Electronic Journal of Graph Theory and Applications **2** (1) (2014), 66–69

Electronic Journal of
Graph Theory and Applications

The Ramsey numbers of fans versus a complete graph of order five

Yanbo Zhang, Yaojun Chen

Department of Mathematics, Nanjing University, Nanjing 210093, P.R. China

ybzhang@163.com, yaojunc@nju.edu.cn

Abstract

For two given graphs F and H, the Ramsey number $R(F,H)$ is the smallest integer N such that for any graph G of order N, either G contains F or the complement of G contains H. Let F_l denote a fan of order $2l+1$, which is l triangles sharing exactly one vertex, and K_n a complete graph of order n. Surahmat et al. conjectured that $R(F_l,K_n)=2l(n-1)+1$ for $l \geq n \geq 5$. In this paper, we show that the conjecture is true for $n=5$.

Keywords: Ramsey number, fan, complete graph
Mathematics Subject Classification : 05C55

1. Introduction

All graphs considered in this paper are finite simple graphs. Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G and $G-S=G[V(G)-S]$, $N_S(v)$ denotes the set of the neighbors of a vertex v contained in S and $d_S(v)=|N_S(v)|$. If $S=V(G)$, we write $N(v)=N_G(v)$, $N[v]=N(v)\cup\{v\}$ and $d(v)=d_G(v)$. Let K_n be a complete graph of order n and mK_n the union of m vertex-disjoint copies of K_n. A fan of order $2l+1$, denoted by F_l, is the join of K_1 and lK_2, that is l triangles sharing exactly one vertex, where the K_1 is called the center of F_l. For notations not defined here, we follow [1]. Let F and H be two given graphs. The Ramsey number $R(F,H)$ is the smallest integer N such that for any graph G of order N, either G contains F or $\overline{G}$ contains H. For a connected graph F of order p, Burr [2] established a general

Received: 14 October 2013, Revised: 10 March 2014, Accepted: 9 April 2014.

lower bound for $R(F,H)$, that is, $R(F,H) \geq (p-1)(\chi(H)-1)+s(H)$, if $p \geq s(H)$, where $\chi(H)$ is the chromatic number of H and $s(H)$ the minimum number of vertices in some color class under all vertex colorings by $\chi(H)$ colors. For the pair F_l and K_n, noting that $\chi(K_n)=n$ and $s(K_n)=1$, we have $R(F_l,K_n) \geq 2l(n-1)+1$ by Burr's lower bound. Gupta et al. showed the equality holds for $n=3$ and established the following.

Theorem 1.1 (Gupta et al. [3]). $R(F_l,K_3)=4l+1$ *for* $l \geq 2$.

Surahmat et al. proved the equality also holds for $n=4$ and obtained the following.

Theorem 1.2 (Surahmat et al. [5]). $R(F_l,K_4)=6l+1$ *for* $l \geq 3$.

Maybe motivated by Theorems 1.1 and 1.2, Surahmat et al. conjectured that the equality holds in a more general case in the same paper, and posed the following.

Conjecture 1 (Surahmat et al. [5]). $R(F_l,K_n)=2l(n-1)+1$ *for* $l \geq n \geq 5$.

Other results on Ramsey numbers of fans versus complete graphs can be found in the dynamic survey [4]. In this paper, we will confirm Conjecture 1 for $n=5$. The main result of this paper is as below.

Theorem 1.3. $R(F_l,K_5)=8l+1$ *for* $l \geq 5$.

2. Proof of Theorem 1.3

Since $4K_{2l}$ contains no F_l and its complement contains no K_5, $R(F_l,K_5) \geq 8l+1$. In the following, we need only to show that $R(F_l,K_5) \leq 8l+1$.

Let G be a graph of order $8l+1$ with $l \geq 5$, we need to show that either G contains an F_l or $\overline{G}$ contains a K_5. Suppose to the contrary that neither G contains an F_l nor $\overline{G}$ contains a K_5.

Let $v \in V(G)$. If $d(v) \leq 2l-1$, then $G-N[v]$ is a graph of order at least $6l+1$. By Theorem 1.2, $\overline{G}-N[v]$ contains a K_4, which implies that $\overline{G}$ contains a K_5, a contradiction. If $d(v) \geq 2l+3$, then a maximum matching M of $G[N(v)]$ contains at least l edges for otherwise $\overline{G}[N(v)-V(M)]$ is a complete graph of order at least 5, which implies that G has an F_l, a contradiction. Therefore, $2l \leq d(v) \leq 2l+2$ for any $v \in V(G)$.

Suppose that G contains a subgraph $H=K_{2l-1}$. Choose $v_0 \in V(G)-V(H)$ such that $d_H(v_0)=\max\{d_H(v) \mid v \in V(G)-V(H)\}$. Obviously, $G-(V(H)\cup\{v_0\})$ is a graph of order $6l+1$. By Theorem 1.2, $G-(V(H)\cup\{v_0\})$ contains an independent set $\{u_1,u_2,u_3,u_4\}$. Since $\overline{G}$ has no K_5, we have $V(H)\cup\{v_0\} \subseteq \cup_{i=1}^{4} N(u_i)$. This implies that $\max\{d_H(u_i) \mid 1 \leq i \leq 4\} \geq \lceil (2l-1)/4 \rceil \geq 3$. By the choice of v_0, we have $d_H(v_0) \geq 3$. If $d_H(v_0) \geq 4$, then there is some u_i having at least two neighbors in $N_H(v_0)\cup\{v_0\}$; if $d_H(v_0)=3$, then $d_H(u_i) \leq d_H(v_0)=3$ for $1 \leq i \leq 4$, which implies that there exists some u_i such that $d_H(u_i) \geq 2$ and $N_H(u_i)\cap N_H(v_0) \neq \emptyset$. In both cases, $G[V(H)\cup\{v_0,u_i\}]$ contains an F_l, a contradiction. Hence, G contains no K_{2l-1}.

By Theorem 1.2, G has an independent set $U=\{u_1,u_2,u_3,u_4\}$. For $1 \leq i \leq 4$, set $X_i=\{v \mid d_U(v)=i,\ v \in V(G)\}$. Obviously,

$$\sum_{i=1}^{4} |X_i| = 8l-3, \tag{1}$$

www.ejgta.org

$$\sum_{i=1}^{4} i|X_i| = \sum_{i=1}^{4} d(u_i). \tag{2}$$

Since $\sum_{i=1}^{4} d(u_i) \leq 8l+8$, by (1) and (2), we have

$$|X_1| \geq 8l-14+|X_3|+2|X_4| \geq 8l-14. \tag{3}$$

Let $X_{1i} = N_{X_1}(u_i)$ for $1 \leq i \leq 4$. Because $\overline{G}$ has no K_5, $G[X_{1i} \cup \{u_i\}]$ is a complete graph. Since G contains no K_{2l-1}, we have $|X_{1i} \cup \{u_i\}| \leq 2l-2$, which implies that $|X_{1i}| \leq 2l-3$ for $1 \leq i \leq 4$. Thus, $|X_1| = \sum_{i=1}^{4} |X_{1i}| \leq 8l-12$. By (3), we have $|X_3|+2|X_4| \leq 2$. By (1),

$$|X_2| \geq 7. \tag{4}$$

Assume without loss of generality that $|X_{11}| \geq |X_{12}| \geq |X_{13}| \geq |X_{14}|$. Then $|X_{11}| = |X_{12}| = 2l-3$, $|X_{13}|+|X_{14}| \geq 4l-8$ and $|X_{14}| \geq 2l-5$. Denote by U_i both the vertex set $X_{1i} \cup \{u_i\}$ and the graph $G[X_{1i} \cup \{u_i\}]$ for $1 \leq i \leq 4$, then U_1, U_2, U_3, U_4 are pairwise vertex-disjoint complete graphs with $|U_1| = |U_2| = 2l-2$, $|U_3|+|U_4| \geq 4l-6$ and $|U_4| \geq 2l-4$.

Let $Y_{ij} = N_{X_2}(u_i) \cap N_{X_2}(u_j)$ for $1 \leq i < j \leq 4$.

Claim 1. If $|U_i| = 2l-2$ for some i with $1 \leq i \leq 4$, then for any $y \in Y_{ij}$, $d_{U_j}(y) \geq 3$ and if $|U_i| = |U_j| = 2l-2$, then $Y_{ij} = \emptyset$.

Proof. Since G contains no K_{2l-1}, $U_i - N(y) \neq \emptyset$. In this case, $G[U_i \cup U_j - N(y)]$ is a complete graph for otherwise any two nonadjacent vertices in $G[U_i \cup U_j - N(y)]$ together with $U \cup \{y\} - \{u_i, u_j\}$ form a K_5 in $\overline{G}$, a contradiction. Since G has no F_l and both U_i and U_j are complete graphs, we have $d_{U_j}(u) \leq 3$ for any $u \in U_i$, which implies that $|U_j - N(y)| \leq 3$. Noting that $|U_j| \geq 2l-4$ and $l \geq 5$, we have $d_{U_j}(y) \geq |U_j| - |U_j - N(y)| \geq (2l-4)-3 \geq 3$.

If $|U_i| = |U_j| = 2l-2$ and $Y_{ij} \neq \emptyset$, then for any $y \in Y_{ij}$, $d_{U_i}(y) + d_{U_j}(y) \leq 2l$ since otherwise $G[N[y]]$ contains an F_l with y as center, a contradiction. Thus we have $|U_i \cup U_j - N(y)| \geq |U_i| + |U_j| - 2l \geq 6$ since $l \geq 5$. By the arguments in the first part, we have $|U_i - N(y)| = |U_j - N(y)| = 3$. Thus, $G[U_i \cup (U_j - N(y))]$ contains an F_l with u as center for any $u \in U_i - N(y)$, a contradiction. Hence $Y_{ij} = \emptyset$. □

If $|U_4| = 2l-2$, then by Claim 1, $X_2 = \cup_{1 \leq i < j \leq 4} Y_{ij} = \emptyset$ which contradicts (4). Hence we have $2l-4 \leq |U_4| \leq 2l-3$.

Assume $|U_3| = 2l-2$. By Claim 1, we have $X_2 = \cup_{1 \leq i < j \leq 4} Y_{ij} = Y_{14} \cup Y_{24} \cup Y_{34}$, that is, $X_2 \subseteq N(u_4)$. If $|U_4| = 2l-3$, then since $\sum_{i=1}^{4} d(u_i) \leq 8l+8$, by (1), (2) and (3), either $|X_2| = 10$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^{4} d(u_i) = 8l+7$ or $|X_2| = 9$, $|X_3| = 1$, $|X_4| = 0$ and $\sum_{i=1}^{4} d(u_i) = 8l+8$. If $|U_4| = 2l-4$, then for the same reason, we have $|X_2| = 11$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^{4} d(u_i) = 8l+8$. Thus we have $|X_2| \geq 9$ in both cases, which implies that $d(u_4) \geq |X_{14}| + |X_2| \geq 2l-5+9 = 2l+4$, a contradiction. Therefore, $|U_3| \leq 2l-3$.

Since $|U_3|+|U_4| \geq 4l-6$ and $|U_4| \leq 2l-3$, we are now left to consider the case when $|U_3| = |U_4| = 2l-3$. Since $\sum_{i=1}^{4} d(u_i) \leq 8l+8$, by (1), (2) and (3), we have $|X_2| = 11$, $|X_3| = |X_4| = 0$ and $\sum_{i=1}^{4} d(u_i) = 8l+8$, which implies $d_{X_2}(u_4) = 6$. Let $N_{X_2}(u_4) = \{y_i \mid 1 \leq i \leq 6\}$. Since $\overline{G}$ contains no K_5, $G[N_{X_2}(u_4)]$ contains at least one edge, say $y_1y_2 \in E(G)$. Since G has no F_l,

$G[\{y_3, y_4, y_5, y_6\}]$ contains no edge. Because $\overline{G}$ has no K_5, we have $|\{y_3, y_4, y_5, y_6\} \cap (N(u_1) \cup N(u_2))| \geq 2$. Assume that $\{y_3, y_4\} \subseteq N(u_1) \cup N(u_2)$. By Claim 1, $d_{U_4}(y_3) \geq 3$ and $d_{U_4}(y_4) \geq 3$, which implies that $d_{X_{14}}(y_3) \geq 2$ and $d_{X_{14}}(y_4) \geq 2$. In this case, there exist $u', u'' \in X_{14}$ such that $u'y_3, u''y_4 \in E(G)$, which implies that $G[U_4 \cup \{y_1, y_2, y_3, y_4\}]$ contains an F_l with u_4 as center, a contradiction.

The proof of Theorem 1.3 is completed.

Acknowledgement

This research was supported by NSFC under grant numbers 11071115, 11371193 and 11101207, and in part by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory*, *Springer*, 2008.

[2] S.A. Burr, Ramsey numbers involving graphs with long suspended paths, *Journal of London Mathematical Society* **24** (1981), 405–413.

[3] S.K. Gupta, L. Gupta and A. Sudan, On Ramsey numbers for fan-fan graphs, *Journal of Combinatorics, Information & System Sciences* **22** (1997), 85–93.

[4] S.P. Radziszowski, Small Ramsey numbers, *The Electronic Journal of Combinatorics*, (2011), DS1.13.

[5] Surahmat, E.T. Baskoro and H.J. Broersma, The Ramsey numbers of fans versus K_4, *Bulletin of the Institute of Combinatorics and its Applications* **43** (2005), 96–102.