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Bounds on weak and strong total domination number in graphs

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Abstract

A set D of vertices in a graph $G = (V, E)$ is a total dominating set if every vertex of G is adjacent to some vertex in D. A total dominating set D of G is said to be weak if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$. The weak total domination number $\gamma_{wt}(G)$ of G is the minimum cardinality of a weak total dominating set of G. A total dominating set D of G is said to be strong if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \leq d_G(u)$. The strong total domination number $\gamma_{st}(G)$ of G is the minimum cardinality of a strong total dominating set of G. We present some bounds on weak and strong total domination number of a graph.

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1. Introduction

We consider finite, undirected, simple graphs. Let G be a graph, with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. By $G[S]$ we denote the *subgraph* induced by the vertices of S. If v is a vertex of V, then the *degree* of v denoted by $d_G(v)$,

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is the cardinality of its open neighborhood. By $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *maximum* and *minimum degree* of a graph G, respectively. A *star* $K_{1,n}$ is a tree of order $n+1$ with at least n vertices of degree 1. A tree T is a *double star* if it contains exactly two vertices that are not leaves. We denote by $S(a, b)$ a double star in which one of the centers has degree a and the other center has degree b. The *corona* $cor(G)$ of a graph G is a graph obtained from G by attaching a leaf to every vertex of G.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S and is a *total dominating set* (td-set) if every vertex in V has a neighbor in S. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (respectively, total dominating set) of G. Total domination was introduced by Cockayne, Dawes and Hedetniemi [4]. Note that every graph without isolated vertices has a td-set, since $V(G)$ is such a set. In [14], Sampathkumar and Pushpa Latha have introduced the concept of weak and strong domination in graphs. A subset $D \subseteq V$ is a *weak dominating set* (wd-set) if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$, where $d_G(v) \geq d_G(u)$. The subset D is a *strong dominating set* (sd-set) if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$, where $d_G(u) \geq d_G(v)$. The *weak* (*strong*, respectively) *domination number* $\gamma_w(G)$ ($\gamma_s(G)$, respectively) is the minimum cardinality of a wd-set (an sd-set, respectively) of G. Strong and weak domination have been studied for example in [5, 6, 9, 10, 12, 13]. For more details on domination in graphs and its variations, see the two books [7, 8].

A large part of extremal graph theory studies the extremal values of graph parameters on families of graphs. Results of *Nordhaus-Gaddum* type study the extremal values of the sum (or product) of a parameter on a graph and its complement, following the classic paper of Nordhaus and Gaddum [11] solving these problems for the chromatic number on n-vertex graphs.

Chellali et al. [3] have introduced the concept of weak total domination in graphs. A total dominating set D of G is said to be *weak* if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$. The *weak total domination number* $\gamma_{wt}(G)$ of G is the minimum cardinality of a weak total dominating set of G.

Problem 1.1 (Chellali et al. [3])**.** *(1) Can you bound* $\gamma_{wt}(G) + \gamma_{wt}(\overline{G})$*?*
(2) What can you say about strong total domination?

The concept *strong total domination* can be defined analogously. A total dominating set D of G is said to be *strong* if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \leq d_G(u)$. The *strong total domination number* $\gamma_{st}(G)$ of G is the minimum cardinality of a strong total dominating set of G. We obtain Nordhause- Gaddum type bounds for weak total domination number as well as for strong total domination number of a graph. We also present sharp upper and lower bounds for the strong total domination number of a tree in terms of order and the number of leaves and support vertices. We abbreviate a weak total dominating set of G as wtd-set, and a strong total dominating set of G as std-set. A wtd-set of minimum cardinality is called a $\gamma_{wt}(G)$-set, and a std-set of minimum cardinality is called a $\gamma_{st}(G)$-set.

2. Useful results

In this section we state some useful results that we need for the next. We begin with the following observation of [7].

Observation 2.1. *For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.*

Proposition 2.1 (Chellali et al. [3])**.** *For any graph G of order n, minimum degree δ and with no isolated vertices, $\gamma_{wt}(G) \leq n + 1 - \delta$.*

The *generalized corona* G^* of a graph $G = (V, E)$ as the graph that is obtained by attaching one or more leaves to each vertex $v \in V$.

Proposition 2.2 (Chellali et al. [3])**.** *A connected graph G of order $n \geq 2$ has $\gamma_{wt}(G) = n$ if and only if G is obtained from a generalized corona of a graph H by adding a set of vertices A (possibly empty) attached to vertices of H so that each vertex of A has degree less than the degree of its neighbors.*

As a consequent, we have the following.

Corollary 2.1. *If $G \neq K_2$ is a graph of order n and $\gamma_{wt}(G) = n$ then G has at least two leaves.*

We next state some useful results on the total domination number of a graph.

Theorem 2.1 (Cockayne et al. [4])**.** *If G has n vertices, no isolates, and $\Delta(G) < n - 1$, then $\gamma_t(G) + \gamma_t(\overline{G}) \leq n + 2$, with equality if and only if G or $\overline{G} = mK_2$.*

Theorem 2.2 (Chellali and Haynes [2])**.** *If T is a nontrivial tree of order $n \geq 3$ and with s support vertices, then $\gamma_t(T) \leq \frac{n+s}{2}$.*

Theorem 2.3 (Chellali and Haynes [1])**.** *If T is a nontrivial tree of order n and with l leaves, then $\gamma_t(T) \geq \frac{n+2-l}{2}$.*

3. Results

We first present Nordhaus-Gaddum type bounds for the weak total domination number as well as strong total domination number, and then present sharp bounds on the strong total domination number in trees.

3.1. Nordhaus-Gaddum type bounds

We obtain sharp upper and lower bounds for $\gamma_{wt}(G) + \gamma_{wt}(\overline{G})$, $\gamma_{wt}(G)\gamma_{wt}(\overline{G})$, $\gamma_{st}(G) + \gamma_{st}(\overline{G})$ and $\gamma_{st}(G)\gamma_{st}(\overline{G})$.

Theorem 3.1. *Let G be a graph of order n. If G and $\overline{G}$ have no isolated vertex then $\gamma_{wt}(G) + \gamma_{wt}(\overline{G}) \leq 2n$. Furthermore, the equality holds if and only if $G = \overline{G} = P_4$.*

Proof. The upper bound is obviously verified, thus we prove the equality part. Assume that $\gamma_{wt}(G) + \gamma_{wt}(\overline{G}) = 2n$. Then $\gamma_{wt}(G) = \gamma_{wt}(\overline{G}) = n$. Since G and $\overline{G}$ have no isolated vertex, we have $\delta(G) \geq 1$, $\delta(\overline{G}) \geq 1$, $\Delta(G) \leq n - 2$, and $\Delta(\overline{G}) \leq n - 2$. By Proposition 2.2, G is obtained from a generalized corona of a graph H by adding a set of vertices A (possibly empty) attached to vertices of H so that each vertex of A has degree less than the degree of its neighbors. Thus $\delta(G) = \delta(\overline{G}) = 1$. If $|V(H)| = 1$ then $\overline{G}$ has an isolated vertex, and if $|V(H)| \geq 3$ then $\delta(\overline{G}) \geq 2$, both of which is a contradiction. Thus $|V(H)| = 2$. Let $V(H) = \{a, b\}$. If a is not adjacent to b then $\delta(\overline{G}) \geq 2$, a contradiction. Thus a is adjacent to b. If both a and b are strong support vertices, then $\delta(\overline{G}) \geq 2$, a contradiction. Thus we may assume, without loss of generality, that there is precisely one leaf adjacent to b. Then b has no neighbor in A. Consequently, $\deg(b) = 2$. Since A is an independent set, we find that G is a double star with centers a and b. If $\deg(a) \geq 2$, then $\overline{G}$ has precisely one leaf, contradicting Corollary 2.1. Thus $\deg(a) = 2$. Consequently $G = P_4$. □

Theorem 3.2. *Let G and $\overline{G}$ be graphs of order n with no isolated vertex. Then $\gamma_{wt}(G)+\gamma_{wt}(\overline{G}) = 2n-1$ if and only if G or $\overline{G}$ is $cor(C_3)$, or is obtained from a star $K_{1,3}$ by subdividing one edge.*

Proof. Since G and $\overline{G}$ have no isolated vertex, we have $\delta(G) \geq 1$, $\delta(\overline{G}) \geq 1$, $\Delta(G) \leq n-2$, and $\Delta(\overline{G}) \leq n-2$. Without loss of generality we may assume that $\gamma_{wt}(G) = n$ and $\gamma_{wt}(\overline{G}) = n-1$. By Proposition 2.2, G is obtained from a generalized corona of a graph H by adding a set of vertices A (possibly empty) attached to vertices of H so that each vertex of A has degree less than the degree of its neighbors. Thus $\delta(G) = 1$. By Proposition 2.1, $\delta(\overline{G}) \leq 2$. Thus $|V(H)| \leq 3$. Assume that $|V(H)| = 3$. Let $V(H) = \{a, b, c\}$. It is obvious that one of the vertices a, b or c has minimum degree in $\overline{G}$. Without loss of generality assume that a is a vertex of minimum degree in $\overline{G}$. If b or c is adjacent to at least two leaves of G, then $\delta(\overline{G}) \leq 3$, a contradiction. Thus each of b and c is adjacent to precisely one leaf of G. Furthermore, a is adjacent to both b and c in G. If a is a strong support vertex in G, then $V(G) - \{a_1, a_2\}$ is a wtds for G, where a_1 and a_2 are leaves adjacent to a. This contradiction implies that a is adjacent to precisely one leaf. Assume that $A \neq \emptyset$. If $|A| > 1$, then $V(G) - A$ is a wtds for G, a contradiction. Thus $|A| = 1$. Let $A = \{a_0\}$. If a_0 is adjacent to both b and c then $V(G) - \{b_1, c_1\}$ is a wtds for G, where b_1 is the leaf adjacent to b, and c_1 is the leaf adjacent to c, a contradiction. Thus we may assume that a_0 is not adjacent to b. Then $V(G) - (A \cup \{b_1\})$ is a wtds for G, where b_1 is the leaf adjacent to b, a contradiction. Thus $A = \emptyset$, and $G = cor(H)$. If $H = P_3$, then $\gamma_{wt}(\overline{G}) < n-1$, a contradiction. Thus $H = C_3$, and so $G = cor(C_3)$.

We next assume that $|V(H)| = 2$. Let $V(H) = \{a, b\}$. If a is not adjacent to b, then G is a disconnected graph with two components $K_{1,1}$ and $K_{1,t}$ for some $t \geq 1$. Then it can be easily seen that $\gamma_{wt}(\overline{G}) < n-1$, a contradiction. Thus a is adjacent to b. We may assume without loss of generality that a is a vertex of minimum degree in $\overline{G}$. Then b is adjacent to at most two leaves. Suppose that b is adjacent to two leaves. Then $V(G) - \{a_1, b_1\}$ is a wtds for G, where a_1 is the leaf adjacent to a, and b_1 is the leaf adjacent to b, a contradiction. Thus b is adjacent to precisely one leaf, and thus $deg(b) = 2$. If $\deg(a) = 2$ then $\gamma_{wt}(\overline{G}) < n-1$, a contradiction. If $\deg(a) \geq 4$, then $V(G) - \{a_1, a_2\}$ is a wtds for G, where a_1 and a_2 are two leaves adjacent to a, a contradiction. Thus $\deg(a) = 3$. Consequently, G is obtained from a star $K_{1,3}$ by subdividing one edge. □

Theorem 3.3. *Let G be a graph of order n. If G and $\overline{G}$ have no isolated vertex then $5 \leq \gamma_{wt}(G) + \gamma_{wt}(\overline{G}) \leq n + \Delta(G) + 2$. These bounds are sharp.*

Proof. For the upper bound, by Proposition 2.1,

$$\begin{aligned}\gamma_{wt}(G)+\gamma_{wt}(\overline{G}) &\leq n+1-\delta(G)+n+1-\delta(\overline{G})\\ &= n+1-\delta(G)+n+1-(n-1-\Delta(G))\\ &= n+1-\delta(G)+2+\Delta(G)\\ &\leq n+\Delta(G)+2.\end{aligned}$$

For the lower bound it is obvious that $\gamma_{wt}(G)\geq 2$ and $\gamma_{wt}(\overline{G})\geq 2$. Assume that $\gamma_{wt}(G)=\gamma_{wt}(\overline{G})=2$. Let $S=\{x,y\}$ be a $\gamma_{wt}(G)$-set, and $D=\{a,b\}$ be a $\gamma_{wt}(\overline{G})$-set. If $a=x$ then y is not dominated by D is $\overline{G}$, a contradiction. Thus $a\neq x$ and thus we may assume that $S\cap D=\emptyset$. Clearly we may assume that a is weakly dominated by x and b is weakly dominated by y. Thus $\deg_G(x)\leq\deg_G(a)$ and $\deg_G(y)\leq\deg_G(b)$. Since S is $\gamma_{wt}(G)$-set, we have

$$\deg_G(a)+\deg_G(b)\geq\deg_G(x)+\deg_G(y)\geq n \tag{1}$$

Since D is a $\gamma_{wt}(\overline{G})$-set we obtain that

$$n\leq\deg_{\overline{G}}(a)+\deg_{\overline{G}}(b)=n-1-\deg_G(a)+n-1-\deg_G(b). \tag{2}$$

(1) and (2) imply that $n\leq\deg_G(a)+\deg_G(b)\leq n-2$, a contradiction.

To see the sharpness of the upper bound consider a path P_4. To see the sharpness of the lower bound consider a double star $S(5,5)$ with centers $\{x,y\}$ and leaves $\{a_i,b_i: i=1,2,3,4\}$, where $\{xa_i: i=1,2,3,4\}\cup\{yb_i: i=1,2,3,4\}\subseteq S(5,5)$. Now add the edges a_1b_2, a_1b_3, a_1b_4, a_2b_1, a_2b_3, a_2b_4, a_3b_1, a_3b_2, a_3b_4, a_4b_1, a_4b_2, a_4b_3, a_1a_2, a_3a_4, b_1b_3, and b_2b_4 to obtain a graph G. It is straightforward to see that $\gamma_{wt}(G)=2$ and $\gamma_{wt}(\overline{G})=3$. □

Similarly we have the following.

Theorem 3.4. *Let G be a graph of order n. If G and $\overline{G}$ have no isolated vertex then $6\leq\gamma_{wt}(G)\gamma_{wt}(\overline{G})\leq n^2$. These bounds are sharp.*

Next we obtain sharp upper and lower bounds for $\gamma_{st}(G)+\gamma_{st}(\overline{G})$ and $\gamma_{st}(G)\gamma_{st}(\overline{G})$. The following are easily verified.

Observation 3.1. *(1) Every std-set of a graph G contains all support vertices of G.*
(2) For any graph G with no isolated vertex, $\gamma_{st}(G)\geq\gamma_t(G)$. Moreover, if G is regular, then $\gamma_{st}(G)=\gamma_t(G)$.

Proposition 3.1. *For paths and cycles, $\gamma_{st}(P_n)=\gamma_{st}(C_n)=\gamma_t(P_n)=\gamma_t(C_n)$.*

For a graph with no isolated vertices, obviously the strong total domination number is bounded above by its order n. Next we improve this upper bound.

Proposition 3.2. *For any graph G of order n, maximum degree Δ and with no isolated vertices, $\gamma_{st}(G)\leq n+1-\Delta$.*

Proof. The result holds if $\Delta(G) = 1$, and thus we assume that $\Delta(G) \geq 2$. Let v be a vertex of V of maximum degree and w be a vertex of $N(v)$. If $N(v) - \{w\}$ has no support vertex, then $S = V - (N(v) - \{w\})$ is a strong total dominating set for G. Therefore, $\gamma_{st}(G) \leqslant |V - (N(v) - \{w\})| = n + 1 - \Delta$. Thus assume that $N(v) - \{w\}$ contains some support vertex. Let T be the set of support vertices of $N(v) - \{w\}$. For each support vertex $x \in N(v) - \{w\}$, let x^* be a leaf adjacent to x. Then $S_1 = ((N(v) - \{w\}) - T) \cup \{x^* : x \in T\}$ is a strong total dominating set for G, and therefore, $\gamma_{st}(G) \leqslant n + 1 - \Delta$. □

Proposition 3.3. *A connected graph G of order $n \geq 2$ has $\gamma_{st}(G) = n$ if and only if $G = K_2$.*

Proof. Let G be a graph with $\gamma_{st}(G) = n$. By Proposition 3.2, we have $\Delta = 1$. Thus the result follows. □

Proposition 3.4. *If G is a connected graph of order $n \geq 3$ then $\gamma_{st}(G) \leq n - 1$ with equality if and only if $G \in \{P_3, C_3\}$.*

Proof. The upper bound follows from Proposition 3.3. Moreover, it is obvious that $\gamma_{st}(P_3) = \gamma_{st}(C_3) = 2 = n - 1$. Let G be a connected graph with $\gamma_{st}(G) = n - 1$. By Proposition 3.2, we find that $\Delta(G) \leq 2$. Since $n \geq 3$ we have $\Delta(G) = 2$. Thus G is a path or a cycle. By Observation 2.1 and Proposition 3.1, we obtain that $n - 1 = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$ and this implies that $n = 3$. Hence $G \in \{P_3, C_3\}$. □

Observation 3.2. *For a graph G, $\gamma_{st}(G) = 2$ if and only if G is a star or a double star.*

Theorem 3.5. *Let G be a graph of order n. If G and $\overline{G}$ have no isolated vertex then $4 \leq \gamma_{st}(G) + \gamma_{st}(\overline{G}) \leq n + 2$. These bounds are sharp.*

Proof. Since G and $\overline{G}$ have no isolated vertex, $\Delta(G) < n-1$ and $\Delta(\overline{G}) < n-1$. Since $\gamma_{st}(G) \geq 2$ and $\gamma_{st}(\overline{G}) \geq 2$, the lower bound follows. To show the sharpness we prove a stronger result. Equality for the lower bound holds if and only if both G and $\overline{G}$ are double-star by Observation 3.2. Thus $G = \overline{G} = P_4$. We next establish the upper bound. By Proposition 3.2,

$$\begin{aligned}
\gamma_{st}(G) + \gamma_{st}(\overline{G}) &\leq n + 1 - \Delta(G) + n + 1 - \Delta(\overline{G}) \\
&\leq n + 1 - \Delta(G) + n + 1 - (n - 1 - \delta(G)) \\
&\leq n + 3 - \Delta(G) + \delta(G) \\
&\leq n + 3.
\end{aligned}$$

If $\gamma_{st}(G) + \gamma_{st}(\overline{G}) = n + 3$ then all inequalities in the above become equalities. In particular, $\Delta(G) = \delta(G)$, and thus G is a regular graph. By Theorem 2.1,

$$n + 3 = \gamma_{st}(G) + \gamma_{st}(\overline{G}) = \gamma_t(G) + \gamma_t(\overline{G}) \leq n + 2,$$

a contradiction. Thus $\gamma_{st}(G) + \gamma_{st}(\overline{G}) \leq n + 2$. To see the sharpness, let $G = mK_2$ for some $m > 1$. □

Similarly we obtain the following.

Theorem 3.6. *Let G be a graph of order n. If G and $\overline{G}$ have no isolated vertex then $4 \leq \gamma_{st}(G)\gamma_{st}(\overline{G}) \leq (n - 2)^2$. Both bounds are sharp for $G = P_4$.*

3.2. Sharp bounds on the strong total domination number in trees

Chellali et al. [3] obtained bounds on the weak total domination number of a tree T in terms of the order and the number of support vertices and leaves of T. Here, we present sharp upper and lower bounds for the strong total domination number of a tree T in terms of the order and the number of support vertices and leaves of T.

Theorem 3.7. *For any tree T of order $n \geq 4$ with l leaves and s support vertices,*

$$\frac{n+2-l}{2} \leq \gamma_{st}(T) \leq \frac{n+s}{2}.$$

These bounds are sharp.

Proof. For the lower bound, by Theorem 2.3, $\gamma_{st}(T) \geq \gamma_t(T) \geq \frac{n+2-l}{2}$. To see the sharpness consider a path P_4. We next establish the upper bound. We proceed by induction on the order n. It is a routine matter to check that if $2 \leq diam(T) \leq 5$ then $\gamma_{st}(T) = \gamma_t(T)$ and thus the result is valid by Theorem 2.2. This establishes the base case. Assume the result is valid for any tree T' of order $n' < n$, and T has n vertices and s support vertices. Let x and y be two leaves with $d(x,y) = diam(T)$. We assume that $diam(T) \geq 6$. We root T at x. Let y_1 be the parent of y, y_2 the parent of y_1, y_3 the parent of y_2, and y_4 the parent of y_3.

Assume first that $\deg(y_2) \geq 3$. Let T_1 be the component of $T - y_1y_2$ containing y_2. Then y_2 is either a support vertex in T_1 or is adjacent to a support vertex. Let S_1 be a minimum std-set for T_1. We may assume that $y_2 \in S$. Then $S_1 \cup \{y_1\}$ is a std-set for T. By the inductive hypothesis

$$\gamma_{st}(T) \leq \gamma_{st}(T_1) + 1 \leq \frac{(n - \deg(y_1)) + s - 1}{2} + 1 = \frac{n - \deg(y_1) + s + 1}{2} < \frac{n+s}{2}.$$

Next assume that $\deg(y_2) = 2$.

If $\deg(y_3) \geq 3$ then we let T_2 be the component of $T - y_2y_3$ containing y_3. Let S_2 be a minimum std-set for T_2. Then $S_2 \cup \{y_1, y_2\}$ is a std-set for T. By the inductive hypothesis

$$\gamma_{st}(T) \leq \gamma_{st}(T_2) + 2 \leq \frac{(n - \deg(y_1) - 1) + s - 1}{2} + 2 = \frac{n - \deg(y_1) + s + 2}{2} \leq \frac{n+s}{2}.$$

Thus we assume that $\deg(y_3) = 2$.

let T_3 be the component of $T - y_3y_4$ containing y_4. Let S_3 be a minimum std-set for T_3. Then $S_2 \cup \{y_1, y_2\}$ is a std-set for T. By the inductive hypothesis

$$\gamma_{st}(T) \leq \gamma_{st}(T_3) + 2 \leq \frac{(n - \deg(y_1) - 2) + s}{2} + 2 = \frac{n - \deg(y_1) + s + 2}{2} \leq \frac{n+s}{2}.$$

To see the sharpness consider a path P_6. □

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