binom{j}{k}, \text{ for } 1 \leq j \leq n-1 \text{ and } \sum_{i=1}^{n} t_i = (k-2)\binom{n}{k}. \tag{6}$$

Proof. The proof of necessity is on the same lines as the proof of Theorem 3.1. Here the necessity follows from conditions (4) and (5). For the proof of sufficiency, suppose that a non-increasing sequence of integers $t_1 \geq t_2 \geq \cdots \geq t_n$ satisfies conditions (6). For each $i = 1, 2, \cdots, n$, define s_i and r_i as in (3), then $t_1 < \binom{n-1}{k-1}$, and

$$t_n = \sum_{i=1}^{n} t_i - \sum_{i=1}^{n-1} t_i > (k-2)\binom{n}{k} - (n-1)\binom{n-1}{k-1} + 2\binom{n-1}{k} = -\binom{n-1}{k-1}.$$

So, $-\binom{n-1}{k-1} < t_i < \binom{n-1}{k-1}$ and hence $s_i > 0$ and $r_i > 0$ for all i. The sequence $[s_i]_1^n$ and $[r_i]_1^n$ are respectively non-increasing and non-decreasing. So, for $1 \leq j \leq n-1$,

$$\sum_{i=1}^{j} s_i = \sum_{i=1}^{j} \frac{1}{2}\left[\binom{n-1}{k-1} + t_i\right]$$

$$< \frac{1}{2}\left[j\binom{n-1}{k-1} + j\binom{n-1}{k-1} - 2\binom{j}{k}\right]$$

$$= j\binom{n-1}{k-1} - \binom{j}{k}$$

and likewise

$$\sum_{i=1}^{j} r_i = \frac{j}{2}\binom{n-1}{k-1} - \frac{1}{2}\sum_{i=1}^{j} t_i > \binom{j}{k},$$

with $\sum_{i=1}^{n} s_i = (k-1)\binom{n}{k}$ and $\sum_{i=1}^{n} r_i = \binom{n}{k}$.
Thus, by conditions (4) and (5), there exists a strong k-hypertournament with $[s_i]_1^n$ and $[r_i]_1^n$ as its score and losing score sequences, and hence $t_i = s_i - r_i$, for $i = 1, 2, \cdots, n$ as its total scores. □

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